Math 1a Practice Final Solutions.

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Note: Each of the questions will be graded out of 10. This exam is probably longer than half the length of the actual final. No guarantees are given as to either the correctness of any of the solutions, or the similarity of the questions to the questions on the real final.

1. Find the local maxima and minima and the intervals of increase and decrease of the function

$$f(x) = \frac{2x-5}{x^2-4}.$$

By the quotient rule,

$$f'(x) = \frac{-2x^2+10x-8}{(x^2-4)^2} = \frac{-2(x-1)(x-4)}{(x^2-4)^2}.$$

The critical numbers are $-2, 1, 2, 4$. By determining the sign of $f'(x)$ between adjacent critical numbers, we find that the intervals of increase are $(1,2), (2,4)$, while the intervals of decrease are $(-\infty,-2), (-2,1), (4,\infty)$. By the first derivative test, $f(1) = 1$ is a local minimum and $f(4) = 1/4$ is a local maximum.

[Grade scheme: Correct $f'(x)$ (3); correct intervals of increase and decrease (1 each); local minimum at $x = 1$ (1); local maximum at $x = 4$ (1); forgetting to compute or incorrectly computing $f(1)$ and/or $f(4)$ (minus 1); any extraneous local maxima or minima (minus 1).]

2. Suppose $x^2 + y^2 = r^2$, $x = A/w$, $dy/dt = -2$, $dr/dt = 2$, and $dw/dt = -1/2$. Find dA/dt when $r = 5$, $A = 4$, $w = 1$, and $y > 0$.

The two equations give $x = 4$ and $y = 3$ (not -3 since $y > 0$). Implicitly differentiating the two equations gives $2x\ dx/dt + 2y\ dy/dt = 2r\ dr/dt$ and $dx/dt = (w\ dA/dt - A\ dw/dt)/w^2$. The first gives $dx/dt = (r\ dr/dt - y\ dy/dt)/x = 4$. The second gives $dA/dt = (w^2\ dx/dt + A\ dw/dt)/w = 2$.

[Grade scheme: Correct values for x and y (1 each); differentiating the equations correctly (2 each); correct values for dx/dt and dA/dt (2 each) (all 4 points if dA/dt is computed correctly by another route).]

3. Evaluate the limit, if it exists (possibly as an infinite limit):

$$\lim_{x\to 1^+} \frac{|x^2-3x+2|}{x^2-1}.$$

Note $x^2 - 3x + 2 = (x-1)(x-2)$. For x slightly larger than 1, $x-1$ is positive and $x-2$ is negative, so x^2-3x+2 is negative. Hence

$$\begin{aligned}
&\lim_{x\to 1^+} \frac{|x^2-3x+2|}{x^2-1} \\
=&\lim_{x\to 1^+} \frac{-(x^2-3x+2)}{x^2-1} \\
=&\lim_{x\to 1^+} \frac{-(x-1)(x-2)}{(x-1)(x+1)} \\
=&\lim_{x\to 1^+} \frac{-(x-2)}{x+1} \\
=&\frac{1}{2}.
\end{aligned}$$

[Grade scheme: Explaining convincingly that x^2-3x+2 is negative for x slightly larger than 1 (3); replacing $|x^2-3x+2|$ with $-(x^2-3x+2)$ (2); computing the resulting limit correctly (5) (using L'Hospital is ok).]

4. Find a and b so that the function

$$f(x) = \begin{cases} x^4 + 3x^3 + 5x^2 - 2 & \text{if } x \le -1 \\ ax + b & \text{if } x > -1 \end{cases}$$

is differentiable at $x = -1$.

For f to be differentiable at -1, it must first be continuous at -1. So $\lim_{x\to -1^+} f(x) = f(-1)$, giving us $-a+b = 1$. Next, f's one-sided derivatives at -1 must agree. Coming from the left, the derivative is $4x^3+9x^2+10x$; plugging in -1 yields -5. Coming from the right, the derivative is a. Hence $a = -5$. Since $-a+b = 1$, $b = -4$.

[Grade scheme: Observing that f is continuous at $x = -1$ (2); correct value for $\lim_{x\to -1^+} f(x)$ (1); correct value for $f(-1)$, or equivalently $\lim_{x\to -1^-} f(x)$ (1); observing that f's one-sided derivatives at $x = -1$ must agree (2); correct values for the one-sided derivatives (1 each); correct values for a and b (1 each).]

5. Differentiate the function $f(x) = x^{x^3} \sin x$.

Set $y = x^{x^3}\sin x$. Taking the ln of both sides of $|y| = |x^{x^3}\sin x|$ to get $\ln|y| = x^3 \ln|x| + \ln|\sin x|$. Take $\frac{d}{dx}$ of both sides to get $\frac{1}{y}\cdot\frac{dy}{dx} = 3x^2\ln|x| + x^2 + \frac{\cos x}{\sin x}$. Multiply both sides by y and plug in $y = x^{x^3}\sin x$ to get

$$\frac{dy}{dx} = x^{x^3}\sin x\left(3x^2\ln|x| + x^2 + \frac{\cos x}{\sin x}\right).$$

Here the absolute values can be removed because $x > 0$ in the domain.

[Grade scheme: Writing $y = x^{x^3}\sin x$ with the intention of applying natural log (2); taking the ln of the absolute value of both sides, with the correct simplification (3); differentiating both sides correctly (3); solving for dy/dx correctly in terms of x and y (1); plugging back in $y = x^{x^3}\sin x$ (1); doing everything without absolute values (minus 1). It's ok to compute the derivative of x^{x^3} via logarithmic differentation and then compute $f'(x)$ by the product rule. In this case, give 8 points for computing $\frac{d}{dx}x^{x^3} = x^{x^3}(3x^2\ln|x| + x^2)$ and 2 points for the correct $f'(x)$.]

