Math 104 - Practice Final.

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The final will contain five questions, one for each week of class since the midterm. Each question will be divided into two parts: a "bookwork" style part (a), and a problem-solving based part (b). You should answer at most three part (a)s and three part (b)s. Grading will then be as for the midterm.

1 Continuity.

1. (a) i. Define what it means for a function $f : [a, b] \to \mathbb{R}$ to be *continuous* and what it means for it to be *bounded*.
   ii. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is bounded.
   iii. State and prove the Intermediate Value Theorem.

   (b) Suppose $h : (0, 1) \to \mathbb{R}$ has the property that for all $x \in (0, 1)$, there exists a $\delta > 0$ such that for all $y \in (x, x + \delta) \cap (0, 1)$, $h(x) \leq h(y)$.
   i. Prove that if h is continuous on $(0, 1)$ then h is increasing.
   ii. Give a counterexample to show that this need not be true if h is not continuous.

2. (a) i. Give a definition of *continuous* that doesn't mention limits.
   ii. State the Intermediate Value Theorem.
   iii. Suppose f is strictly increasing on $[a, b]$. Let $c \in [a, b]$. Prove that $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist. [Hint: consider $\sup\{f(x) : a \leq x < c\}$ and $\inf\{f(x) : c < x \leq b\}$.]
   iv. Prove that f is continuous (using your definition of continuous above) at c if and only if $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing.
   i. Prove that $f[a, b] = [f(a), f(b)]$.
   ii. Prove that f is injective.
   iii. Deduce that f has an inverse function $g : [f(a), f(b)] \to [a, b]$ and show g is strictly increasing and continuous.

3. (a) i. Define what it means to say f *is continuous at* a.
   ii. Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous at c and $f(c) > 0$, then there is some $\delta > 0$ such that $f(x) > 0$ for all x such that $|x - c| < \delta$.
   iii. State and prove the Intermediate Value Theorem.

   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous, with $f(a) < f(b)$. Suppose that for every $y \in \mathbb{R}$ there is at most one $x \in [a, b]$ such that $f(x) = y$.
   i. Prove that for all $z \in (a, b)$, $f(a) < f(z) < f(b)$.
   ii. Deduce that f is strictly increasing on $[a, b]$.

2 Differentiation.

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$. Throughout this question, you may assume without proof that the continuous image of a closed bounded interval is bounded and attains its bounds.

   i. Define what it means for f to have a *local maximum* at c.

   ii. Prove that if f has a local maximum at c and is differentiable at c, then $f'(c) = 0$.

   iii. State and prove Rolle's theorem.

   (b) i. Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiably and that $f(x+1) = f(x)$ for all x. Prove that there is a sequence (y_n) such that $f^n(y_n) = 0$.

   ii. If $f(x) := e^{\sin(2\pi x)}$, deduce that there is some $r \in [0,1]$ such that for all $N \in \mathbb{N}$ there is a $k > N$ such that $f^{(k)}$ has a root in $(r - 0.000104, r + 0.000104)$.

2. (a) For this question, you may assume that the continuous image of a closed bounded set is bounded and attains its bounds.

   i. If f has a local minimum or local maximum at c and is differentiable at c, what is the value of the derivative? [No proof required.]

   ii. State and prove Rolle's Theorem.

   iii. State and prove the Mean Value Theorem.

   (b) Let the function $h : \mathbb{R} \to \mathbb{R}$ be differentiable at every point and suppose that $h(0) = 0$, $h(2) = 0$ and $|h'(x)| \leq 1$ for all x. Prove that:

   i. $h(1) \leq 1$;

   ii. $h(1) \neq 1$.

3. (a) i. Define what it means to say that f is differentiable at a.

   ii. Prove that if f is differentiable at a, then it is continuous at a.

   iii. Prove that if f and g are differentiable at a then so is fg (the product, not the composite).

   (b) i. Give an example of a function which is continuous everywhere and differentiable everywhere except 0.

   ii. Let f be defined by

   $$f(x) := \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

   Show that f is everywhere differentiable, but its derivative is not continuous at 0.

   iii. Let g be defined by

   $$g(x) := x^2 \chi_{\mathbb{Q}}(x).$$

   Show that g is differentiable at 0 but not differentiable at any other point.

3 Integration.

1. All integrals referred to Riemann integrals.

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions where g has the property that $g(x) \geq 0$. You may assume that their images have sups and infs and other standard theorems about continuous functions (week four stuff).

i. Explain how the integral $\int_a^b f$ is defined.

ii. Prove that, if $m \leq f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g.$$

iii. Deduce that for some $z \in [a, b]$,

$$\int_a^b fg = f(z) \int_a^b g.$$

(b) Suppose that $\phi : [0, 1] \to \mathbb{R}$ is continous. You may use without proof the FTC.

i. Prove that

$$\lim_{n\to\infty} \int_{1/\sqrt{n}}^1 \phi(x) \frac{n}{1 + (nx)^2} \, dx = 0.$$

ii. Give an example to show that

$$\lim_{n\to\infty} \int_0^1 \phi(x) \frac{n}{1 + (nx)^2} \, dx$$

need not be 0.

