Math 1B - Quiz Solutions.

Adam Booth.

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Quiz 1 – 1/25/06.

1. (Basic skills, 4 points.) Fill in the blanks to make true identities in the following four equations:

$$\sin^2 x + \underline{\cos^2 x} = 1$$
$$\tan^2 x + \underline{1} = \sec^2 x$$
$$\tfrac{1}{2}\sin \underline{2x} = \sin x \, \cos x$$
$$\sin^2 x = \frac{1 \underline{-} \cos 2x}{2}$$

2. (Idea of Integration by Parts, 3 points.)

   (a) Give the formula for integration by parts:

   $$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx.$$

   (b) Pick what to take as u and what to take as $\frac{dv}{dx}$ in these examples:

      i. $\int x \sin x \, dx$;
      $u = x$, $\frac{dv}{dx} = \sin x$ (x is simplifying).

      ii. $\int e^{3x} \sin(x+1)\, dx$;
      Either way around works (they are both cyclic).

      iii. $\int \arctan(x)\, dx$.
      $u = \arctan(x)$, $\frac{dv}{dx} = 1$. (arctan is simplifying).

3. (Integration by Parts Calculation, 3 points.) First make a substitution and then use Integration by Parts to evaluate

$$\int e^{\sqrt{x}}\, dx.$$

We could either make the substitution $t = \sqrt{x}$ or $t = e^{\sqrt{x}}$. I'll show you how to do the former.

If $t = \sqrt{x}$, then $dt = \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2t}\, dx$. So, $dx = 2t\, dt$. Hence, we have

$$\int e^{\sqrt{x}}\, dx = \int 2te^t\, dt.$$

We now use integration by parts, taking $u = 2t$, $\frac{dv}{dx} = e^t$. This gives $\frac{du}{dt} = 2$ and $v = e^t$. Hence,

$$\begin{aligned}\int 2te^t \, dt &= 2te^t - \int 2e^t \, dt \\ &= 2te^t - 2e^t + C \\ &= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.\end{aligned}$$

Quiz 2 – 2/1/06.

1. (Basic skills, 3 points.) Compute the following limits. Be sure to distinguish between “does not exist” and ”infinity”.

   (a) $\lim_{x\to\infty} \sin x$.
   Does not exist.

   (b) $\lim_{x\to\infty} \frac{x}{e^x}$.
   0: “Exponentials beat powers of x.” To check this, one could use L’Hospital’s rule.

   (c) $\lim_{x\to\infty} \frac{x}{\ln x}$.
   ∞: “Powers of x beat logs.” To check this, one could use L’Hospital’s rule.

2. (Idea of trig integrals, 3 points.) For each of the integrals below, give a substitution you could use to evaluate it. Also give the most important trig identity you’d use in your calculations.

   (a) $\int \sin^3 x \cos x \, dx$.
   Use $\sin^2 x = 1 - \cos^2 x$, then substitute $u = \cos x$.

   (b) $\int \sec^4 x \, dx$.
   Use $\sec^2 x = 1 + \tan^2 x$, then substitute $u = \tan x$.

   (c) $\int \frac{x^3}{\sqrt{2-x^2}}$.
   Substitute $x = \sqrt{2}\sin\theta$ and then use $1 - \sin^2\theta = \cos^2\theta$.

3. (Integration by Partial Fractions Calculation, 4 points.) Use the technique of partial fractions to evaluate

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\, dx.$$

The fraction is bottom-heavy and the denominator is as factored as we can, so we begin by finding the partial fractions decomposition. Its form will be

$$\frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}.$$

To evaluate the constants, we mutliply through by the denominator to get

$$-x^3 + 2x^2 - x + 1 \equiv A(x^2+1)^2 + (Bx + C)x(x^2+1) + (Dx + E)x.$$

Substituting $x = 0$ allows us to find that $A = 1$. We also note that the only way we get an x^3 term on the right hand side is $C \cdot x \cdot 1$, so $C = -1$. Now, we know A and C, we can find the others either by looking at coefficients or substituting in random numbers. Equating x^4 coefficients gives $0 = A + B$, so $B = -1$ and equating x coefficients give $C + E = -1$, so $E = 0$. Considering any other coefficient, or plugging in any other value for x will let us find that $D = 1$. So,

$$\begin{aligned}\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\, dx &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\, dx - \int \frac{dx}{x^2+1} + \int \frac{x}{(x^2+1)^2}\, dx \\ &= \ln(x) - \tfrac{1}{2}\ln(x^2+1) - \arctan(x) - \frac{1}{2(x^2+1)} + K\end{aligned}$$

