Math 104 - Homework Problem Sets.

Adam Booth.

Summer 2007.

Week 1 – Axioms for the real numbers; sequences.

1. Use mathematical induction to prove the following:
   (a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$;
   (b) $3^{4n} - 1$ is divisble by 80;
   (c) $n! > 2^n$ for $n > 4$. [Think: what should your base case be here?]

2. Prove carefully from the axioms that the following hold for elements of $\mathbb{R}$: [Two-column proofs or paragraph proofs are fine; you may quote any result which has been proved in class.]
   (a) If $ab = ac$ and $a \neq 0$, then $b = c$; [This is referred to as the *cancellability* of multiplication.]
   (b) If $ab = 0$, then $a = 0$ or $b = 0$; [One nice way of doing this is by contradiction. This property of $\mathbb{R}$ is called being an *integral domain*.]
   (c) $a \geq b$ iff $a + c \geq b + c$;
   (d) If $a < b$ and $c < 0$, then $ac > bc$.

3. This question is meant to help you appreciate an algebraic way in which $\mathbb{R}$ is special. (Much of the course is devoted to analytic ways in which $\mathbb{R}$ is special). Show that there does not exist a subset $P \subseteq \mathbb{C}$ satisfying the *formal properties of positive numbers* on page 11 of Burn.

4. This question is about the absolute value function.
   (a) Show that $||a| - |b|| \leq |a| + |b|$. Also state a 'better bound' on the left hand side [hint: look at the three results summarized on p. 21 of Burn]. Give examples of values for a and b which show that the left hand side, the better bound and the right hand side can all be same, or that some can be different.
   (b) Prove that $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$;
   (c) Prove a similar identity for minimum.

5. Some more practice with absolute values.
   (a) Find all $x \in \mathbb{R}$ satisfying the following inequalities:
      i. $4 < |x + 2| + |x - 1| < 5$;
      ii. $|2x + 1| \leq |x - 1|$.
   (b) Find and sketch the set of all pairs (x, y) satisfying:

i. $|x| - |y| \geq 2$;
ii. $1 \leq |xy| < 2$.

6. For each of the following sequences, say whether or not it's monotonic (and if so, which of the four monotonicity properties it has) and whether it's bounded, unbounded or neither. If it's not monotonic, give a monotonic subsequence.

   (a) $a_n = n$.
   (b) $b_n = 4$.
   (c) $c_n = (-1)^n n$
   (d) $d_n = \lfloor 600/n \rfloor$. [Note that, unlike Burn, I'm using 'floor notation' – $\lfloor \cdot \rfloor$ – for the integer function, which he writes $[\cdot]$. This is because I want to still be able to use square brackets as an alternative to (parentheses) when this helps make a formula easier to read, and because floor notation is more common and has the advantage of reminding you that we're rounding down, not up.]
   (e) $a_n = 100 - 20n + n^2$.

7. Show that the following sequences are null. Do some using the definition (ie. by finding an N for each ϵ) and some using the properties of null sequences listed on page 45.

   (a) $a_n = \frac{1}{2n+1}$.
   (b) $b_n = \frac{n}{n^2+1}$.
   (c) $c_n = 2^{-n}$.
   (d) $d_n = n^{-1} - n^{-2}$.

8. Adapt the method of question 3.40 from Burn to show that $\frac{2^n}{n!}$ is null.

9. Further adapt the method of question 3.40 to show that $\frac{2^n}{n^2} \to \infty$.

Week 2 – Completeness.

1. Using the algebra of limits to calculate limits.

   (a)
   $$\frac{4\sqrt{n^4+1}+1}{n^2+2n+4}.$$

   (b)
   $$\frac{3^n n^2 + 5^n n^3}{5^n\sqrt{n^6+1}+n^{100}}.$$

   (c)
   $$\sqrt{n+1}-\sqrt{n}.$$

2. Calculating a hard limit.

   (a) Prove that
   $$(1+t)^n \geq \frac{n(n-1)}{2}t^2$$
   for all $t \geq 0$. [Hint: Binomial Theorem.]

   (b) Hence prove that
   $$(1+\frac{2}{\sqrt{n}})^n \geq n \geq 1$$
   for sufficiently large n.

   (c) Deduce that $n^{1/n} \to 1$ as $n \to \infty$.

3. ...And another one.

