Math 104 - Homework Solutions.

Adam Booth.

Summer 2007.

Week 1 – Axioms for the real numbers; sequences.

1. Use mathematical induction to prove the following:

   (a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$;
   Let $P(n)$ be the proposition in the question. $P(1)$ is just $1 = 1^2$ which is true. Now, suppose $P(n)$ holds. We must prove $P(n+1)$. But

$$\begin{aligned} 1 + 3 + 4 + \cdots + (2n-1) + (2n+1) &= n^2 + 2n + 1 && \text{inductive hypothesis} \\ &= (n+1)^2 && \text{completing the square.} \end{aligned}$$

   (b) $3^{4n} - 1$ is divisible by 80;
   Let $P(n)$ be the statement in the question. $P(1)$ is the statement $81 - 1 = 80$ is divisible by 80, which is true. Now, suppose $P(n)$, ie. that $3^{4n} - 1$ is divisible by 80. It is enough to show that $(3^{4(n+1)} - 1) - (3^{4n} - 1)$ is divisible by 80. But,

$$\begin{aligned} (3^{4(n+1)} - 1) - (3^{4n} - 1) &= 3^{4n}(3^4 - 1) \\ &= 3^{4n} \cdot 80. \end{aligned}$$

   (c) $n! > 2^n$ for $n > 4$. [Think: what should your base case be here?]
   Let $P(n)$ be the statement in the question. The base case is $P(5)$, which is the statement $120 > 32$, which is true. Now, suppose $P(n)$, ie. $n! > 2^n$. Then,

$$\begin{aligned} (n+1)! &= (n+1)n! \\ &> (n+1)2^n && \text{inductive hypothesis} \\ &> 2 \cdot 2^n && n+1 > 2, \text{ as } n > 4 \\ &= 2^{n+1}. \end{aligned}$$

2. Prove carefully from the axioms that the following hold for elements of $\mathbb{R}$: [Two-column proofs or paragraph proofs are fine; you may quote any result which has been proved in class.]

   (a) If $ab = ac$ and $a \neq 0$, then $b = c$; [This is referred to as the *cancellability* of multiplication.]

As $a \neq 0$, its reciprocal a^{-1} exists. Then we have,

$$\begin{aligned}
ab &= ac && \text{given} \\
a^{-1}(ab) &= a^{-1}(ac) && \text{properties of equality} \\
(a^{-1}a)b &= (a^{-1}a)c && \text{M2, associativity} \\
1b &= 1c && \text{M4} \\
b1 &= c1 && \text{M1, commutativity} \\
b &= c && \text{M3, identity}
\end{aligned}$$

(b) If $ab = 0$, then $a = 0$ or $b = 0$; [One nice way of doing this is by contradiction. This property of $\mathbb{R}$ is called being an *integral domain*.]

Suppose towards a contradiction that $ab = 0$, but $a \neq 0$ and $b \neq 0$. Then a^{-1} and b^{-1} exist. We argue as follows to get a contradiction with our assumption that $b \neq 0$, proving the result.

$$\begin{aligned}
ab &= 0 && \text{given} \\
a^{-1}(ab) &= a^{-1}0 && \text{properties of equality} \\
b &= 0 && \text{LHS: as in part (a). RHS: proved in class.}
\end{aligned}$$

(c) $a \geq b$ iff $a + c \geq b + c$;

Recall that we have already shown this result for $<$. To extend it to $\geq$, we proceed by cases. Suppose $a \geq b$.

Case I: $a = b$. Then $a + c = b + c$ by substitution, so $a + c \geq b + c$.

Case II: $b < a$. Then, $b + c < a + c$, by a result proved in class (Burn, 2.10). Hence, $a + c \geq b + c$.

(d) If $a < b$ and $c < 0$, then $ac > bc$.

$a < b$, so $b - a$ is positive. $c < 0$, so $-c$ is positive. Hence, $(b - a)(-c) = ac - bc$ is positive. Hence, $ac > bc$.

3. This question is meant to help you appreciate an algebraic way in which $\mathbb{R}$ is special. (Much of the course is devoted to analytic ways in which $\mathbb{R}$ is special). Show that there does not exist a subset $P \subseteq \mathbb{C}$ satisfying the *formal properties of positive numbers* on page 11 of Burn.

   Suppose, towards a contradiction, that such a set, P, exists. Consider i. $i \neq 0$, so either i or $-i$ is in P. Either way $(-i) \cdot (-i) = i \cdot i = -1 \in P$. Hence, $i \in P$ iff $(-1)i = -i \in P$ (as $-1 \in P$ and P is closed under multiplication). But, this contradicts (P3) from the sheet, or property 1 from Burn (p. 11).

4. This question is about the absolute value function.

   (a) Show that $||a| - |b|| \leq |a| + |b|$. Also state a 'better bound' on the left hand side [hint: look at the three results summarized on p. 21 of Burn]. Give examples of values for a and b which show that the left hand side, the better bound and the right hand side can all be same, or that some can be different.

$$\begin{aligned} ||a|-|b|| &\leq |a-b| && \text{Burn, 2.63} \\ &\leq |a|+|b| && \text{Triangle inequality} \end{aligned}$$

The 'better bound' is given by Burn, 2.63: $|a-b|$. All three expressions are equal to 0 when $a=b=0$; they can be made different by taking $a=1$, $b=-2$, for then we have

$$\begin{aligned} ||a|-|b|| &= 1 \\ |a-b| &= 3 \\ |a|+|b| &= 3 \end{aligned}$$

(b) Prove that $\max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$;
We proceed by cases. Case I: $a \geq b$. Then, $\max\{a,b\} = a$. Also, $|a-b| = a-b$ as $a-b \geq 0$, so the right hand size simplifies to

$$\begin{aligned} \frac{1}{2}(a+b) + \frac{1}{2}(a-b) &= \left(\frac{1}{2}+\frac{1}{2}\right)a + \left(\frac{1}{2}-\frac{1}{2}\right)b \\ &= a \end{aligned}$$

Case II: $a < b$. Then, $\max\{a,b\} = b$, and $|a-b| = b-a$. By a similar calculation to the above, the right hand side simplifies to b, as required.

(c) Prove a similar identity for minimum.
The identity is $\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$. The proof is just as above.

5. Some more practice with absolute values.

(a) Find all $x \in \mathbb{R}$ satisfying the following inequalities:

i. $4 < |x+2| + |x-1| < 5$;
We break into cases, to allow ourselves to eliminate $|\cdot|$-signs.
Case I: $x \geq 1$. Then, $x-1, x+2 \geq 0$. So, we have

$$\begin{aligned} 4 <& 2x+1 < 5 \\ 3 <& 2x < 4 \\ \frac{3}{2} <& x < 2 \end{aligned}$$

These are all consistent with the case we're in, so we get $(\frac{3}{2}, 2)$ in the set of solutions.
Case II: $x \in [-2, 1)$. Then, $x-1 < 0$, $x+2 \geq 0$. So, our inequality becomes

$$\begin{aligned} 4 <& x+2+1-x < 5 \\ 4 <& 3 < 5 \end{aligned}$$

This has no solutions, so there are no solutions in this case.

Case III: $x < -2$. Then, $x - 1, x + 2 < 0$. So, we have

$$\begin{aligned} 4 < -2 - x + 1 - x &< 5 \\ 4 < -1 - 2x &< 5 \\ 5 < -2x &< 6 \\ \frac{-5}{2} > x &> -3 \end{aligned}$$

These are all consistent with the case we're in, and we've now exhausted all cases, so the set of solutions is $(-3, \frac{-5}{2}) \cup (\frac{3}{2}, 2)$.

ii. $|2x + 1| \leq |x - 1|$.
We again break into cases.
Case I: $x \geq 1$. Then, $x - 1, 2x + 1 \geq 0$. So, our inequality becomes

$$\begin{aligned} 2x + 1 &\leq x - 1 \\ x &\leq -2. \end{aligned}$$

However, this is inconsistent with the case we're in ($x \geq 1$), so this case yields no solutions.
Case II: $x \in [\frac{-1}{2}, 1)$. The $2x + 1 \geq 0$, $x - 1 < 0$. So, we have

$$\begin{aligned} 2x + 1 &\leq 1 - x \\ 3x &\leq 0 \\ x &\leq 0. \end{aligned}$$

Intersecting this with the case we're in gives that $[\frac{-1}{2}, 0]$ is part of the set of solutions.
Case III: $x < \frac{-1}{2}$. Then, $x - 1, 2x + 1 < 0$. So, we have

$$\begin{aligned} -2x - 1 &\leq 1 - x \\ -2 &\leq x. \end{aligned}$$

This gives $[-2, \frac{-1}{2}]$ as part of the set of solutions.
We have now exhausted all cases, so we take the union of what we got to get the complete set of solutions, which is $[-2, 0]$.