[We’ve already used C to mean something else in this calculation, so we follow Leibniz and use K (from the German *Konstant*) above.]

Quiz 3 – 2/8/06.

1. ('Basic' skills, 3 points.) Evaluate the following integral:

$$\int \frac{x+8}{x^2+2x+4}.$$

$$\begin{aligned}
\int \frac{x+8}{x^2+2x+4}\,dx &= \int \frac{x+1+7}{(x+1)^2+3}\,dx && v = x+1 \\
& && dv = dx \\
&= \int \frac{v+7}{v^2+3}\,dv \\
&= \int \frac{v}{v^2+3}\,dv + \int \frac{7}{v^2+3}\,dv && u = v^2+3,\ v = \sqrt{3}\tan\theta \\
& && du = 2v dv,\ dv = \sqrt{3}\sec^2\theta d\theta \\
&= \tfrac{1}{2}\int \frac{du}{u} + \tfrac{7}{3}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta \\
&= \tfrac{1}{2}\ln|u| + \tfrac{7}{3}\arctan\frac{v}{\sqrt{3}} + C \\
&= \tfrac{1}{2}\ln(x^2+2x+9) + \tfrac{7}{3}\arctan\frac{x+1}{\sqrt{3}} + C
\end{aligned}$$

2. (Ideas from improper integrals, 3 points.)

   (a) If f is a continuous function, give the definition of $\int_0^\infty f(x)\,dx$.

   $$\int_0^\infty f(x)\,dx := \lim_{t\to\infty}\int_0^t f(x)\,dx.$$

   (b) If g is continuous on $(3,4]$, give the definition of $\int_3^4 g(x)\,dx$.

   $$\int_3^4 g(x)\,dx := \lim_{t\to 3+}\int_t^4 g(x)\,dx.$$

   (c) Give a clear statement of the comparison theorem. It is OK if your statement only applies to Type I integrals, as in the book.
   If $f(x)$ and $g(x)$ are functions satisfying $0 \le f(x) \le g(x)$ for all $x \in [a,\infty]$, then if $\int_0^\infty g(x)\,dx$ converges then so too does $\int_0^\infty f(x)\,dx$. Conversely, if $\int_0^\infty f(x)\,dx$ diverges, then so too does $\int_0^\infty g(x)\,dx$.

3. (Improper integral examples, 4 points.)

   (a) Verify that $\int_1^\infty \frac{1}{x}\,dx$ diverges and $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

   $$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}\ln(x)]_1^t = \infty.$$

   $$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\frac{-1}{x}]_1^t = 1.$$

(b) Using the comparison test and the integrals from (a), test the following integrals for convergence:

i. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$.
$0 \le \sin^2 x \le 1$, so we have $0 \le \sin^2 x/x^2 \le 1/x^2$. We showed above that $\int_1^\infty x^{-2}\,dx$ converges, so by the comparison theorem, $\int_1^\infty \sin^2 x/x^2\,dx$ does too.

ii. $\int_1^\infty \frac{x+\ln x}{x^2}\,dx$.
For any $x \ge 1$, we have $\ln x > 0$, so $(x+\ln x)/x^2 > x/x^2 = x^{-1} > 0$. But, $\int_1^\infty x^{-1}\,dx$ diverges so the integral we're interested in does too.

Quiz 4 – 2/24/06.

1. (Basic skills, 3 points.) Test the following integral for convergence. [Remember, log grows slower than any power of x.]

