Math 104 - Practice Midterm questions.

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Here are some practice midterm questions. I'll also use this space to explain the structure of the midterm.

The midterm will contain three questions, two on sequences and one on series. Each question will have two parts: a part (a) which will be "book work"; and a part (b) which will involve "unseen material". These practice questions are structured the same way to let you see what I mean.

Each half question will be graded out of 10. To compute your letter grade for the midterm, I will first find your total points for part (a)s and for part (b)s. Your numerical score will then be the lower of these plus half the higher. This is to ensure that you show competence in both skills areas: the reproduction of standard material; and problem solving.

This gives a total of 45 points available. I expect the following scores would gain the following letter grades: 30, A; 21, B; 14, C; 9, D. [Note: these are not cut-offs, but rather typical scores.] Given this grading scheme, you should be careful to not spend too much of your time on either part (a)s or (b)s, but try and spread your time evenly between them. For instance, scoring 10 points on (a)s and 10 on (b)s would probably gain a C, but if all of those 20 points were gained in, say, (a)s, a D would be more likely. Also, note that it's possible to be awarded an A by writing perfect answers to two questions and ignoring the other one (not that I recommend ignoring a whole question).

1. (a) Let (a_n) be a sequence of real numbers.
   - i. Define
     - A. (a_n) is *bounded.*
     - B. (a_n) is *increasing.*
     - C. (a_n) is a *Cauchy sequence.*
   - ii. For each term in (i) give, with brief reasons, an example a sequence which satisfies the definition and one which doesn't.
   - iii. What is a subsequence of the sequence (a_n)?
   - iv. According to your definition, is it possible for the sequence (z_n) to have two different subsequences, if $z_n = 0$ for all n?

   (b) For this part of the question, you may assume the version of the completeness axiom that states that any non-empty set which is bounded above has a least upper bound. If you wish to use any other equivalent formulation, you must prove it.
   - i. Prove that every bounded sequence has a monotone subsequence.
   - ii. Prove that every Cauchy sequence is bounded.
   - iii. Under what circumstances is a monotone sequence convergent? [You need not give a proof.]

iv. Prove that any Cauchy sequence is convergent.

2. (a) Let $(a_n), (b_n)$ be sequences of real numbers; $a_n \to a$ and $b_n \to b$ as $n \to \infty$; suppose also that $b_n \neq 0$ for all n and $b \neq 0$.
   i. Define what it means to say that $a_n \to n$ as $n \to \infty$.
   ii. Prove that $a_n + b_n \to a + b$.
   iii. Prove that $a_n b_n \to ab$.
   iv. Prove that $a_n / b_n \to a/b$

   (b) Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a.$$

3. (a) i. Prove that if $t \in [0, 1)$ then (t^n) is a null sequence.
   ii. If (a_n) is a sequence of reals, define what it means to say that $\sum a_n$ is convergent?
   iii. Prove, from first principles, that $\sum t^n$ is convergent for $t \in [0, 1)$, but not for $t \geq 1$.
   iv. What is a *Cauchy sequence*? Is such a sequence convergent?
   v. If $(c_n), (d_n)$ are sequences satisfying $|c_n| \leq d_n$ and $\sum d_n$ is convergent, prove that $\sum c_n$ is. [You may not use any tests for convergence in your answer unless you prove them.]

   (b) Find the interval of convergence for the following power series.
   i. $\sum (n + 1)x^n$.
   ii. $\sum s(n) \left(\frac{x}{\sqrt{2}}\right)^n$, where $s(n)$ is the nth coefficient of $\sqrt{2}$ in base 2. [You may assume $\sqrt{2}$ is irrational.]

4. (a) i. Define the what it means for M to be the supremum of a set of real numbers.
   ii. State the version of the completeness property which is in terms of suprema of certain sets.
   iii. Show, using this version of the completeness property, that if (a_n) is a bounded monotonic sequence of real numbers then it is convergent.

   (b) The sequence (a_n) is defined by

$$a_0 = 1$$
$$a_{n+1} = \frac{2}{5}(a_n^2 + 1).$$

   i. Prove that if (a_n) converges to a limit a, then a is $\frac{1}{2}$ or 2.
   ii. Prove that $(a_{n+1} - \frac{1}{2})$ and $(a_n - \frac{1}{2})$ have the same sign; as do $(a_{n+1} - 2)$ and $(a_n - 2)$.
   iii. Prove that $a_{n+1} < a_n$.
   iv. Deduce that $a_n \to \frac{1}{2}$.

5. You may use any test you like in this question, provided it is clearly stated.

   (a) Let (a_n) be a sequence of reals.
   i. Define what it means for (a_n) to be *Cauchy*. Are such sequences convergent?

ii. What does it mean to say the series $\sum a_k$ is *convergent*? *Absolutely convergent*? [You may refer to the notion of convergence for sequences in your answer.]

iii. Prove that absolutely convergent series are convergent.

iv. Give an example of a series of real numbers which is convergent but not absolutely convergent.

(b) i. Suppose (a_k) and (b_k) are sequences of real numbers such that the series $\sum a_k^2$ and $\sum b_k^2$ converge. Prove that $\sum a_r b_r$ converges and that

$$\sum a_r b_r \leq \frac{1}{2}\left(\sum a_k^2 + \sum b_k^2\right).$$

ii. Show, by giving an example, that the same is not true for sequences of complex numbers.

6. (a) i. Define the terms *bounded* and *convergent* for a sequence (a_n) of real numbers.

ii. What does it mean to say that S is the supremum of a set X?

iii. Prove that convergent sequences are bounded.

(b) Let (a_n) be a bounded sequence of reals and define two sequences (b_n) and (c_n) by

$$b_n = \inf\{a_n, a_{n+1}, \ldots\}$$
$$c_n = \sup\{a_n, a_{n+1}, \ldots\}$$

i. Prove that $b_n \leq b_{n+1}$, $c_n \geq c_{n+1}$ and $b_m \leq c_n$ for all m, n.

ii. Prove that if (a_n) converges then the sequences (b_n) and (c_n) are also convergent and have the same limit.

iii. Show, by giving an example, that it is possible for (b_n) and (c_n) to converge without (a_n) converging.

7. (a) Let (a_n) be a sequence of real numbers.

i. State the Bolzano-Weierstrass Theorem.

ii. Define what it means to say that (a_n) is *Cauchy*.

iii. Using the Bolzano-Weierstrass Theorem as your only version of the completeness principle, prove that (a_n) is Caucy if and only if it is convergent.

(b) i. What does it mean to say that the series $\sum a_n$ converges.

ii. Suppose now that (a_n) and (b_n) are sequences of complex numbers for which $\sum |a_n|$ and $\sum |b_n|$ converge. Prove that $\sum |a_n + b_n|$ converges.

iii. Show, by giving examples, that it is possible for $\sum |a_n + b_n|$ to converge without $\sum |a_n|$ or $\sum |b_n|$ converging.

8. (a) i. Let $x \in (0, 1)$. Prove that (x^n) converges to 0 and that $\sum x^n$ converges. Find its limit.

ii. Let (a_n) be a sequence of positive real numbers. Prove that if the sequence $\frac{a_{n+1}}{a_n}$ converges to a number less than 1, then the series $\sum a_n$ converges.

iii. State, without proof, the corresponding result for when the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ converges to a number greater than 1.

(b) Use the above to explain why the following series converge or diverge.

   i. $\sum \frac{(4n)!}{(n!)^2}$

   ii. $\sum \frac{n!}{n^n}$.