(b) Find and sketch the set of all pairs (x, y) satisfying:

i. $|x| - |y| \geq 2$;
Consider each quadrant separately. In the first quadrant, we are graphing the inequality $x - y \geq 2$, etc. For the graph, see separate figures page.

ii. $1 \leq |xy| < 2$.
Again, think about each quadrant separately.

6. For each of the following sequences, say whether or not it's monotonic (and if so, which of the four monotonicity properties it has) and whether it's bounded, unbounded or neither. If it's not monotonic, give a monotonic subsequence.

(a) $a_n = n$.

Monotonic (strictly increasing). Neither bounded nor unbounded (it has a lower bound – 0, say – but no upper bound).

(b) $b_n = 4$.

Monotonic (both increasing and decreasing). It is bounded (above and below by 4).

(c) $c_n = (-1)^n n$.

It is not monotonic. The even terms form a monotonic (strictly increasing) subsequence. It is unbounded.

(d) $d_n = \lfloor 600/n \rfloor$. [Note that, unlike Burn, I'm using 'floor notation' – $\lfloor\cdot\rfloor$ – for the integer function, which he writes $[\cdot]$. This is because I want to still be able to use square brackets as an alternative to (parentheses) when this helps make a formula easier to read, and because floor notation is more common and has the advantage of reminding you that we're rounding down, not up.]

It is monotonic (decreasing, but not strictly so). It is bounded (above by 600, below by 0).

(e) $a_n = 100 - 20n + n^2$.

It is not monotonic, however the tail subsequence starting at $n = 11$ is strictly increasing. It is neither bounded nor unbounded.

7. Show that the following sequences are null. Do some using the definition (ie. by finding an N for each ϵ) and some using the properties of null sequences listed on page 45.

(a) $a_n = \frac{1}{2n+1}$.

I'll do this one by the definition. Let $\epsilon > 0$ be given. Note that the following inequalities are equivalent:

$$\frac{1}{2n+1} < \epsilon$$
$$1 < 2\epsilon n + \epsilon$$
$$\frac{1-\epsilon}{2\epsilon} < n$$

So, take N to be some integer greater than $\frac{1-\epsilon}{2\epsilon}$. Then, for $n \geq N$, we must have $\frac{1}{2n+1} < \epsilon$. This does it, as the sequence is positive.

(b) $b_n = \frac{n}{n^2+1}$.

This one I'll do using the theorems. First, note the indentity

$$\frac{n}{n^2+1} = \frac{1}{n + \frac{1}{n}}.$$

$\frac{1}{n}$ is a null sequence, as $a'_n = n$ is a sequence tending to infinity (Burn 3.31). But, $\frac{1}{n+\frac{1}{n}} < \frac{1}{n}$ as $\frac{1}{n}$ is positive. Hence, we have $\frac{1}{n+\frac{1}{n}}$ is a null sequence by the squeeze theorem.

(c) $c_n = 2^{-n}$.

2^n is an sequence tending to infinity, so 2^{-n} is a null sequence by (Burn 3.31).

(d) $d_n = n^{-1} - n^{-2}$.
$0 \leq n^{-1} - n^{-2} \leq n^{-1}$, so by the squeeze theorem and question 29, (d_n) is null.

8. Adapt the method of question 3.40 from Burn to show that $\frac{2^n}{n!}$ is null.

   Following 3.40, we start by calculating

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} \\ = \frac{2}{n}.$$

   Hence, we can observe that for $k = 3$, if $n \geq k$, then $\frac{a_{n+1}}{a_n} < \frac{3}{4}$. As in 3.40, we then see that (a_n) is eventually dominated by $\left(\left(\frac{3}{4}\right)^n a_3\right)$. Hence, by the squeeze theorem, (a_n) is null.

9. Further adapt the method of question 3.40 to show that $\frac{2^n}{n^2} \to \infty$.

   Again, we calculate the ratio of successive terms:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)^2}\frac{n^2}{2^n} \\ = \frac{2n^2}{(n+1)^2}$$

   Instead of showing this is eventually less than one, we show it's eventually greater. For $n > 3$, $\frac{a_{n+1}}{a_n} > \frac{9}{8}$. So, eventually (a_n) dominates $\left(\left(\frac{9}{8}\right)^n a_3\right)$, which tends to infinity. Hence, (a_n) does too.

Week 2 – Completeness.

1. Using the algebra of limits to calculate limits.

   (a)

$$\frac{4\sqrt{n^4+1}+1}{n^2+2n+4}.$$

   We start by dividing top and bottom by n^2, to get

$$\frac{4\sqrt{1+\frac{1}{n^4}}+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{4}{n^2}}.$$

   Recalling that $\frac{1}{n^k} \to 0$ (for $k > 0$), and using algebra of limits we get that this has limit

$$\frac{4\sqrt{1+0}+0}{1+0+0}.$$

   This is 4.

(b)

$$\frac{3^n n^2 + 5^n n^3}{5^n\sqrt{n^6+1} + n^{100}}.$$

Pulling out $5^n n^3$ from top and bottom gives

$$\frac{\left(\frac{3}{5}\right)^n \frac{1}{n} + 1}{\sqrt{1+\frac{1}{n^6}} + \left(\frac{1}{5}\right)^n n^{97}}.$$

"Exponentials beat powers," so the second term of the denominator has limit 0. Using the algebra of limits, then, we get the limit of the whole thing being

$$\frac{0+1}{\sqrt{1+0}+0}.$$

This is 1.

(c)

$$\sqrt{n+1} - \sqrt{n}.$$

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &\to 0.\end{aligned}$$

2. Calculating a hard limit.

(a) Prove that

$$(1+t)^n \geq \frac{n(n-1)}{2}t^2$$

for all $t \geq 0$. [Hint: Binomial Theorem.]

The right hand side is one of the terms in the binomial expansion of the left hand side. All the terms are positive, so the left hand side is greater than the right.

(b) Hence prove that

$$(1+\frac{2}{\sqrt{n}})^n \geq n \geq 1$$

for sufficiently large n.

Substituting $t = \frac{2}{\sqrt{n}}$ gives

$$\begin{aligned}\left(1+\frac{2}{\sqrt{n}}\right)^n &\geq \frac{n(n-1)}{2}\frac{4}{n} \\ &= 2(n-1) \\ &\geq n && \text{for } n > 1.\end{aligned}$$

$n \geq 1$ is true (for $n \geq 1$).[1]

[1] This sounds silly, but it's what you have to say.

(c) Deduce that $n^{1/n} \to 1$ as $n \to \infty$.
Raising each sequence to the power $\frac{1}{n}$ in the previous inequality gives

$$1 + \frac{2}{\sqrt{n}} \geq n^{1/n} \geq 1.$$

Now, applying the sandwich theorem gives the desired result, as $(1 + \frac{2}{\sqrt{n}}) \to 1$.

3. ...And another one.