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$\ln(x) < x^{1/2}$ for all $x \geq 1$, so $\frac{\ln(x)}{x^2} < \frac{1}{x^{3/2}}$. $\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent so, by comparison theorem (for integrals), so is $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$. [You can compare $\ln(x)$ with any power of x between 0 and 1.]

2. (Ideas about sequences and series, 3 points.)

   (a) Give a clear definition of what it means to say that $\sum_{n=1}^\infty a_n = 17$. [Your target audience should be some-one who understands Σ-notation and what it means to say that a sequence converges, but has never seen series before.]

   The sequence $(\sum_{i=1}^n a_i)$ converges and has limit 17 (as $n \to \infty$).

   (b) Give an example of a sequence (a_n) such that (a_n) is convergent, but $\sum_{n=1}^\infty a_n$ isn't.

   The constant sequence $a_n = 1$ converges (to 1), but the sequence of partial sums, $s_n = n$, diverges. Hence, $\sum_{n=1}^\infty a_n$ diverges. [Many other examples work here.]

3. (Series calculations, 4 points.)

   (a) Find a nice concise form (no Σs or $\cdots$s) for the nth partial sum of

$$\sum_{n=1}^\infty \ln\left(\frac{n+1}{n}\right).$$

   $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$. And,

$$\begin{aligned}\sum_{i=1}^n [\ln(i+1) - \ln(i)] &= \cancel{\ln(2)} - \ln(1) + \cancel{\ln(3)} - \cancel{\ln(2)} + \cdots + \ln(n+1) - \cancel{\ln(n)} \\ &= \ln(n+1).\end{aligned}$$

   (b) Hence determine whether or not the above series converges. If it does, evaluate it.

   $\ln(n+1) \to \infty$ as $n \to \infty$, so the series diverges.

Quiz 5 – 3/1/06.

1. (Basic skills, 3 points.) Find all (real) values of x satisfying the following inequality:

$$x^2 + x > 6$$

$$\begin{aligned} &x^2 + x > 6 \\ \Leftrightarrow &x^2 + x - 6 > 0 \\ \Leftrightarrow &(x-2)(x+3) > 0 \end{aligned}$$

which is true exactly when $(x-2)$ and $(x+3)$ have the same sign. So, we either have $x > 2$ and $x > -3$ or $x < 2$ and $x < -3$. But, if $x > 2$ then $x > -3$ is automatically true. Similarly, if $x < -3$ then $x < 2$ is automatically true. Hence, we can describe the solution set as $\{x : x < -3 \text{ or } x > 2\}$.

2. (Test statements, 3 points.) Give a clear statement of either the theorem which gives us the integral test **or** the alternating series theorem. [Your choice which. It should be clear which you are stating! You shouldn't state both.]

You were only meant to do one, but here's both.

Integral Test Theorem. Suppose f is a positive continuous decreasing function on $[1, \infty]$ and for all $n \in \mathbb{N}$, $f(n) = a_n$. The $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ does.

Alternating Series Test. Suppose (a_n) is a decreasing series with $\lim a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

3. (Test problem, 4 points.) Test the following series for convergence. [Hint: there are probably a few ways of doing this. One is to use **both** of the comparison tests for series, one after the other.]

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) \ln(n).$$

Let $a_n := \sin\left(\frac{1}{n^2}\right) \ln(n)$. I first do a limit comparison test with $b_n := \frac{\ln(n)}{n^2}$ (I got the idea for this because we have a function of n which converges to 0 inside a sine).

$$\begin{aligned} \lim \frac{a_n}{b_n} &= \lim \frac{\sin\left(\frac{1}{n^2}\right)}{n^{-2}} \\ &= \lim_{x \to 0} \frac{\sin(x^2)}{x^2} && \text{Putting } x = 1/n. \\ &= \lim_{x \to 0} \frac{\cancel{2x}\cos(x^2)}{\cancel{2x}} && \text{L'Hospital} \\ &= 1. \end{aligned}$$

So, the Limit Comparison Test applies and I see that $\sum a_n$ converges if and only if $\sum b_n$ does. I now test $\sum b_n$ for convergence. I am going to compare this with $c_n := 1/n^{3/2}$. (Why? Well, first I guessed that $\sum b_n$ converged, by ignoring the log. I then knew that

I had to find something bigger than it that converged, so I used the fact that $\ln(n) < n^p$ for any $p > 0$ with a suitably small p to get convergence.)