   (a) Prove that for every pair of $n, r \in \mathbb{N}$ with $n \geq r$,
   $$\frac{n!}{r!} \geq r^{n-r}.$$

   (b) Deduce that for a given $r \in \mathbb{N}$, for sufficiently large n,
   $$n! \geq \left(\frac{1}{2}r\right)^n.$$

   (c) Hence show that $(n!)^{1/n} \to \infty$ as $n \to \infty$

4. Some practice thinking about density (a very important topic in more advanced analysis or topology).

   (a) Show that the union of two dense sets is dense.

   (b) Show that the *shift* of any dense set is dense, where the shift of a set A by c is defined as $\{x+c : x \in A\}$. [If we wanted to be ridiculously formal, we could write this $+[A \times \{c\}]$. More usually, though, it's written $A+\{c\}$ or even $A+c$].

   (c) Show that the intersection of two dense sets need not be dense.

   (d) Show that no dense set can be finite. Can it be countable?

5. Some practice thinking about countability.

(a) Show that the union of two countable sets is countable.

(b) By constructing a function from $\mathbb{N}$ onto $\mathbb{N}^2$ (ie. the set of pairs of counting numbers), show that the union of countably many countable sets is countable.

(c) Hence, show that the set of *algebraic numbers* (ie. those numbers which are the solution to some polynomial with integer coefficients) is countable. [You may assume that $\mathbb{Z}^k$ is countable for any k.]

6. Convergence of subsequences. Let (a_n) be some sequence and define $d_n := a_{2n+1}$ (o**d**d), $e_n := a_{2n}$ (**e**ven), $t_n := a_{3n}$ (**t**riples). Prove the following from the definitions, not using results about subsequences from class or the book.

   (a) If $a_n \to l$, then $d_n, e_n, t_n \to l$.

   (b) If $d_n, e_n \to l$, then $a_n \to l$.

   (c) If d_n, e_n and t_n are all convergent, then so is a_n. Give an example to show that it's not enough just to have d_n and e_n convergent.

7. Determine whether the sets below are bounded above or below and, where possible, find their suprema and infima:

   (a) $\{2^n : n \in \mathbb{N}\}$;

   (b) $\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$;

   (c) $\{\sin(x) : x \in \mathbb{R}\}$.

8. Seeing how suprema interact with other concepts.

   (a) Suppose $A \subseteq B \subseteq \mathbb{R}$ and B is bounded above. Prove carefully (assuming the appropriate version of the completeness principle) that A and B have suprema and that $\sup(A) \leq \sup(B)$.

   (b) Suppose, $A, B \subseteq \mathbb{R}$ are bounded above. Show that $\sup(A \cup B)$ exists and satisfies $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

   (c) Is it always true that $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$?

9. This questions tries to give a reason why taking convergence of decimals as our central completeness axiom may not have been the best idea. We have seen that a whole host of properties all hold in $\mathbb{R}$ and all fail in $\mathbb{Q}$. What about it $\mathbb{N}$? Decide which of the completeness principles (decimals, bounded monotonic subseqences, intersection of nested closed intervals, bounded sequences having convergent subsequences, cluster points, Cauchy sequences, l.u.b.s) hold in $\mathbb{N}$.

Week 3 – Series.

Note: this problem set makes reference to some functions we haven't formally defined yet. For these, you should state clearly what properties of the functions you're using. Somewhere during this problem set, you may like to use that $\sum \frac{1}{n\log(n)}$ is divergent. [I can't think of a way to prove this other than the integral test.]

1. Evaluate the following series, or show that they're divergent.

   (a)
   $$\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^{4n+1}.$$

   (b)
   $$\sum \log\left(\frac{n+1}{n}\right).$$

   (c)
   $$\sum \sin(n).$$

2. We saw last week that there are sequences which list all the rationals between 0 and 1 without repetition. Show that the sum of any such series is divergent.

3. For which of the following sequences is $\sum a_n$ convergent? [Hint: you might need to use that the sequence $\left(\frac{n+1}{n}\right)^n$ is convergent and converges to something greater than 1.]

   (a) $\frac{n+1}{n^2-n+22}$.
   (b) $\frac{\log(n)}{n}$.
   (c) $\frac{\log(n)}{n^2}$.
   (d) $(2+\frac{1}{n})^{-n}$.
   (e) $\frac{n!}{n^n}$.
   (f) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$.

4. For which of the following sequences is $\sum a_n$ convergent? Is it absolutely or conditionally convergent?

   (a) $(-1)^n \frac{\log(n)}{n}$.
   (b) $(-1)^{t(n)}2^{-n}$, where t is defined by
   $$t(n) := \begin{cases} 0 & n = 3m \text{ or } n = 3m+1 \\ 1 & n = 3m+2. \end{cases}$$
   (c) $\frac{\sin(n)}{n^2}$.
   (d) $(-1)^n b_n$ where b_n is given by $b_1 = 10^{-6}$, $b_{n+1} = \sin(b_n)/2$.