(a) Prove that for every pair of $n, r \in \mathbb{N}$ with $n \geq r$,

$$\frac{n!}{r!} \geq r^{n-r}.$$

We proceed by induction on n. The base case is $n = r = 1$, which is $1 \geq 1$, which is true.
Now, suppose we have the result for n. Let $r \leq n+1$. We break into two cases.
Case I: $r < n+1$. Then, we have

$$\begin{aligned} \frac{(n+1)!}{r!} & (n+1)\frac{n!}{r!} & \\ & \geq (n+1)r^{n-r} & \text{inductive hypothesis} \\ & \geq r \cdot r^{n-r} & \text{case assumption} \\ & = r^{(n+1)-r}. & \end{aligned}$$

Case II: $r = n+1$. Then, we have $\frac{(n+1)!}{(n+1)!} \geq (n+1)^0$, as both sides are 1.

(b) Deduce that for a given $r \in \mathbb{N}$, for sufficiently large n,

$$n! \geq \left(\frac{1}{2}r\right)^n.$$

Multiplying both sides of the inequality from (a) by $r!$ gives

$$\begin{aligned} n! & \geq r^n \frac{r!}{r^r} & \\ & = \left(\frac{1}{2}r\right)^n \frac{2^n r!}{r^r} & \\ & \geq \left(\frac{1}{2}r\right)^n & \text{for } n \text{ large enough.} \end{aligned}$$

(c) Hence show that $(n!)^{1/n} \to \infty$ as $n \to \infty$
Raising both sides of the previous inequality to the power $\frac{1}{n}$ we get that for any given $r \in \mathbb{N}$, for sufficiently large n, $(n!)^{1/n} \geq \frac{1}{2}r$. Hence, $\lim(n!)^{1/n} \geq \frac{1}{2}r$ for all $r \in \mathbb{N}$. Hence, it's infinite.

4. Some practice thinking about density (a very important topic in more advanced analysis or topology).

(a) Show that the union of two dense sets is dense.

Let A, B be dense sets, (c, d) an interval. Then, as A is dense, there is an $a \in A \cap (c, d)$. Hence, $a \in (A \cup B) \cap (c, d)$. So, $A \cup B$ is dense.[2]

(b) Show that the *shift* of any dense set is dense, where the shift of a set A by c is defined as $\{x + c : x \in A\}$. [If we wanted to be ridiculously formal, we could write this $+[A \times \{c\}]$. More usually, though, it's written $A + \{c\}$ or even $A + c$].

Let A be dense, (c, d) be an interval, $b \in \mathbb{R}$. I'll show that $A + \{b\}$ is dense. Consider the interval $(c-b, d-b)$. A is dense, so there's some $a \in A \cap (c-b, d-b)$. Then, $a + b \in A + \{b\}$ and $a + b \in (c, d)$. So, $A + \{b\}$ is dense.

(c) Show that the intersection of two dense sets need not be dense.

Consider $\mathbb{Q}$ and $\mathbb{Q} + \{\sqrt{2}\}$. $\mathbb{Q}$ is dense (as shown in class) and so $\mathbb{Q} + \sqrt{2}$ is dense too as it's a shift of a dense set. However, $\mathbb{Q} \cap (\mathbb{Q} + \{\sqrt{2}\})$ isn't dense, in fact it's empty, as I'll now show.

Suppose $x \in \mathbb{Q} \cap (\mathbb{Q} + \{\sqrt{2}\})$. Then, we have

$$x = \frac{p}{q} = \frac{r}{s} + \sqrt{2}$$

for $p, q, r, s \in \mathbb{Z}$. Then $\sqrt{2} = \frac{ps-qr}{qs} \in \mathbb{Q}$, but $\sqrt{2}$ is not rational. Contradiction.

(d) Show that no dense set can be finite. Can it be countable?

Let A be dense. For each $n \in \mathbb{N}$, let $a_n \in A \cap (n, n+1)$. Then, each a_n is in A and different and there are infinitely many of them. So, A cannot be finite.

Dense sets can be countable. $\mathbb{Q}$ is an example.

5. Some practice thinking about countability.

(a) Show that the union of two countable sets is countable.

Let A and B be countable, with A consisting of the terms of the sequence (a_n) and B consisting of the terms of the sequence (b_n). Now, define a sequence (c_n) by[3]

$$c_n := \begin{cases} a_m & n = 2m-1 \\ b_m & n = 2m \end{cases}$$

This sequence contains all the elements of $A \cup B$. It may contain some repetitions, but that's OK (as we're going for 'countable', rather than 'countably infinite'. This result is still true for 'countably infinite', but requires more fiddly numerology.)

(b) By constructing a function from $\mathbb{N}$ onto $\mathbb{N}^2$ (ie. the set of pairs of counting numbers), show that the union of countably many countable sets is countable.

Any member of $\mathbb{N}$ can be written in a unique way as $2^{\tau(n)}(2\delta_n - 1)$ for integers τ_n and δ_n. Also, all pairs (k, l) arise this way, as $k = \tau_{(2^k(2l-1))}$ and $l = \delta_{2^k(2l-1)}$. So, $f : n \mapsto (\tau_n, \delta_n)$ is an onto function from $\mathbb{N}$ to $\mathbb{N}^2$. Now, let A_i, $(i = 1, 2, \dots)$,

[2]Note that this argument never used the density of B – the union of a dense set with any other set is dense.

[3]I'll follow Burn here, and start my sequences with $n = 1$. If you start with $n = 0$, the same idea works, but the details will be slightly different. Math culture note: these differences are what mathematicians mean when they use the word 'numerology'.

be a countable family of countable sets, all of whose members are exhaustively listed by sequences (a^i_n). We build a new sequence, (b_n) by

$$b_n := a^{\tau(n)}_{\delta(n)}.$$

As f is onto, this catches every element of $\bigcup_{i\in\mathbb{N}} A_i$.

(c) Hence, show that the set of *algebraic numbers* (ie. those numbers which are the solution to some polynomial with integer coefficients) is countable. [You may assume that $\mathbb{Z}^k$ is countable for any k.]

Note that $\mathbb{A} = \bigcup_{n\in NN} A_n$, where A_n is the set of solutions to polynomials of degree n with integer coefficients. Hence, by (b), it is enough to show that each A_n is countable.

Note that $A_n = \bigcup_{(c_0,\dots,c_n)\in\mathbb{Z}^n} B_{(c_0,\dots,c_n)}$, where $B_{(c_0,\dots,c_n)}$ is the set of solutions of $c_0+c_1x+\cdots c_nx^n = 0$. Now, there are at most n solutions to any polynomial of order n, so if we can show that this is really a countable union, we'll be done by (b).

But, it is, by the hint. Let (s_k) be a sequence enumerating $\mathbb{Z}^n$. Then, $A_n = \bigcup_{n\in\mathbb{N}} B_{s_n}$, so we're done.

6. Convergence of subsequences. Let (a_n) be some sequence and define $d_n := a_{2n+1}$ (o**d**d), $e_n := a_{2n}$ (**e**ven), $t_n := a_{3n}$ (**t**riples). Prove the following from the definitions, not using results about subsequences from class or the book.

(a) If $a_n \to l$, then $d_n, e_n, t_n \to l$.

I'll show this for e_n; d_n and t_n are similar. Let $\epsilon > 0$ be given. Then there is some N such that for $n > N$, $|a_n - l| < \epsilon$. Hence, for $n > N/2$,

$$\begin{aligned}|e_n - l| &= |a_{2n} - l| \\ &< \epsilon \qquad\qquad \text{as } 2n > N.\end{aligned}$$

(b) If $d_n, e_n \to l$, then $a_n \to l$.

Let $\epsilon > 0$ be given. Then, there are N_1, N_2 such that for $n > N_1$, $|d_n - l| < \epsilon$ and for $n > N_2$, $|e_n - l| < \epsilon$. Let $N = \max\{N_1, N_2\}$. Then, if $n > N$, then $2n > N_1$ and $2n+1 > N_2$, so $|a_n - l| < \epsilon$, so (a_n) is convergent.

(c) If d_n, e_n and t_n are all convergent, then so is a_n. Give an example to show that it's not enough just to have d_n and e_n convergent.