$\frac{\ln(n)}{n^2} < \frac{1}{n^{3/2}}$, and $\sum \frac{1}{n^{3/2}}$ converges, so $\sum b_n$ does too. Hence, $\sum a_n$ converges.

(The parenthetical comments aren't needed to make your response good, but they're the kind of thought processes you should be going through to work out what to write).

Quiz 6 – 3/8/06.

1. (Basic skills, 3 points.) If $g(x) := x^2$, then $2x^2 = 2g(x)$, $\sin^2 x = g(\sin(x))$ and $2x = g'(x)$. If $f(x) = \frac{1}{1+x}$, then express the following functions in terms of f in the same way as I expressed the three functions in terms of g above.

   (a) $\frac{1}{2-x} = \frac{1}{1+(1-x)} = f(1-x)$. (There are other ways to do this one.)

   (b) $\frac{1}{1+x^2} = f(x^2)$.

   (c) $\arctan(x) = \int f(x^2)\,dx$.

2. (Interval of convergence, 4 points.) Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

   We use ratio test:

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|x^{n+1}n|}{|x^n(n+1)|} \\ &= |x|\frac{n}{n+1} \\ &= |x|\end{aligned}$$

   So, it converges for $|x| < 1$ and diverges for $|x| > 1$. I need to check the endpoints.

   If $x = -1$, the series is $\sum_{n=1}^{\infty}(-1)^n/n$ which converges by Alternating Series Test. If $x = 1$, I get the harmonic series which is divergent.

   Hence, the interval of convergence is $[-1, 1]$

3. (Bad arguments, 4 points.) There is an error in each of the following arguments. Point out what it is in each case. [You don't need to tell me whether the conclusion is right or wrong, or to try to correct the argument, just tell me where the argument slips up.]

   (a) Test $\sum_{n=1}^{\infty} \ln(n)$ for convergence: $\int_1^{\infty} \ln(x)\,dx = \lim_{t\to\infty}[x(\ln(x)-1)]_1^t = \infty$. So, by integral test, $\sum_{n=1}^{\infty} \ln(n)$ diverges.

   The Integral Test only applies to decreasing functions.

   (b) Test $\sum_{n=1}^{\infty}(-1)^n \sin^2(n)/n$ for convergence: The term sequence is alternating (as $\sin^2(n)/n$ is always positive) and $\sin^2(n)/n \to 0$ as $n \to \infty$. Hence, by Alternating Series Test, sequence converges.

   We'd need $(\sin^2(n)/n)$ to be decreasing to apply the Alternating Series Test, but it isn't.

   (c) Test the sequence $\sum_{n=0}^{\infty} a_n$ for convergence, where a_n is given by $a_0 = 1$, $a_{n+1} = \sin(a_n)/2$: We first note that the sequence (a_n) converges to 0, because sin is bounded above by 1, so $a_n \leq 1/2^n$ and all the terms are positive. We now use the ratio test.

$$
\begin{aligned}
\lim \frac{a_{n+1}}{a_n} &= \lim \frac{\sin(a_n)}{2a_n} \\
&= \lim \frac{\cos(a_n)}{2} && \text{By L'Hospital} \\
&= \frac{1}{2} < 1
\end{aligned}
$$

So, the series converges.

You can't apply L'Hospital's rule directly to sequences like that (what would it mean to differentiate a sequence?).

Quiz 7 – 3/15/06.

Adam Booth

1. (Basic skills: review of 11.1, 3 points.) Do examples exist of sequences with the following combinations of properties? If yes, give an example; if no, explain why.

   (a) Unbounded, monotone, convergent.

   Impossible: all convergent sequences are bounded.

   (b) Bounded, non-monotone, convergent.

   There are many examples. Here's a particularly simple one: $(0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots)$, ie.

   $$a_n := \begin{cases} 1 & n = 2 \\ 0 & n \neq 2. \end{cases}$$

   (c) Both increasing and decreasing.