2. (a) Suppose $g : [a, b] \to \mathbb{R}$ is continuous

i. Explain how the (proper) Riemann integral is defined.

ii. If $g(x) \geq 0$, prove that $\int_a^b g \geq 0$.

iii. Suppose that $c > 0$. Prove that $\int_a^b cg = c \int_a^b g$.

iv. State the form of the FTC which talks about the integral of a derivative.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

i. Explain why

$$\int_0^1 \left(\int_0^1 (f'(x)f(y) - f(x)f'(y))^2 \right) dx) \, dy \geq 0.$$

ii. Hence, prove that

$$2 \int_0^1 (f')^2 \int_0^1 f^2 \geq \frac{1}{2}(f(1)^2 - f(0)^2).$$

3. (a) i. Explain how to define the Riemann integral.

ii. State and prove the test theorem.

iii. Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable.

(b) Define the *ruler function* on $[0, 1]$ by

$$f(x) := \begin{cases} 0 & x \notin \mathbb{Q} \\ 0 & x = 0 \\ \frac{1}{q} & x = p/q \end{cases}$$

where p, q have no common factors.

i. Show that the lower sum of f is zero.

ii. Show that there are no more than $m(m-1)/2$ points $x \in [0, 1]$ such that $f(x) > 1/m$.

iii. Hence, show how to find an upper step function with integral $< \epsilon$ for any given $\epsilon > 0$.

iv. Is the ruler function Riemann integrable?

4 Uniform Convergence.

1.a.i) Define what it means to say $f_n \to f$ uniformly.

ii) State precisely what it means to say that *uniform convergence preserves continuity.*

iii) Prove that uniform convergence preserves continuity.

b. Given a function $f : [1, \infty) \to \mathbb{R}$, define a sequence of functions f_n with the same domain by

$$f_n(x) := f(x)\chi[n, n+1)(x).$$

i) What's the pointwise limit of the sequence (f_n)? ii) Show that the sequence converges uniformly if and only if $f(x) \to 0$ as $x \to \infty$. iii) Show that the series $\sum f_n$ also converges uniformly if and only if $f(x) \to 0$ as $x \to \infty$.

2. a. i) Define what it means for a sequence of functions to converge uniformly. ii) Prove that a power series with radius of convergence $R > 0$ converges uniformly on any interval $[-a, a]$ where $0 < a < R$. iii) If f_n are Riemann integrable on $[b, c]$ and converge unformly to f, prove that f is Riemann integrable on $[b, c]$. iv) What is the value of $\int_{-b}^{c} f$? [No proof required.] b. i) If $h(t) = \sum a_n t^n$ is a power series, find a power series representation for $(x - t)h(t)$ with the same radius of convergence. [Be clear about what the coefficient of t^n is.]

ii) Hence, find a power series representation of $\int_0^x (x - t)h(t)\, dt$ with the same radius of convergence. [Be clear about what the coefficient of t^n is.]

iii) Hence, find all power series solutions to the integral equation

$$h(x) = 1 - \int_0^x (x - t)h(t)$$

subject to $h(0) = 0$, $h'(0) = 1$.

5 Metric Spaces.

1. (a) i. Define what it means to say that $\langle X, d\rangle$ is a *metric space.*

ii. Define what $B_\epsilon(x)$ means (relative to a fixed metric space).

iii. Define what it means for a sequence (x_n) to *converge* in a given metric space.

iv. Define what it means for two metrics on the same space to be *equivalent.*

v. Prove that equivalent metrics have the same convergent sequences.

(b) Define a function from $(\mathbb{R}^2)^2$ to $\mathbb{R}$ by

$$d_m((x, y), (u, v)) := \max\{|x - u|, |y - v|\}.$$

i. Prove that $\langle \mathbb{R}^2, d_m\rangle$ is a metric space.

ii. Show that (x_n, y_n) is a d_m-convergent sequence if and only if (x_n) and (y_n) are convergent sequences of reals (in the usual sense).

iii. Draw a picture of $B_1((0,0))$.

iv. Prove that d_m is equivalent to the Euclidean metric on $\mathbb{R}^2$.

2. (a) i. Define what it means for $\langle X, d\rangle$ to be a *metric space.*

ii. Define what it means for a set $Y \subseteq X$ to be *open.*

iii. Define what it means for two metrics on the same set to be *equivalent.*

iv. Prove that two equivalent metrics have the same open sets.

(b) Let X be some finite set.

i. Prove that, in any metric on X, all singletons are open.

ii. Deduce that all metrics on X are equivalent.

iii. Show that the following function from $(\mathbb{N} \cup \{0\})^2$ to $\mathbb{R}$ is a metric but that it is not equivalent to the discrete metric on $\mathbb{N} \cup \{0\}$.

$$d_r(n,m) := \begin{cases} \left|\frac{1}{n} - \frac{1}{m}\right| & n, m \neq 0 \\ \frac{1}{n} & m = 0, n \neq 0 \\ \frac{1}{m} & n = 0, m \neq 0 \\ 0 & n = m = 0. \end{cases}$$

3. (a) i. Define what it means for $\langle X, d\rangle$ to be a metric space.

ii. Define what *closed* and *open* subsets of a metric space are.

iii. Prove a set is closed if and only if its complement is open.

iv. Define what a *compact* subset of a metric space is.

v. In the reals, as set is compact if and only if it's closed and bounded. Is this true in general for metric spaces?

(b) Let B be the set of bounded sequences of real numbers, L be the set of convergent sequences of real numbers and L_0 the set of sequences of reals that converge to 0. You may assume without proof that $L \subseteq B$. Define a function d from B^2 to $\mathbb{R}$ by

$$d((x_n), (y_n)) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

i. Show that $\langle B, d\rangle$ is a metric space.

ii. Show that L_0 is a closed subset of B. Is it compact?

iii. Is L closed? Is it compact?