Suppose $d_n \to d$, $e_n \to e$, $t_n \to t$. Let (u_n), (v_n) be subsequences given by $u_n := a_{6n}$, $v_n := a_{6n-3}$. Then, (u_n) is a subsequence of (e_n) and (t_n), and (v_n) is a subsequence of (d_n) and (t_n).

So, $u_n \to e$ and $u_n \to t$. Limits are unique, so $e = t$. $v_n \to d$ and $v_n \to t$, so $d = t$. Hence, $e = d$. Hence, by (b), $a_n \to e$.

For the counterexample to the strengthening, consider $a_n = (-1)^n$. Then, $d_n = -1$ for all n so is convergent and $e_n = 1$ for all n so is convergent too. However, a_n is not convergent.

7. Determine whether the sets below are bounded above or below and, where possible, find their suprema and infima:

(a) $\{2^n : n \in \mathbb{N}\}$;

This set is bounded below, but not above. Its infimum is 2.

(b) $\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$;

This set is bounded above and below. Its infimum is -1 and its supremum is $\frac{3}{2}$.

(c) $\{\sin(x) : x \in \mathbb{R}\}$.

This set is bounded above and below. Its infimum is -1 and its supremum is 1.

8. Seeing how suprema interact with other concepts.

(a) Suppose $A \subseteq B \subseteq \mathbb{R}$ and B is bounded above. Prove carefully (assuming the appropriate version of the completeness principle) that A and B have suprema and that $\sup(A) \leq \sup(B)$.

B has a supremum by completeness principle VI, as it is bounded above. A is also bounded above, as the following argument shows: let $a \in A$, $a > \sup(B)$. As $A \subseteq B$, $a \in B$. So, $a \leq \sup(B)$. Contradiction.

So, A has a supremum, by completeness principle VI. The above argument aslo shows that $\sup(A) \leq \sup(B)$.

(b) Suppose, $A, B \subseteq \mathbb{R}$ are bounded above. Show that $\sup(A \cup B)$ exists and satisfies $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

Let $M = \max\{\sup(A), \sup(B)\}$. Then I claim, M is an upper bound for $A \cup B$: let $x \in A \cup B$. Then, either $x \in A$ or $x \in B$. Hence, $x \leq \sup(A)$ or $x \leq \sup(B)$. Either way, $x \leq M$. So, M is an upper bound and so $A \cup B$ has a supremum by completeness principle VI.

I must now show that M is least among upper bounds, to see that it is the supremum. Suppose $N < M$ is also an upper bound for $A \cup B$. Then it is an upper bound for A and B too. As $N < M$, either $N < \sup(A)$ or $N < \sup(B)$. Either way, this is a contradiction, as the suprema are the least upper bounds.

(c) Is it always true that $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$?

No, this is not always true. Let $A = \{0, 1\}$, $B = \{0, 2\}$. Then, $\sup(A) = 1$, $\sup(B) = 2$, but $\sup(A \cap B) = 0$.

9. This questions tries to give a reason why taking convergence of decimals as our central completeness axiom may not have been the best idea. We have seen that a whole host of properties all hold in $\mathbb{R}$ and all fail in $\mathbb{Q}$. What about it $\mathbb{N}$? Decide which of the completeness principles (decimals, bounded monotonic subseqences, intersection of nested closed intervals, bounded sequences having convergent subsequences, cluster points, Cauchy sequences, l.u.b.s) hold in $\mathbb{N}$.

The decimals property fails in $\mathbb{N}$: even most finite decimals fail to exist. However, the others still hold.

I: Let (a_n) be a bounded montonic sequence in $\mathbb{N}$. Then, it's eventually constant, so it converges.

II: Let $[a_n, b_n]$ be a sequence of nested closed intervals. Then (a_n) is bounded above (by b_1) and increasing, so has a limit (by I). (b_n) is decreasing, so has a limit b. By standard arguments, we have $a \leq b$.

III: Every bounded sequence of counting numbers is constant on some subsequence, so convergent.

IV: There are no infinte bounded sets of counting numbers, so this is vacuously true.

V: Suppose (a_n) is Cauchy. Take $\epsilon = \frac{1}{2}$. Then, there is some N such that for all $n, m > N$, $|a_n - a_m| < \frac{1}{2}$. So, $a_n = a_m$. That is, every Cauchy sequence is eventually constant, so converges.

VI: Suppose A is a non-empty set of counting numbers bounded above, but with no least upper bound. Then, if M is an upper bound, $M - 1$ must be too, or else M would be least. Iterating this argument M times, we get that 0 is an upper bound, so A is empty. Contradiction.

Hence, the decimals property is a bad way to characterize completeness, if we're interested in generalizing to other spaces, as it relies on them having a sufficiently similar arithmetic to $\mathbb{R}$. Analytic properties shouldn't depend on arithmetic. The other properties are actually all equivalent in any ordered metric space. Our official completenes property will be V, as it makes sense in any metric space (not necessarily ordered)[4]. We'll meet the concept of metric spaces in week 8.

Week 3 – Series.

1. Evaluate the following series, or show that they're divergent.

   (a)
   $$\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^{4n+1}.$$

   This is a geometric series with first term $2^5/3^4$ and common ratio $2^4/3^4$, so it converges and has sum $\frac{\frac{2^5}{3^5}}{1-\frac{2^4}{3^4}} = \frac{32}{65}$.

   (b)
   $$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right).$$

   For this question we will need the following property of the logarithm: $\log(p/q) = \log(p) - \log(q)$. For, using that, we get

   $$\begin{aligned} s_n &= (\log(n+1) - \log(n)) + (\log(n) - \log(n-1)) + \cdots + (\log(2) - \log(1)) \\ &= \log(n+1) + (-\log(n) + \log(n)) + \cdots + (-\log(2) + \log(2)) - \log(1) \\ &= \log(n+1) \end{aligned}$$

   $\log(n+1) \to \infty$, so this sum is divergent.

   (c)
   $$\sum \sin(n).$$

   $(\sin(n))$ is not a convergent sequence, so the series can't be convergent.

[4]So does II, once you've worked out the right generalization of closed interval.

2. We saw last week that there are sequences which list all the rationals between 0 and 1 without repetition. Show that the sum of any such series is divergent.

   Suppose such a series was convergent, then the sequence would satisfy $a_n \to 0$. So, there would be an N such that for all $n > N$, $a_n < \frac{1}{2}$. But, the sequence lists all rationals between 0 and 1, so all the numbers between $\frac{1}{2}$ and 1 get listed. So, they must all get listed within the first N terms, so there are only N of them. Contradiction.

3. For which of the following sequences is $\sum a_n$ convergent? [Hint: you might need to use that the sequence $\left(\frac{n+1}{n}\right)^n$ is convergent and converges to something greater than 1.]

   (a) $\frac{n+1}{n^2-n+22}$.
   We do a limit comparison with $\frac{1}{n}$. Recall that $\sum \frac{1}{n}$ diverges, so if we get the limit coming out to something in between 0 and infinity, we'll have shown the series diverges.

$$\frac{n+1}{n^2-n+22}\frac{n}{1} = \frac{1+\frac{1}{n}}{1-\frac{1}{n}+\frac{22}{n^2}} \to 1$$

   (b) $\frac{\log(n)}{n}$.
   We use $\frac{\log(n)}{n} > \frac{1}{n}$ for n sufficiently large. So, by the first comparison test, the series diverges. [Here, we're using that $\log(n) > 1$ for $n > 2$.]

   (c) $\frac{\log(n)}{n^2}$.
   We use the property $\log(n) < n^{1/2}$. This gives that $\frac{\log(n)}{n^2} < \frac{n^{1/2}}{n^2} = n^{-3/2}$, which converges. So, the series converges.

   (d) $(2+\frac{1}{n})^{-n}$.
   We use the root test. $|a_n|^{1/n} = \frac{1}{2+1/n} \to \frac{1}{2}$. So, it converges.

   (e) $\frac{n!}{n^n}$.
   We use the ratio test.

$$\begin{aligned}\frac{(n+1)!}{(n+1)^{(n+1)}}\frac{n^n}{n!} &= (n+1)\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}\\ &= \left(\frac{n}{n+1}\right)^n\\ &\to e^{-1}\end{aligned}$$

   $e^{-1} < 1$, so this converges.