   The only sequences that are both increasing and decreasing, the way Stewart defines these terms, are the constant sequences, eg. $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots)$.

2. (Ideas about power series, 3 points.) Suppose the interval of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is $(p, q]$.

   (a) Give a formula for the radius of convergence in terms of a and q. Also, write p in terms of a and q.

   The interval is symmetric about a, so $R = q - a$ and $p = a - R = a - (q - a) = 2a - q$.

   (b) Do you have enough information to say what the radius of convergence of $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ is? If so, give it in terms of a and q; if not, explain why not.

   Yes, this series is the previous one differentiated, so it has the same radius of convergence: $p - a$.

   (c) Do you have enough information to say what the interval of convergence of $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ is? If so, give it in terms of a and q; if not, give two (different) possible intervals.

   No, differentiating can change the interval of convergence. Two possibilities are (p, q) and $(p, q]$.

3. (Power series calculation, 4 points.) Let $f(x) := \frac{1}{(1+x^2)^2}$.

   (a) Find a power series representation for f (centred at 0).

   Note that

   $$f(x) = \frac{-1}{2x}\frac{d}{dx}\left(\frac{1}{1+x^2}\right).$$

   Using the formula for the sum of a geometric series, we can get a series for the thing being differentiated above:

   $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

   Differentiating this gives,

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1}$$

Thus,

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{2n-2}.$$

(b) What's the interval of convergence of your power series?

The radius of convergence is 1, as it was for the geometric series and nothing we've done changes that. We must check end-points.

$x = -1$: We have $\sum_{n=1}^{\infty}(-1)^{n-1}n(-1)^{2n-2}$ which diverges (by test for divergence).

$x = 1$: We have $\sum_{n=1}^{\infty}(-1)^{n-1}n$ which diverges (by test for divergence too).

Hence, the interval is $(-1, 1)$.

(c) Hence, evaluate the 100th derivative of f at 0.

The 100th derivative of f evaluated at 0 will be $c_{100}100!$, where c_{100} is the coefficient of x^{100}. The x^{100} term comes up for $n = 51$, so $c_{100} = 51$. Hence, $f^{(100)}(0) = 51(100!)$[1].

[1] Which, if you're interested, is 4 759 636 987 641 151 786 766 661 181 669 601 725 026 514 381 483 462 694 898 241 158 656 097 599 654 725 696 056 014 662 783 982 432 598 938 593 962 188 411 670 810 445 756 760 064 000 000 000 000 000 000 000 000. There was a bonus point available for any one who got this.

Math 1B – Quiz 8 – 4/7/06.

1. (Basic skills: Integration, 3 points.) Evaluate the following integral:

$$\int \frac{dx}{x^2 - 1}.$$

We use partial fractions.

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$
$$\therefore 1 = A(x-1) + B(x+1)$$

Substituting $x = 1$ and $x = -1$, we get $A = \frac{-1}{2}$, $B = \frac{1}{2}$.

$$\therefore \int \frac{dx}{x^2-1} = \frac{-1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C.$$

2. (Ideas about differential equations, 3 points.) Consider the differential equation $\frac{dy}{dx} + x^2 + y^2 = 1$.

   (a) Is this equation separable?
   No, separable equations have to be able to be written as $\frac{dy}{dx} = f(y)g(x)$ and this can't.

   (b) Suppose $y_p(x)$ is a solution of the o.d.e which has a critical point at $x = 1$. What's the value of $y_p(1)$?
   At a critical point, $\frac{dy}{dx} = 0$, so as y_p is a solution of the o.d.e, we must have

$$0 + 1^2 + y_p(1)^2 = 1.$$

   Hence, $y_p(1) = 0$

   (c) Is the critical point a local max, local min or inflection point?
   By implicit differentiation, and substitution of the o.d.e, we have

$$\frac{d^2y}{dx^2} + 2x + 2y\frac{dy}{dx} = 0$$
$$\therefore \frac{d^2y}{dx^2} = -2x - 2y(1 - x^2 - y^2)$$
$$= -2$$

   So, it's a local max.