Optional, hard. As in (d), but without the /2.

Optional, hard. [1] $\frac{\sin(n)}{n}$.

[1]When I say hard, I mean it. I've thought about each of these for about five minutes, and can't do them. If you can, let me know!

5. Find the interval of convergence of the following power series.

   (a) $\sum n^{2007} x^n$.
   (b) $\sum x^{\binom{n}{k}}$. [Your answer may depend on k].
   (c) $\sum \frac{x^n}{3^n n^2}$.
   (d) $\sum 2^n x^{n!}$.

6. Give a proof or counterexample to each of the following.

   (a) If $n^2 a_n \to 0$, then $\sum a_n$ converges.
   (b) If $n a_n \to 0$, then $\sum a_n$ converges.
   (c) If $\sum a_n$ converges, then so does $\sum a_n^2$.
   (d) If $\sum a_n$ converges absolutely, then so does $\sum a_n^2$.

7. Towards a proper definition of the exponential function.

   (a) Show that $\sum_{r=n+1}^{\infty} \frac{n!}{r!}$ is convergent and that its sum is less than $\frac{1}{n}$. [Hint: compare with a geometric series.]
   (b) Define e to be $\sum_{r=0}^{\infty} \frac{1}{r!}$. Show that

   $$0 < e - \sum_{r=0}^{n} \frac{1}{r!} < \frac{1}{n(n!)}.$$

   (c) Deduce that e is irrational. [Hint: if a number is rational, it can be written in the form $p/n!$ for some $p \in \mathbb{Z}$, $n \in \mathbb{N}$.]

   Note: we now have three potential definitions for the exponential function (the power series, one based on a differential equation, and one which puts together this definition of e with a proper definition of one number to the power of another – which we're also yet to give). Tieing together the first two is straightforward and rather similar to the next exercise. We won't be in a position to think about tieing the third in until next week, when we have a better understanding of continuous functions.

8. sin and cos are defined to be the unique solutions of the differential equation $y'' + y = 0$ subject to the initial conditions [$y(0) = 0$, $y'(0) = 1$ in the case of sine], [$y(0) = 1, y'(0) = 0$ in the case of cosine].

   (a) Use these differential equations to find power series for sine and cosine, using the fact (to be proved later) that you can differentiate a power series termwise on the interior of its interval of convergence. [Given we haven't developed the notion of derivative yet, this question definitely doesn't fit with the formal model of developing analysis. Think of it as 'speculative' or something, if you like.]
   (b) Multiply the series for sine and cosine and use your knowledge of trig identities to get a closed form for

   $$\sum_{k=1}^{n} \frac{1}{(2k-1)!(2(n-k-1))!}.$$

Week 4 – Continuity.

1. Calculating limits.

   (a) Find the limit of the following function as $x \to 1$ from the left and the right. Do one of these using the sequences definition and one using the neighborhood definition. Does $\lim_{x\to 1} f(x)$ exist? What relation does it have to $f(1)$?

   $$f(x) := \begin{cases} 3 - x & x > 1 \\ 1 & x = 1 \\ 2x & x < 1. \end{cases}$$

   (b) Use the algebra of limits to find $\lim_{x\to 0} g(x)$, for the following $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. [Note that the limit can be defined at a point not in the domain.]

   $$g(x) := \frac{(1+x)^2 - 1}{x}.$$

   (c) Recall,

   $$\chi_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

   Show that $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{x\to 0} xf(x)$ does.

2. Some preservation results.

   (a) Prove that if f is continuous at x_0, then $|f|$ is too.

   (b) Is the reverse true?

   (c) Use part (a) together with a result from the first problem set to prove that if f and g are continuous functions, then so are $\max\{f, g\}$ and $\min\{f, g\}$.

3. Examples.

   (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at the points of the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

   (b) Find a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere except at the points of the set $\{\frac{1}{n} : n \in \mathbb{N}\}$.

4. Some applications of the IVT.

   (a) Show that every polynomial of odd degree has a root in $\mathbb{R}$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose $f(x) \neq B$ for all $x \in \mathbb{R}$. Show that it's either the case that for all x, $f(x) < B$ or for all x, $f(x) > B$.

5. Some applications of the fact that the continuous image of a closed bounded interval is a closed bounded interval.

   (a) Suppose $f(x) \to \infty$ as $x \to \pm\infty$ for some continuous f. Show that f has a global minimum.