   (f) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$.

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)}$$

   This converges by limit comparison with $n^{-3/2}$.

4. For which of the following sequences is $\sum a_n$ convergent? Is it absolutely or conditionally convergent?

(a) $(-1)^n \frac{\log(n)}{n}$.

We need to use three facts about log: it's (eventually) positive; $\frac{\log(n)}{n}$ is decreasing; and tends to zero. This gives us precisely the data we need to apply the alternating series test and see that the series is convergent. However, it's not absolutely convergent, as we showed in the previous question. Hence, it's conditionally convergent.

(b) $(-1)^{t(n)}2^{-n}$, where t is defined by

$$t(n) := \begin{cases} 0 & n = 3m \text{ or } n = 3m+1 \\ 1 & n = 3m+2. \end{cases}$$

Grouping consecutive positive terms together, we get that this is

$$(2+1) - \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{8}) - \tfrac{1}{16} + (\tfrac{1}{32} + \tfrac{1}{64}) - \tfrac{1}{128} + \cdots = 3 - \tfrac{1}{2} + \tfrac{3}{8} - \tfrac{1}{16} + \tfrac{3}{64} - \tfrac{1}{128} + \cdots$$

This passes the conditions for the alternating series test, so is convergent.

(c) $\frac{\sin(n)}{n^2}$.

$|\sin(n)/n^2| \le \frac{1}{n^2}$, so the series is absolutely convergent by comparison and hence convergent.

(d) $(-1)^n b_n$ where b_n is given by $b_1 = 10^{-6}$, $b_{n+1} = \sin(b_n)/2$.

We first show that $b_n \to 0$. We have the identity $\lim b_{n+1} = \lim \sin(b_n)/2$. Suppose $\lim b_n = b$. Then $b = \sin(b)/2$. The only roots of this equation are 0 are two numbers whose absolute values are somewhere between 0 and $\pi/2$, reasonably far from 0 (compared with the starting value of 10^{-6}). At each step, the sequence decreases, so the limit must be 0.

So, the series converges by the alternating series test. To see whether it converges absolutely or not, we do the ratio test:

$$\begin{aligned} |\frac{a_{n+1}}{a_n}| &= \frac{b_{n+1}}{b_n} \\ &= \frac{\sin(b_n)}{2b_n} \\ &\to \frac{1}{2} && \text{as } b_n \to 0. \end{aligned}$$

(We're using L'Hospital here, which we haven't proved, but you can take that as a 'property of sine'.) We conclude the sequence is absolutely convergent.

Optional, hard. As in (d), but without the /2.

The argument that it's convergent still goes through, but the argument for absolute convergence doesn't (we get 1 as the limit which is inconclusive). I don't know another way to get this.

Optional, hard. [5] $\frac{\sin(n)}{n}$.

My guess is that it converges, but not absolutely. The intuition is this: the sequence $(\sin(n))$ is quite 'random'[6]. In particular, it's positive and negative about half the time each and doesn't tend to get smaller or larger in absolute value over time. Hence, the series should behave quite like $\sum \frac{(-1)^n}{n}$ and have the same properties. I can't quite manage to make this into a precise proof, though.

5. Find the interval of convergence of the following power series.

(a) $\sum n^{2007} x^n$.
(n^{2007}) is a convergent sequence, so we may replace the lim sup by just lim. $1/\lim n^{2007/n} = 1/\left(\lim n^{1/n}\right)^{2007} = 1$, so the radius of convergence is 1. The center of the interval is 0, so the endpoints to test are ± 1. But, substituting either of these in for x gives a divergent sequence of terms, so the series is divergent. Hence, the interval of convergence is $(-1, 1)$.

(b) $\sum x^{C_k^n}$. [Your answer may depend on k].
The coefficient sequence[7] isn't convergent (note: it isn't constant!). The convergent subsequences split into two classes: those which are all 0 after a finite point and those that are all 1 after a finite point. Hence, the possible limits of convergent subsequences are 0 and 1, so $\limsup \sqrt[n]{a_n} = 1$. So, the radius of convergence is 1 and the endpoints to test are ± 1. But substituting these in for x, we get term sequences which don't tend to 0, so the series is divergent. Hence, the interval of convergence is again $(-1, 1)$.

(c) $\sum \frac{x^n}{3^n n^2}$.
The coefficient sequence is convergent, so we calculate

$$\begin{aligned}\lim \sqrt[n]{a_n} &= \lim \sqrt[n]{\frac{1}{3^n n^2}} \\ &= \frac{1}{3}\left(\lim n^{1/n}\right)^{-2} \\ &\to \frac{1}{3}.\end{aligned}$$

So, the radius of convergence is 3. The endpoints to test are ± 3. The series converges with either one of these, as $\sum \frac{1}{n^2}$ is absolutely convergent. Hence, the interval of convergence is $[3, 3]$.

(d) $\sum 2^n x^{n!}$.
The sequence of terms, once taken to the nth root has 0s and $2^{1/(n-1)!}$ terms. Taking the limit of the latter, we get that the lim sup is 1. The series doesn't converge at either endpoint by the null sequence test, so the interval of convergence is $(1, 1)$.

---

[5] When I say hard, I mean it. I've thought about each of these for about five minutes, and can't do them. If you can, let me know!

[6] It's not very random. In particular, if you know the last six terms, you can make a guess at much better than 50% odds of whether the next term will be positive or negative. You might like to try this out down the bar if you want to pay your student loans off.

[7] I guess I mean the sequence of nth roots of coefficients here, but since they're all 0 or 1, it doesn't matter.

6. Give a proof or counterexample to each of the following.

   (a) If $n^2 a_n \to 0$, then $\sum a_n$ converges.
   This is true. If $n^2 a_n \to 0$, then $n^2|a_n| \to 0$. Take $\epsilon = 1$. Then, there is some N such that, for $n > N$, $n^2|a_n| < 1$. So, we have $|a_n| < 1/n^2$ on a tail, so by the first comparison test, $\sum |a_n|$ converges. But, this means that $\sum a_n$ is absolutely convergent and, so, convergent.

   (b) If $na_n \to 0$, then $\sum a_n$ converges.
   This is false. Consider $a_n := 1/(n \log(n))$. Then $na_n = 1/\log(n) \to 0$, but $\sum a_n$ diverges.

   (c) If $\sum a_n$ converges, then so does $\sum a_n^2$.
   Not true. Take $a_n := (-1)^n/\sqrt{n}$. Then $\sum a_n$ converges (alternating series test), but $\sum a_n^2$ doesn't (harmonic series).

   (d) If $\sum a_n$ converges absolutely, then so does $\sum a_n^2$.
   This is true. $\sum a_n$ converges, so $a_n \to 0$. So, taking $\epsilon = 1$, there's some N such that for $n > N$, $|a_n| < 1$. On this tail, $|a_n^2| < |a_n|$, so $\sum a_n^2$ converges by the first comparison test.

7. Towards a proper definition of the exponential function.

   (a) Show that $\sum_{r=n+1}^{\infty} \frac{n!}{r!}$ is convergent and that its sum is less than $\frac{1}{n}$. [Hint: compare with a geometric series.]
   Each term of this series is less than the corresponding term of a geometric series with first term $\frac{1}{n+1}$ and common ratio $\frac{1}{n+1}$. (To be formal here, you should use induction. Informally, we can see that we start off at $\frac{1}{n+1}$ and each time we divide by something greater than or equal to $n+1$.) Hence, the sum is bounded by

$$\frac{a}{1-r} = \frac{\frac{1}{n+1}}{\frac{n}{n+1}} = \frac{1}{n}.$$

   (b) Define e to be $\sum_{r=0}^{\infty} \frac{1}{r!}$. Show that

$$0 < e - \sum_{r=0}^{n} \frac{1}{r!} < \frac{1}{n(n!)}.$$

   Note that $e - \sum_{r=0}^{n} \frac{1}{r!} = \sum_{r=n+1}^{\infty} \frac{1}{r!}$, so it's clearly positive. The other inequality comes from dividing the bound from part (a) by $n!$ on both sides.