3. (Solving separable equations, 4 points.) Solve the following o.d.e. (You should make sure to find all solutions).

$$\frac{dy}{dx} = \frac{y^2}{x^2+1}.$$

We first note that there's a constant solution: $y \equiv 0$. Searching now for solutions for which y is never 0 (so we can divide by it):

$$
\begin{aligned}
\frac{1}{y^2}\frac{dy}{dx} &= \frac{1}{x^2+1} \\
\therefore \int \frac{dy}{y^2} &= \int \frac{dx}{x^2+1} \\
\therefore \frac{-1}{y} &= \arctan(x) + C \\
\therefore y &= \frac{-1}{\arctan(x) + C}
\end{aligned}
$$

These are all the solutions.

Quiz 9 – 4/12/06.

1. (Basic skills: Integration, 3 points.) Evaluate the following integral:

$$\int x^3 e^{x^2}\, dx.$$

   We first perform the substitution $t = x^2$ (and hence, $dt = 2xdx$), giving

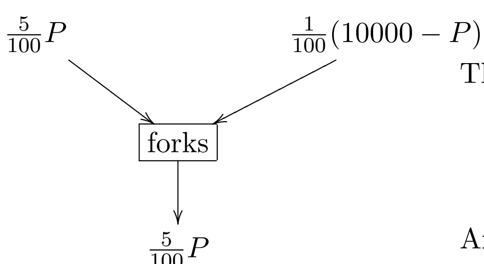
$$\int x^3 e^{x^2}\, dx = \frac{1}{2}\int te^t\, dt.$$

   We can now do integration by parts, with $u = t$, $\frac{dv}{dt} = e^t$, to give

$$\begin{aligned}\frac{1}{2}\int te^t\, dt = \frac{1}{2}\left(te^t - \int e^t\, dt\right) \\ &= \frac{1}{2}\left(te^t - e^t + C\right) \\ &= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2} + C\right)\end{aligned}$$

2. (Mixing problems, 3 points.) A certain large restaurant chain currently owns 10 000 metal forks. From now on, each time one breaks, they plan on replacing it with a plastic fork. No other new forks will be bought. Every day 1% of the metal forks and 5% of the plastic forks break. Set up, but do not solve, a differential equation for the number of plastic forks at time t, with an initial condition. Make sure your notation is clear.

   Let $P(t)$ be the number of plastic forks owned by the chain t days after the start of the experiment. Let's draw a picture of how plastic forks enter and leave the stock of forks:



   This gives us the differential equation

$$\frac{dP}{dt} = \frac{1}{100}(10000 - P).$$

   And we have the initial condition $P(0) = 0$.

3. (Solving separable equations, 4 points.) One of the following ODEs is separable and the other is linear. Say which is which and then solve only the linear one.

$$\frac{dy}{dx} = \sin(x)y + \sin(2x). \tag{1}$$

$$x^2\frac{dy}{dx} = y^2. \tag{2}$$

(1) is linear and (2) is separable, so we solve (1).
$P(x) = -\sin(x)$ and $Q(x) = \sin(2x)$. So, we have,

$$I(x) = e^{\cos x}.$$

Hence,

$$\begin{aligned}
\int Q(x)I(x)\,dx &= \int \sin(2x)e^{\cos x}\,dx \\
&= \int 2\sin x\cos x e^{\cos x}\,dx && u = \cos x \\
& && du = -\sin x\,dx \\
&= -2\int ue^{u}\,du \\
&= -2(ue^{u} - e^{u}) + C \\
&= 2e^{\cos x} - 2\cos x e^{\cos x} + C
\end{aligned}$$

So, all the solutions are of the form:

$$y(x) = 2 - 2\cos x + Ce^{-\cos x}.$$

Quiz 10 – 4/18/06.