   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that for each $x \in [a, b]$, there is a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that f has a root in $[a, b]$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and additive, ie. satisfies the identity

$$f(x+y) = f(x) + f(y).$$

Prove that there is some $c \in \mathbb{R}$, such that $f(x) = cx$. Is the assumption of continuity necessary? [Hint (for first part): first show that $f(0) = 0$, $f(-x) = x$ and use induction to show that $f(nx) = nf(x)$ for $n \in \mathbb{N}$. Then show that $f(rx) = rf(x)$ for rational r. Deduce the result from this.]

7. Suppose $f : [a,b] \to [c,d]$ is continuous, $1:1$, and that $f(a) = c$, $f(b) = d$. Show that f must be strictly increasing.

Week 5 – Differentiation.

1. (a) Show that $f(x)$ is differentiable at $x = 1$, where

$$f(x) := \begin{cases} 2x & x \geq 1 \\ x^2 + 1 & x < a \end{cases}.$$

   (b) Show that $g(x) := |x|$ is not differentiable at $x = 0$.

2. If n is a negative integer and $x \neq 0$ show from the definition that $\frac{d}{dx}(x^n) = nx^{n-1}$. [You will want to use induction and adapt the proof of the product rule.]

3. Show that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^5 + f(x) + x = 0.$$

   [Hint: consider f^{-1}.]

4. (a) What are the (real) roots of $\phi(x) := 1 + x + x^2 + \cdots + x^{2m-1}$? What happens to the sign of $\phi(x)$ as x varies?

   (b) Prove that the function

$$f_m(x) := 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{2m}}{2m}$$

   has no (real) roots.

5. An alternative generalized MVT.

   (a) Let $f, g, h : [a, b] \to \mathbb{R}$ be differentiable functions. Set

$$F(x) := \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

   Show that there is a $c \in [a, b]$ such that $F'(c) = 0$.

   (b) By a suitably cunning choice of h, use (a) to prove a generalized MVT: if f, g are differentiable and $g'(x)$ is never 0, then there is a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

   .

6. Suppose $f : [a, b] \to \mathbb{R}$ has derivatives of all orders and that $f(a) = f(b) = f'(a) = f'(b) = 0$. Show that there is some $c \in (a, b)$ such that $f'''(c) = 0$. [Note: there are three $'$s there, not two.]

7. Calculate the following using L'Hospital's rule.

   (a) $\lim_{x \to 1} \frac{x^2+2x-3}{2x^3-3x^2+4x-3}$.

   (b) $\lim_{x \to 0} \frac{x}{\tan(x)}$.

   (c) $\lim_{x \to 0} \frac{1}{x^2} - \frac{1}{x \sin(x)}$.

8. Use Taylor's Theorem to obtain an approximation to $\sqrt{5}$ for which the error is at most 2^{-9}.

9. Some more general applications of Taylor's Theorem. In this question you may assume that f and g have as many continuous derivatives as you require.

   (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ on the interval $[0, 2]$. Show that $|f'(x)| \leq 2$ on this interval. [Hint: Consider the Taylor expansions of $f(0)$ and $f(2)$ about $x \in [0, 2]$ with remainder involving f''.]

   (b) Suppose $g'(0) = g'(2) = 0$. Show that there is a $c \in [0, 2]$ such that

   $$|f''(c)| \geq |f(2) - f(0)|.$$

   [Hint: Note that $|f(2) - f(0)| \geq |f(2) - f(1)| + |f(1) - f(0)|$. (Why?)]

Week 6 – Integration.

1. (a) Suppose $f(x) \geq 0$ for all x, f is continuous and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.
   (b) Show that the hypotheses of $f(x) \geq 0$ and f continuous were required.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   (a) Suppose f^2 is Riemann-integrable. Does this imply f is?
   (b) Suppose f^3 is Riemann-integrable. Does this imply f is?
   (c) Suppose we were considering $\int_1^\infty f$. Would your answer to (a) or (b) change?

3. This question is about how to 'adjoin' certain types of functions to the class of Riemann integrable functions. If you've seen field theory in abstract algebra (note: this has nothing to do with the sorts of fields one encounters in multivariate calculus), this is rather parallel to the idea of adjoining a root of an equation there.
   (a) Let $R(\chi_{\mathbb{Q}})$ denote the set of functions that can be written in the form
   $$f(x) = \begin{cases} g(x) & x \in \mathbb{Q} \\ h(x) & x \notin \mathbb{Q} \end{cases}$$
   for g, h Riemann integrable functions. For $c \in [0, 1]$, define the c-integral, defined on all functions in $R(\chi_{QQ})$ as
   $$^c\int_a^b f := c\int_a^b g + (1-c)\int_a^b h$$
   where the integral on the left-hand side is the normal Riemann integral. Show that the c-integral is well defined and satisfies the conditions in Burn.
   (b) Given a function F whose range on every bounded set is bounded, show how to define a c-integral on $\dot{R}(F)$, the set of functions which can be written as
   $$f(x) = g(x) + \lambda F(x)$$
   where g is Riemann integrable and $\lambda \in \mathbb{R}$.