   (c) Deduce that e is irrational. [Hint: if a number is rational, it can be written in the form $p/n!$ for some $p \in \mathbb{Z}$, $n \in \mathbb{N}$.]
   Following the hint, suppose $e = p/n!$ (if you can't see why the claim in the hint is true, it would be a good exercise, in the style of 4.1-3, to prove it). Then, we have

$$0 < p - \sum_{r=0}^{n} \frac{n!}{r!} < \frac{1}{n}.$$

   But, this is impossible, as the quantity in the middle is an integer. Contradiction.

8. sin and cos are defined to be the unique solutions of the differential equation $y'' + y = 0$ subject to the initial conditions $[y(0) = 0,\ y'(0) = 1$ in the case of sine$]$, $[y(0) = 1, y'(0) = 0$ in the case of cosine$]$.

(a) Use these differential equations to find power series for sine and cosine, using the fact (to be proved later) that you can differentiate a power series term-wise on the interior of its interval of convergence. [Given we haven't developed the notion of derivative yet, this question definitely doesn't fit with the formal model of developing analysis. Think of it as 'speculative' or something, if you like.]

Set $y(x) := \sum_{n=0}^{\infty} c_n x^n$. Differentiating and re-indexing so as to make them easier to add we get,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ y'(x) &= \sum_{n=0}^{\infty} c_n n x^{n-1} && \text{differentiating term-by-term} \\ &= \sum_{n=1}^{\infty} c_n n x^{n-1} && \text{ignoring first term, as it's 0} \\ &= \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n && \text{re-indexing.} \\ y''(x) &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n && \text{by the same process as above.} \end{aligned}$$

So, we have

$$0 = y''(x) + y(x) = \sum_{n=0}^{\infty} [c_n + c_{n+2}(n+2)(n+1)]x^n.$$

This gives us the recurrence relation

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)}.$$

For sine, we have that $c_0 = 0$, so $c_{2n} = 0$ for all n. For the odd terms we have $c_1 = 1$, $c_3 = \frac{-1}{2\cdot 3}$, $c_5 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$, etc. In general: $c_{2n-1} = \frac{(-1)^{n+1}}{(2n-1)!}$.
For cosine, we have $c_1 = 0$, so all the odd coefficients are 0. For the evens, we have $c_0 = 1$, $c_2 = \frac{-1}{2}$, $c_4 = \frac{1}{2\cdot 3\cdot 4}$, etc. In general $c_{2n} = \frac{(-1)^n}{(2n)!}$.

(b) Multiply the series for sine and cosine and use your knowledge of trig identities to get a closed form for

$$\sum_{k=1}^{n} \frac{1}{(2k-1)!(2(n-k-1))!}.$$

Let

$$c_n := \frac{(-1)^{n+1}}{(2n-1)!}$$
$$b_n := \frac{(-1)^n}{(2n)!}$$

Then,

$$\sin(x) = \sum_{n=1}^{\infty} c_n x^{2n-1}$$
$$\cos(x) = \sum_{n=1}^{\infty} b_n x^{2n}$$

Set

$$\sin(x)\cos(x) = \sum_{n=0}^{\infty} d_n x_n.$$

Using the formula, we then see that $d_{2n} = 0$ as we only get terms as the product of an odd power of x and an even power (odd + even = odd). For the odd terms, (ignoring signs for the moment)

$$|d_{2n-1}| = \sum_{k=1}^{n} |c_{2k-1}||b_{2(n-k-1)}|$$
$$= \sum_{k=1}^{n} \frac{1}{(2k-1)!(2(n-k-1))!}$$

This is the sum we're interested in. But, we know that $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$. Substituting into the power series for sine, gives

$$\sin(2x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} 2^{2n-1} x^{2n-1}.$$

Hence,

$$\sum_{k=1}^{n} \frac{1}{(2k-1)!(2(n-k-1))!} = \frac{4^{n-1}}{(2n-1)!}.$$

Week 4 – Continuity.

1. Calculating limits.

   (a) Find the limit of the following function as $x \to 1$ from the left and the right. Do one of these using the sequences definition and one using the neighborhood definition. Does $\lim_{x\to 1} f(x)$ exist? What relation does it have to $f(1)$?

$$f(x) := \begin{cases} 3-x & x>1 \\ 1 & x=1 \\ 2x & x<1. \end{cases}$$

Let's do the left limit doing the sequences definition. Suppose $a_n \to 1$ with $a_n < 1$ for all n. Then, $f(a_n) = 2a_n \to 2$ by AOL. So, $\lim_{x\to 1-} f(x) = 2$.

Now for the right limit. I claim the right limit is 2. Let $\epsilon > 0$ be given. Take $\delta = \epsilon$. Then, for $x \in (1, 1+\delta)$, $f(x) \in (2, 2+\delta) = (2, 2+\epsilon)$, which proves it.

Hence, the limit is 2. This bears no relation to $f(1)$.

(b) Use the algebra of limits to find $\lim_{x\to 0} g(x)$, for the following $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. [Note that the limit can be defined at a point not in the domain.]

$$g(x) := \frac{(1+x)^2 - 1}{x}.$$

$$\begin{aligned} \lim_{x\to 0} \frac{(1+x)^2-1}{x} &= \lim_{x\to 0} \frac{2x+x^2}{x} \\ &= \lim_{x\to 0}(2+x) \\ &= 2. \end{aligned}$$

(c) Recall,

$$\chi_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show that $\lim_{x\to 0} \chi_{\mathbb{Q}}(x)$ does not exist, but $\lim_{x\to 0} x\chi_{\mathbb{Q}}(x)$ does.

Suppose $\lim_{x\to 0} \chi_{\mathbb{Q}}(x) = l$. Take $\epsilon = \frac{1}{4}$. Then, there is some $\delta > 0$ such that for all $x \in (-\delta, \delta)$, $|x - l| < \frac{1}{4}$. Pick $r, s \in (-\delta, \delta)$, $r \in \mathbb{Q}$, $s \notin \mathbb{Q}$. Then,

$$\begin{aligned} 1 &= |\chi_{\mathbb{Q}}(r) - \chi_{\mathbb{Q}}(s)| \\ &= |(\chi_{\mathbb{Q}}(r) - l) - (\chi_{\mathbb{Q}}(s) - l)| \\ &\le |\chi_{\mathbb{Q}}(r) - l| + |\chi_{\mathbb{Q}}(s) - l| && \text{Triangle inequality.} \\ &< \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} && \text{Contradiction.} \end{aligned}$$

However, $\lim_{x\to 0} x\chi_{\mathbb{Q}}(x) = 0$. To see this, let $\epsilon > 0$ be given. Take $\delta = \epsilon$. Then, if $|x| < \delta$, we also have $|x\chi_{\mathbb{Q}}(x)| \le |x| < \epsilon$.

2. Some preservation results.

(a) Prove that if f is continuous at x_0, then $|f|$ is too.

Suppose $\lim_{x\to x_0} f(x) = y_0 = f(x_0)$. I claim $\lim_{x\to x_0} |f(x)| = |y_0|$. Let $\epsilon > 0$ be given. Take δ to be the δ that could be taken in a proof that $\lim_{x\to x_0} f(x) = y_0$. Then, if $x \in (x_0 - \delta, x_0 + \delta)$, then

$$\begin{aligned} ||f(x)| - |y_0|| &\leq |f(x) - y_0| && \text{The other Triangle inequality.} \\ &< \epsilon \end{aligned}$$

(b) Is the reverse true?

No. Let $f(x) = 2\chi_{\mathbb{Q}}(x) - 1$. This isn't continuous anywhere (see Q1), but $|f|$ is the constant function 1, which is continuous.

(c) Use part (a) together with a result from the first problem set to prove that if f and g are continuous functions, then so are $\max\{f, g\}$ and $\min\{f, g\}$.