1. (Basic skills: Integration, 3 points.) Evaluate the following integral:

$$\int \frac{dx}{(1+x^2)^{3/2}}.$$

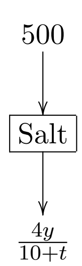
We use the substitution $x = \tan\theta$ (as suggested in section 7.3 of Stewart), giving $dx = \sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^{3/2}} &= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\, d\theta \\
&= \int \frac{d\theta}{\sec\theta} \\
&= \int \cos\theta\, d\theta \\
&= \sin\theta + C \\
&= \sin(\arctan x) + C \\
&= \frac{x}{\sqrt{x^2+1}} + C.
\end{aligned}$$

2. (Mixing problems, 4 points.) A tank initially contains 10 liters of pure water. It is drained out at the rate of 4 liters per minute and, at the same time, filled with brine containing 100 grams of salt per liter at a rate of 5 liters per minute. The tank is kept well mixed. Set up, but do not solve, a differential equation with an initial condition involving the amount of salt in the tank at time t. There should only be two variables (and some numbers) in your ODE. Make sure your notation is clear. [Hint: you will have to work out how much water is in the tank at time t first.]

First, I set up my notation (you're not mind-readers, so you shouldn't be expected to guess what my symbols stand for). I use t for time in minutes (with the experiment starting at $t = 0$) and y for the mass of salt in the tank in grams.

$(100g/l) \cdot (5l/min)$ is the number of grams of salt entering the tank per minute. Note that at time t there's $10 + t$ liters of water in the tank (as 5 liters enter and 4 liters exit each minute). Hence the amount exiting, which is the number of grams of salt per liter of water times by the number of liters of water exiting per minute, is $\frac{y}{10+t} \cdot 4$.



Hence the ODE is

$$\frac{dy}{dt} = 500 - \frac{4y}{10+t}$$

and the initial condition is

$$y(0) = 0.$$

3. (Complex numbers, 3 points.) Draw the fourth roots of 1 on an Argand Diagram. By connecting the points, a polygon is formed. What is the area of this polygon?

Bonus question: Let A_n be the area of the polygon formed by joining the dots of the nth roots of unity. What is $\lim_{n\to\infty} A_n$?

There are four key things to realize (or, remember from when I told you in section on Monday) about the fourth roots of unity:

(a) There are four of them.

(b) They all lie on the circle with center O and radius 1.

(c) They are equally spaced.

(d) 1 is one of them.

With this information, you can easily plot them on a graph. I'm having some difficulty getting a good sketch for you, so I'll leave that as an exercise. Looking at the graph, we can read off that the roots are 1, i, -1, $-i$.

Connecting the dots, we get a square of side $\sqrt{2}$. It therefore has area 2. (There's a better way of seeing this, which involves cutting up the square into triangles and re-arranging into a rectangle, but I'll show you that in section on Friday).

For the bonus part: As we increase n, points (b)-(d) above remain true, but the number of roots becomes n. Hence, the polygon gets closer and closer to a circle with center O and radius 1, which has area π. This was in fact one of the first ways of calculating π.

Quiz 11 – 4/25/06.

1. (Basic skills: Σ notation, 3 points.) Write the following expressions as a single sum in Σ-notation:

   (a) $1 + 2^2 + 3^3 + 4^4 + \cdots$

   $$\sum_{n=1}^{\infty} n^n$$

   (b) $\sum_{n=0}^{\infty} a_n + 2\sum_{n=0}^{\infty} b_n$

   $$\sum_{n=0}^{\infty}(a_n + 2b_n)$$

   (c) $\sum_{n=0}^{\infty} c_n x^n + \frac{d}{dx}\sum_{n=1}^{\infty} d_n x^n$

   $$\begin{aligned} \frac{d}{dx}\sum_{n=1}^{\infty} d_n x^n &= \sum_{n=1}^{\infty} n d_n x^{n-1} \\ &= \sum_{n=0}^{\infty}(n+1)d_{n+1}x^n \\ \therefore \sum_{n=0}^{\infty} c_n x^n + \frac{d}{dx}\sum_{n=1}^{\infty} d_n x^n &= \sum_{n=0}^{\infty}(c_n + (n+1)d_{n+1})x^n. \end{aligned}$$

2. (Undetermined coefficients, 3 points.) Suppose you are trying to solve an ODE of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = G(x)$ for the following Gs. What would you guess as a particular solution when doing the method of undetermined coefficients? How would you change your guess if what you write down turns out to be a solution of the complementary equation?

