Math 1a - revision notes for the second midterm.

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This is the bare minimum of stuff you need to know and be able to do for the second midterm. A great lecturer I once had[1] told me, "It's not enough to read the notes, you have to hear the music." This may sound corny, but it's true. If you read these notes, please read them actively. This means writing summaries as you go along and doing exercises.

1 Differentiation techniques.

This is the stuff that is mostly covered in chapter three of Stewart's book. You should already know the product rule and quotient rule and how to differentiate powers and exponentials from the first midterm.

1.1 The chain rule.

This is covered in Stewart's section 3.5. There are three rules in calculus that tell you how to differentiate functions more complicated from ones you already know how to deal with: the quotient rule, product rule and chain rule. What the quotient rule and product rule apply to is kind of obvious from the name. Not so with the chain rule.

The chain rule applies to composite functions: functions of a function. If we have $f(x) = g(h(x))$ then the chain rule tells us:

$$f'(x) = g'(h(x))h'(x)$$

Exercise 1.1. *Cover up the above formula and write down what $r'(x)$ is in terms of q and p, where $r(x) = q(p(x))$ and q, p are functions.*

The way I remember the chain rule is that I draw brackets in sensible places and say to myself, "Differentiate g as if we were differentiating with respect to the thing inside the brackets ($h(x)$), then differentiate $h(x)$. Multiply the two things I ended up with."

Exercise 1.2. *Differentiate the following functions: $e^{(x^2)}$, $\log(x^-1)$, e^{1+x^e}.*

Now, do some exercises from Stewart's 3.5.

1.2 Implicit differentiation.

Implicit differentiation means differentiating an equation with more than one variable in it. To do this, we need to apply all our differentiation rules. If we're differentiating with respect to x, we pretend all other variables are functions of x (though they're not): this allows us to use the chain rule.

[1] Peter Neumann, possibly the best math lecturer in the world and (incidentally) the first person to say to me "Let $\varepsilon > 0$ be given..."

Take the following example: we have $x^2 + y^3 = e^x$ and we differentiate with respect to x. We get the following:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(e^x)$$
$$2x + 3yy' = e^x$$

Where y' means $\frac{dy}{dx}$.

Exercise 1.3. *Label the above calculation with which rule has been used at which step*

We could then solve the second line for y' to find out what this was.

Exercise 1.4. *Do this. What does it represent?*

A big use of implicit differentiation is in finding the derivatives of inverse functions. Consider say, log. log is defined as the inverse of the exponential function. So, if $y = \log x$, then $e^y = x$. If we want to find y', we use implicit differentiation on this second equation and then solve for y':

$$y'e^y = 1$$
$$y' = 1/e^y$$
$$y' = 1/x$$

Exercise 1.5. *Label the above calculation with what has been done at each step.*

1.3 Trigonometric functions.

You need to know the derivatives of sine and cosine.

Exercise 1.6. *What are they?*

You need to have written down on your sheet of notes the derivatives of the other trigonometric functions. Of course, these can all be derived from the derivative of sine and cosine.

Exercise 1.7. *Recall:* $\tan(x) = \frac{\sin(x)}{\cos(x)}$. *Use you answers to the previous exercise and the quotient rule to find the derivative of the tangent function.*

There are also two limits you need to know:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$
$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

If you have to evaluate any limit involving trig functions, convert the trig functions into sines and cosines and try and manipulate the fractions to get them into a form where you can use either these rules or earlier limit laws.

If you don't recall how to do this, look back over the notes we made in class when we had exercises like this.

Exercise 1.8. *Use implicit differentiation to find the derivate of* $\arcsin(x)$.

1.4 Logarithmic differentiation.

We found the derivative of the log function above. By combining this knowledge with the technique of implicit differentiation, we can differentiate a whole new class of functions.

Exercise 1.9. *The following are all functions of x. Are they power functions, exponential functions or neither:* 2^x, x^3, x^2x^5, $(e^x)^x$, x^x.

Any function which is of the form (some function of x) to the power of (some function of x) has to be handled using logarithmic differentiation. The simplest example is x^x. If we give something a name, sometimes we can do something to it, so let's say $y = x^x$. The first step of logarithmic differentiation is in the name: we take logs. So,

$$\log y = x \log x$$

The second step is in the name too, differentiate (implicitly):

$$\frac{y'}{y} = \log x + 1$$

Then, solve for y':

$$y' = (1 + \log x)y$$

But, we're not finished. We only introduced y to make the problem easier. It wasn't in the original question. But that's OK, we know that $y = x^x$, so we can substitute in to get the answer.

$$y' = (1 + \log x)x^x$$

Exercise 1.10. *Find the derivate of* $(1 + x^2)^{(1+x)}$.

1.5 Hyperbolic functions.

The hyperbolic functions are defined by:

$$\cosh x := \frac{e^x + e^{-x}}{2} \qquad\qquad \sinh x := \frac{e^x - e^{-x}}{2}$$

Exercise 1.11. *Find their derivatives*

By analogy with the trig functions, things like tanh and arcsech can be defined. The derivatives of the non-arc ones are found using quotient rule. The derivatives of the arc ones can be found used implicit differentiation.

2 Simple applications.

Stewart's book is a bit strange, in that he ends Chapter 3 (Differentiation rules) with a short section on applications, rather than including that in Chapter 4 (Applications of Differentiation).

All I can say about related rates is the following:

- If it's a word problem, draw a picture.

- Once you've got a picture, create a formula.
- In the first formula you write down, ONLY put in numbers if they represent things that don't change while the process occurs.
- Differentiate.
- Now put in whatever numbers you need to.

The best way to learn about these problems is to do them.

The other important application is to linear approximations. This means finding the tangent line at a point. We can work this out as a formula: the tangent line to $y = f(x)$ at a point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a)$$

Exercise 2.1. *Do lots of exercises from 3.10 and 3.11 in Stewart. It's the only way to learn these.*

3 Other stuff.

There is lots of other stuff to know for the second midterm. Lots of it resides in chapter 4 of Stewart's book. I've been through chapter 3 for you and done two things:

- Summarized the information (possibly in a different order).
- Worked out some exercises to do.

There is only one exercise for this section:

Exercise 3.1. *Do this process for Chapter 4, sections 1-5.*

Good luck!