We saw in sheet 1 that

$$\max\{f, g\} = \frac{f + g + |f - g|}{2}$$
$$\min\{f, g\} = \frac{f + g - |f - g|}{2}$$

Putting this together with (a) and using the algebra of limits gets us the result.

3. Examples.

(a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at the points of the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

Define f by

$$f(x) := \begin{cases} n & x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that f is discontinuous at each $1/n$ and continuous at each other point other than 0. Why is it discontinuous at 0? Consider two sequences, $a_n = 1/n$, $b_n = \sqrt{2}/n$. Then $\lim_{n\to\infty} f(a_n) = \infty$, $\lim_{n\to\infty} f(b_n) = 0$ and a_n and b_n both tend to 0.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at the points of the set $\{\frac{1}{n} : n \in \mathbb{N}\}$.

We need to adapt the previous function to make it continuous at 0. We saw that the problem there came by having the limit along the points of discontinuity come out differently to other limits. So, if we don't do that, we'll be OK. Say, take

$$g(x) := \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

4. Some applications of the IVT.

(a) Show that every polynomial of odd degree has a root in $\mathbb{R}$.

Without loss of generality, assume the coefficient of the highest order term in p, our polynomial, has coefficient 1 (if not, divide through by the coefficient to get another polynomial of the same order with the same roots).

Then, $f(x) \to \infty$ as $x \to \infty$, so there is some M such that for $x \geq M$, $f(x) \geq 1$. Pick the least such M, so as $f(M) = 1$. Similarly, there is some N such that $f(N) = -1$ as $f(x) \to -\infty$ as $x \to -\infty$. Apply the IVT with $a = N$, $b = M$.

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose $f(x) \neq B$ for all $x \in \mathbb{R}$. Show that it's either the case that for all x, $f(x) < B$ or for all x, $f(x) > B$.

Suppose there's an x and a y such that $f(x) < B$ and $f(y) > B$. Then, by the IVT, there must be some $z \in [x, y]$ such that $f(z) = B$. Contradiction.

5. Some applications of the fact that the continuous image of a closed bounded interval is a closed bounded interval.

(a) Suppose $f(x) \to \infty$ as $x \to \pm\infty$ for some continuous f. Show that f has a global minimum.

From the limit information we have, we know that there are N, M such that $f(x) > f(0)$ if $x < N$, or $x > M$. As we must have $N < 0 < M$, $N \neq M$ and $[N, M]$ is non-empty. Applying the theorem to $[N, M]$ we see that $f[[N, M]]$ is some interval, $[Y, Z]$, say.

I claim Y is the global minimum. Suppose $f(x) < Y$. Then, $x \notin [N, M]$ as $f[[N, M]] = [Y, Z]$ so $x < N$ or $x > M$. But, then, $f(x) > f(0) \in [Y, Z]$. Contradiction.

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that for each $x \in [a, b]$, there is a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that f has a root in $[a, b]$.

Suppose $f[[a, b]] = [Y, Z]$ and $0 \notin [Y, Z]$. Without loss of generality, suppose $Y > 0$. There must be some x such that $f(x) = Y$. Then, there is a y such that $f(y) \leq \frac{1}{2}Y$. But, then $f(y) \notin [Y, Z]$. Contradiction.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and additive, ie. satisfies the identity

$$f(x + y) = f(x) + f(y).$$

Prove that there is some $c \in \mathbb{R}$, such that $f(x) = cx$. Is the assumption of continuity necessary? [Hint (for first part): first show that $f(0) = 0$, $f(-x) = x$ and use induction to show that $f(nx) = nf(x)$ for $n \in \mathbb{N}$. Then show that $f(rx) = rf(x)$ for rational r. Deduce the result from this.]

Following the hint we first note that

$$f(0) = f(0 + 0) = f(0) + f(0) = 2f(0).$$

So, we must have $f(0) = 0$. Also,

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x).$$

So, $f(x) = -f(x)$. Now for the induction. Note that the base case is clear. Now suppose that $f(nx) = nf(x)$. Then we have

$$\begin{aligned} f((n+1)x) &= f(nx + x) \\ &= f(nx) + f(x) && \text{by property of } f \\ &= nf(x) + f(x) && \text{inductive hypothesis} \\ &= (n+1)f(x). \end{aligned}$$

To see that the claim continues to hold for all rational multiples, note that we need only show that $f(x/n) = f(x)/n$ and our previous work will give the result. This is the same as showing that $nf(t/n) = f(t)$ for all t, but substituting $x = t/n$ into the previous result gives us this.

So, we see that for any rational r, we have $f(r) = rf(1)$. Now, let $y \in \mathbb{R}$ be given. Pick a sequence (y_n) from $\mathbb{Q}$ converging to y (say, the truncated decimals). By continuity,

$$\begin{aligned} f(y) &= \lim_{n\to\infty} f(y_n) \\ &= \lim_{n\to\infty} y_n f(1) \\ &= yf(1). \end{aligned}$$

Then, taking $c = f(1)$ gives us the result.

7. Suppose $f : [a, b] \to [c, d]$ is continuous, $1 : 1$, and that $f(a) = c$, $f(b) = d$. Show that f must be strictly increasing.

   Suppose not. Then, we can find $a' < b'$ such that $f(a') \geq f(b')$. $f(a') = f(b')$ is impossible, as f is $1 : 1$. So, $f(a') > f(b')$. Let $m = \frac{1}{2}(f(a')+f(b'))$. Then, there is an $x_0 \in [a', b']$ such that $f(x_0) = m$, by the IVT. But, $m < f(a')$, so $m \in [f(a), f(a')]$, so the IVT also tells us that there is an $x_1 \in [a, a']$ such that $f(x_1) = m$. But, $x_0 \neq a' \neq x_1$ as $f(a') \neq m$, so $x_0 \neq x_1$ but $f(x_0) = f(x_1)$. Contradiction with f $1 : 1$.

Week 5 – Differentiation.

1. (a) Show that $f(x)$ is differentiable at $x = 1$, where

$$f(x) := \begin{cases} 2x & x \geq 1 \\ x^2 + 1 & x < a \end{cases}.$$

I show the limit exists by taking the left and right limits separately and showing they're equal.

$$\begin{aligned} \lim_{h\to 0+} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0+} \frac{2h}{h} \\ &= 2 \\ \lim_{h\to 0-} \frac{f(1+h) - f(1)}{h} &= \lim_{h\to 0-} \frac{2h + h^2}{h} \\ &= 2 \end{aligned}$$

(b) Show that $g(x) := |x|$ is not differentiable at $x = 0$.
This time, I show the limit doesn't exist, as the 1-sided limits are different.

$$\begin{aligned} \lim_{h\to 0+} \frac{f(h) - f(0)}{h} &= \frac{h}{h} \\ &= 1 \\ \lim_{h\to 0-} \frac{f(h) - f(0)}{h} &= \frac{-h}{h} \\ &= -1. \end{aligned}$$

2. If n is a negative integer and $x \neq 0$ show from the definition that $\frac{d}{dx}(x^n) = nx^{n-1}$. [You will want to use induction and adapt the proof of the product rule.]

   This is equivalent to showing that for n a positive integer, $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$. We start with the base case, for $n = 1$:

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{x-(x-h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2}.
\end{aligned}$$

   Now, for the inductive step, suppose $\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x^{n+1}} &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^n}\frac{1}{x+h} - \frac{1}{x^n}\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^n}\frac{1}{x+h} - \frac{1}{x^n}\frac{1}{(x+h)} + \frac{1}{x^n}\frac{1}{(x+h)} - \frac{1}{x^n}\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{1}{x+h}\frac{{\frac{1}{x+h}}^n - \frac{1}{x^n}}{h} + \frac{1}{x}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \frac{1}{x}\frac{-n}{x^{n+1}} + \frac{1}{x^n}\frac{-1}{x^2} \\
&= \frac{-(n+1)}{x^{n+2}}.
\end{aligned}$$

3. Show that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^5 + f(x) + x = 0.$$

   [Hint: consider f^{-1}.]