   (a) $G(x) := x^2$.

   $$y_p = Ax^2 + Bx + C.$$

   (b) $G(x) := e^x$.

   $$y_p = Ae^x.$$

   (c) $G(x) := x \sin 3x$.

   $$y_p = (Ax + B)\sin 3x + (Cx + D)\cos 3x.$$

   If any of these are solutions of the complementary equation, multiply by x until they aren't any more.

3. (IVPs, 3 points.) Find α such that f approaches 0 as $x \to \infty$, where f is the solution to the initial-value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$.

Let's start by solving the equation. The auxiliary equation is:

$$r^2 - r - 2 = 0$$

which has solutions $r = 2, -1$. Hence the general solution is:

$$y = C_1 e^{2x} + C_2 e^{-x}.$$

If $\lim_{x\to\infty} y$ is to be 0, we need $C_1 = 0$. Hence,

$$y' = -C_2 e^{-x}$$

so, substituting in the condition $y'(0) = 2$, we get that $C_2 = -2$. Hence, $\alpha = y(0) = -2$.

Quiz 12 – 5/3/06.

1. (Basic skills: Geometric Series, 3 points.)

   (a) Evaluate the series $\sum_{n=0}^{\infty} x^n$.

   $$\frac{1}{1-x}$$

   (b) Hence, find a power series centered at 0 for the function $\frac{1}{(1-x)^2}$

   $$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\frac{1}{1-x}\\ &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n\\ &= \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

2. (Applications of 2nd order ODEs, 3 points.) The motion of a particle on a damped spring under forced vibrations is modelled by the following ODE:

   $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

   (a) What physical things do m, c, k and F represent?
   m is the mass of the particle; c, the damping constant; k, the spring constant; F the external force.

   (b) Suppose you were solving this ODE by the method of undetermined coefficients. If $F(t) = \sin(t)$, there are two cases for what you should put for x_p. What are the two possibilities? Give a condition on m, c, k for which possibility you will need.

   We will need either $A\sin t + B\cos t$ or $At\sin t + Bt\cos t$. We will need the second if and only if $c^2 - 4mk = -1$ (which will make $A\sin t + B\cos t$ a solution of the homogeneous part).

3. (2nd order non-homogeneous ODEs, 4 points.) Solve the following ODE.

   $$y'' + 3y' + 2y = \sin(e^x).$$

First, we solve the homogeneous part: $y'' + 3y' + 2y = 0$. This has auxilliary equation $r^2 + 3r + 2 = (r+1)(r+2) = 0$, so we take solutions $y_1 = e^{-x}$, $y_2 = e^{-2x}$.

We may then calculate $W = y_1y_2' - y_1'y_2 = -e^{-3x}$. Hence,

$$
\begin{aligned}
u_1 &= -\int \frac{Gy_2}{aW}\,dx \\
&= \int \frac{e^{-2x}\sin(e^x)}{e^{-3x}}\,dx \\
&= \int e^x \sin(e^x)\,dx && u = e^x \\
&= \int \sin(u)\,du \\
&= -\cos(u) + C_1 \\
&= -\cos(e^x) + C_1
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \int \frac{Gy_1}{aW}\,dx \\
&= -\int e^{2x}\sin(e^x)\,dx && t = e^x \\
&= -\int t\sin(t)\,dt && u = t, \frac{dv}{dt} = \sin(t) \\
&= t\cos t - \int \cos t\,dt \\
&= t\cos t - \sin t + C_2 \\
&= e^x\cos(e^x) - \sin(e^x) + C_2
\end{aligned}
$$

Hence the general solution is

$$
\begin{aligned}
y &= (-\cos(e^x) + C_1)e^{-x} + (e^{-x}\cos(e^x) - \sin(e^x) + C_2)e^{-2x} \\
&= -e^{-2x}\sin(e^x) + C_1 e^{-x} + C_2 e^{-2x}.
\end{aligned}
$$