6. If $(x+y)^2 + (y-4)^2 = 25$, express dy/dx in terms of x and y.

   Take $\frac{d}{dx}$ of both sides to get $2(x+y)(1+dy/dx) + 2(y-4)dy/dx = 0$. Move all multiples of dy/dx to one side to get $2(x+2y-4)dy/dx = -2(x+y)$. Thus

$$\frac{dy}{dx} = -\frac{x+y}{x+2y-4}.$$

   [Grade scheme: Computing $\frac{d}{dx}(x+y)^2$ correctly (2); computing $\frac{d}{dx}(y-4)^2$ correctly (2); computing $\frac{d}{dx}25$ correctly (2); solving for dy/dx correctly (4).]

7. What happens when you apply Newton's method to the equation $x^2 + 2x + 4 = 0$ with initial guess $x_0 = 0$? Why would we expect the method to not find a solution of the equation? [Hint: What are the solutions?]

   $f(x) = x^2 + 2x + 4 = 0$, so $f'(x) = 2x + 2$. So, the iteration is

$$x_{n+1} = x_n - \frac{x_n^2 + 2x_n + 4}{2x_n + 2}$$

   Generating the first few values we observe: $x_0 = 0$; $x_1 = -2$; $x_2 = 0$; .... So, we get "bouncing" between two values, we don't get closer to any root.

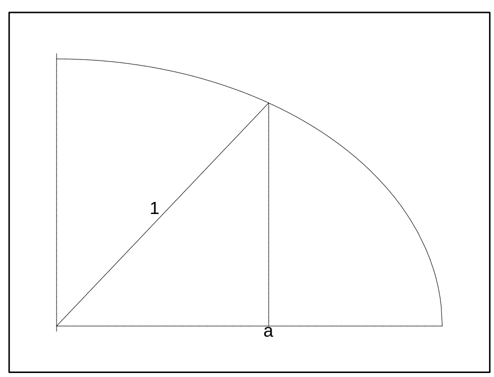
   We expect the method not to work, as $x^2+2x+4 = 0$ has no roots: the discriminant of the quadratic is $4 - 16 < 0$.

   [Grade scheme: Describing the bouncing (5); saying that the equation has no root (5). If bouncing isn't described, 2 points are available for correctly finding the iteration.]

8. **By considering areas**, evaluate $\int_0^a \sqrt{1-x^2}\,dx$ when $0 < a < 1$ (in terms of a).

   We break the area up into a triangle and a slice of a circle. The triangle has base a and height $\sqrt{1-a^2}$, so has area $a\sqrt{1-a^2}/2$. The angle of the slice is the same as the top angle of the triangle (by a fact about geometry sometimes called the Z-rule), which is $\arcsin(a)$ by trig. Hence, the total area is

$$\frac{1}{2}a\sqrt{1-a^2} + \frac{1}{2}\arcsin(a).$$



   [Grade scheme: a good sketch (2); splitting the area into a triangle and a slice (2); correctly finding the area of the triangle (2); correctly finding the area of the slice (4).]

9. Find $\frac{d}{dx}\int_1^{x^2+1} \cos(\frac{1}{t})\,dt$.

   We use chain rule and Part II of the FTC to get:

$$2x\cos(\frac{1}{x^2+1})$$

[Grade scheme: having $\cos(\frac{1}{x^2+1})$ written down (4); having multiplied this by something (1); this thing being $2x$ (4); having nothing else written as part of the final derivative (1).]

10. Use a substitution to find $\int \frac{x^2}{(x+1)^{10}}\,dx$.

We use the substitution $u = x + 1$. Hence, $du = dx$. So, the given integral is equal to

$$\begin{aligned}\int \frac{(u-1)^2}{u^{10}}\,du &= \int (u^{-8} - 2u^{-9} + u^{-10})\,du \\ &= \frac{-1}{7}u^{-7} + \frac{2}{8}u^{-8} - \frac{1}{9}u^{-9} + C \\ &= \frac{-1}{7}(x+1)^{-7} + \frac{2}{8}(x+1)^{-8} - \frac{1}{9}(x+1)^{-9} + C\end{aligned}$$

[Grade scheme: picking the right substitution (4); correctly determining du (2); integrating correctly (3 or 2 if the $+C$ has been missed off); substituting back into 'x's (1).]

11. By recognizing it as a Riemann sum, evaluate

$$\lim_{n\to\infty} \frac{\pi}{n}(\sin(\pi/n) + \sin(2\pi/n) + \sin(3\pi/n) + \cdots + \sin(\pi))$$

Comparing this to the formula of a Riemann sum, we recognize this as equal to

$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= [-\cos(x)]_0^\pi \\ &= 1 + 1 = 2\end{aligned}$$

[Grade scheme: getting correct limits in the definite integral (2); getting the function write (3); evaluating correctly (5, take off 2 for an arithmetic error).]

12. State the Mean Value Theorem for integrals. For $\int_0^1 (x^2 + A)\,dx$, where A is some constant, find a c which satisfies the conclusion.

The MVT for integrals states that if f is a continuous function then for any $a < b$ there is a c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In the case given, $b - a = 1$ and $\int_0^1 x^2 + A\,dx = 1/3 + A$, so $c = 1/\sqrt{3}$ works.

[Grade scheme: the statement of the MVT involving the existence of a c in $[a, b]$ (2); having f continuous as a condition (2); having the conclusion right (2); correctly finding the integral (2); correctly finding c (2); if $-1/\sqrt{3}$ is suggested (and not rejected) as a possible value for c, deduct one mark.]