   Suppose $g = f^{-1}$. Then, $x^5 + x + g(x) = 0$. This gives us a definition of g and we can see that it's differentiable (it's a polynomial). It's onto (it's odd) and $g'(x) = x^4 + 1 > 0$, so it's 1:1 so the inverse exists and is differentiable.

4. (a) What are the (real) roots of $\phi(x) := 1 + x + x^2 + \cdots + x^{2m-1}$? What happens to the sign of $\phi(x)$ as x varies?

   We can spot that we can take a factor of $(1 + x)$ out to give

$$(1 + x)(1 + x^2 + \cdots + x^{2m-2}).$$

   The second factor is always a sum of positive terms, whatever the sign of x, so is always positive. Hence, the only root is at $x = -1$ and ϕ is negative to the left of here and positive to the right.

(b) Prove that the function

$$f_m(x) := 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{2m}}{2m}$$

has no (real) roots.

This is a polynomial of even order and the coefficient of the largest power of x is positive, so it's positive for large x. We see where it has turning points.

$$\frac{d}{dx}(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{2m}}{2m}) = 1 + x + x^2 + \cdots + x^{2m-1}$$

We showed that the only root of this is at $x = -1$. At $x = -1$, $f_m(x) = (1-1) + (\frac{1}{2} + \frac{-1}{3}) + \cdots + \frac{1}{2m}$ which is a sum of non-negative terms, some of which are positive, so is positive. Hence, f_m has no roots.

5. Another generalized MVT.

(a) Let $f, g, h : [a, b] \to \mathbb{R}$ be differentiable functions. Set

$$F(x) := \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

Show that there is a $c \in (a, b)$ such that $F'(c) = 0$.

Note that $F(a)$ and $F(b)$ are 0 (determinants of matrices with linearly dependent rows are 0). Hence, by the MVT, there is a $c \in (a, b)$ such that $F'(c) = 0$

(b) By a suitably cunning choice of h, use (a) to prove a generalized MVT: if f, g are differentiable and $g'(x)$ is never 0, then there is a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Pick $h = 1$. Then, we have

$$F(x) = f(x)[g(a) - g(b)] - g(x)[f(a) - f(b)] + f(a)g(b) - f(b)g(b).$$

So,

$$F'(x) = f'(x)[g(a) - g(b)] - g'(x)[f(a) - f(b)].$$

$F'(c) = 0$, so we have

$$0 = f'(c)[g(a) - g(b)] - g'(c)[f(a) - f(b)].$$

This easily rearranges to what we want.

6. Suppose $f : [a, b] \to \mathbb{R}$ has derivatives of all orders and that $f(a) = f(b) = f'(a) = f'(b) = 0$. Show that there is some $c \in (a, b)$ such that $f'''(c) = 0$. [Note: there are three $'$s there, not two.]

By applying the MVT to f, we get that there is a $c_0 \in (a, b)$ such that $f'(c_0) = 0$. Now, apply the MVT to f' to get that there is a $c_{0,0} \in (a, c_0)$ such that $f''(c_{0,0}) = 0$ and a $c_{0,1} \in (c_0, b)$ such that $f''(c_{0,1}) = 0$. Hence, applying the MVT to f'', there is a $c \in (c_{0,0}, c_{0,1})$ (and hence in (a, b)) such that $f'''(c) = 0$.

7. Calculate the following using L'Hospital's rule.

   (a) $\lim_{x\to 1} \frac{x^2+2x-3}{2x^3-3x^2+4x-3}$.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2+2x-3}{2x^3-3x^2+4x-3} &= \lim_{x\to 1} \frac{2x+2}{6x^2-6x+4} \\ &= 1.\end{aligned}$$

   (b) $\lim_{x\to 0} \frac{x}{\tan(x)}$.

$$\begin{aligned}\lim_{x\to 0} \frac{x}{\tan(x)} &= \lim_{x\to 0} \frac{1}{\sec^2(x)} \\ &= \lim_{x\to 0} cos^2(x) \\ &= 1.\end{aligned}$$

   (c) $\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x\sin(x)}$.

$$\begin{aligned}\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x\sin(x)} &= \lim_{x\to 0} \frac{\sin(x)-x}{x^2\sin(x)} \\ &= \lim_{x\to 0} \frac{\cos(x)-1}{x^2\cos(x)+2x\sin(x)} \\ &= \lim_{x\to 0} \frac{-\sin(x)}{(2-x^2)\sin(x)+4x\cos(x)} \\ &= \lim_{x\to 0} \frac{-\cos(x)}{(6-x^2)\cos(x)-6x\sin(x)} \\ &= \frac{-1}{6}.\end{aligned}$$

8. Use Taylor's Theorem to obtain an approximation to $\sqrt{5}$ for which the error is at most 2^{-9}.

   We use Taylor's theorem with the Lagrange form of the reminder. Let $f : [4, 5] \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Note that this function is infinitely differentiable, as we've kept away from 0. Then, we have

$$f(5) = f(4) + f'(4) + \frac{1}{2!}f''(4) + \cdots + \frac{1}{n!}f^{(n)}(x)$$

   for some $c \in (a, b)$. Since the maximum absolute value of $f^{(n)}(c)$ occurs at $c = 4$ (for $n > 0$), we'll calculate these until we get a small enough value.

n	$f^{(n)}(x)$	$\frac{1}{n!}f^{(n)}(4)$
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$\frac{-1}{4}x^{-3/2}$	2^{-6}
3	$\frac{3}{8}x^{-5/2}$	2^{-9}

   So, we see, it's enough to go up to the $n = 2$ term to get a small enough error. This gives us

$$2 + \frac{1}{4} - \frac{1}{64} = \frac{143}{64}.$$

As a check, $\left(\frac{143}{64}\right)^2$ is 4.99 to 3 s.f.

9. Some more general applications of Taylor's Theorem. In this question you may assume that f and g have as many continuous derivatives as you require.

   (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ on the interval $[0, 2]$. Show that $|f'(x)| \leq 2$ on this interval. [Hint: Consider the Taylor expansions of $f(0)$ and $f(2)$ about $x \in [0, 2]$ with remainder involving f''.]
   Let $x \in (0, 2)$. Then, we have

$$f(0) = f(x) - xf'(x) + \frac{x^2}{2} f''(c_1) \qquad (I)$$
$$f(1) = f(x) + (2 - x)f'(x) + \frac{(2 - x)^2}{2} f''(c + 2) \qquad (II)$$

   for $c_1 \in (0, x)$, $c_2 \in (x, 2)$. Finding $(II) - (I)$, we see that

$$f(1) - f(0) = 2f'(x) + \frac{(2 - x)^2}{2} f''(c_2) - \frac{x^2}{2} f''(c_1).$$

   Re-arranging and applying the triangle inequality, gives

$$\begin{aligned} |2f'(x)| &\leq 1 + 1 + \frac{1}{2}|(2 - x)^2 - x^2| \\ &= 2 + |2 - 2x| \\ &\leq 4. \end{aligned}$$

   Hence, $|f'(x)| \leq 2$.

   (b) Suppose $g'(0) = g'(2) = 0$. Show that there is a $c \in [0, 2]$ such that

$$|f''(c)| \geq |f(2) - f(0)|.$$

   [Hint: Note that $|f(2) - f(0)| \geq |f(2) - f(1)| + |f(1) - f(0)|$. (Why?)]

Using that the first derivatives at 0 and 2 are 0, we get

$$\begin{aligned} f(1) &= f(2) + 2f''(c_1) \\ &= f(0) + 2f''(c_2) \end{aligned}$$

for $c_i \in (i, i + 1)$. Applying the hint, we get that

$$|f(2) - f(1)| \leq 2\left(|f''(c_1)| + |f''(c_2)|\right).$$

Hence, the largest out of $|f''(c_i)|$ for $i = 1, 2$ must be at least $|f(2) - f(1)|$. But, this was exactly what we were trying to prove.