4. This question is about applying the techniques of integration theorems slightly more generally than you usually would in a calculus class.
   (a) If f is a 1:1 continuously differentiable function with no zeroes, find a closed form expression for
   $$\int_b^a \frac{f'}{f}.$$
   (b) Suppose f and g both have infinitely many continuous derivatives and $f^{(n)} = f$, $g^{(m)} = g$ (where $n, m > 0$ are the least such) and the least common multiple of n and m is odd. Find a closed form expression for
   $$\int fg.$$

5. A proof of the Hölder inequality (fundamental to functional analysis – a bringing together of linear algebra and analysis – and applications). Throughout this question, assume that $\frac{1}{p} + \frac{1}{q} = 1$ and that $p, q > 0$. The special case $p = q = 2$ is the Schwarz inequality and is the most used form of the inequality.

   (a) Prove Young's inequality: for $u, v \geq 0$,

   $$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

   [Hint: Write $uv = e^{\log(a)+\log(b)}$ and the fact that the exponential function is convex, ie. for $t \in (0,1)$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.]

   (b) Prove that if $f, g : [a,b] \to \mathbb{R}$ are positive integrable functions, such that $\int_a^b f^p = \int_a^b g^q = 1$, then

   $$\int_b^a fg \leq 1.$$

   (c) Prove the Hölder inequality for arbitrary integrable functions $f, g : [a,b] \to \mathbb{R}$:

   $$|\int_a^b fg| \leq \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}.$$

6. Let f be positive and continuous on $[1, \infty)$. Suppose that

   $$F(x) := \int_1^x f \leq [f(t)]^2$$

   for $x \in [1, \infty)$. Show that $f(x) \geq \frac{1}{2}(x-1)$ in the same range. [Hint: Evaluate the integral $\int_1^x [F(t)]^{-1/2} F'(t)\, dt$ and see how the inequality is relevant to it.]

7. Prove that, for $y \in (0, \frac{1}{2})$,

   $$\int_{\frac{1}{2}-y}^{\frac{1}{2}+y} \frac{2x-1}{x(1-x)}\, dx = 0.$$

   Can you deduce that $\int_0^1 \frac{2x-1}{x(1-x)}\, dx$ exists as an improper integral?

8. The Riemann zeta function is defined as

   $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

   Prove that

   $$\zeta(s) = s \int_1^{\infty} \frac{\lfloor x \rfloor}{x^{s+1}}.$$

   [The Rieman zeta function is the source of one of the most famous open problems in math: The Riemann Hypothesis. This states that all zeroes of the zeta function have real part $\frac{1}{2}$, or are negative integers.]

Week 7 – Uniformity.

1. Which of the following sequences of functions is converges uniformly on $[0,1]$?
   (a) x^n
   (b) $(x/2)^n$
   (c) $\sin(nx)$.

2. (a) Prove that if $f_n : [a,b] \to \mathbb{R}$ are continuous, decreasing in n and $f_n \to f$ which is also continuous on $[a,b]$, then $f_n \to f$ uniformly.
   (b) Is the same true for functions on the whole real line?

3. Consider

$$f_n(x) := \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2(\pi/x) & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases}$$

   (a) Show that (f_n) converges to a continuous function, but not uniformly.
   (b) By considering $\sum f_n$, show that a series can converge absolutely on the whole real line without converging uniformly.

4. Show that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

   is continuous on $\mathbb{R}$.

5. Let (x_n) be a sequence of distinct points from $(0,1)$ and $\sum c_n$ be an absolutely convergent series. Show that $\sum c_n \chi_{(x_n,\infty)}(x)$ is continuous at every $x \neq x_n$.

6. Consider

$$\sum_{n=1}^{\infty} \frac{\lfloor nx \rfloor}{n^3}$$

   (a) Show that f is discontinuous on a countable dense set.
   (b) Show that f is Riemann-integrable on every bounded interval.

7. Consider the following two initial value problems

$$y' = (1-2x)y \qquad y(0) = 1 \qquad (1)$$
$$y' = x^2 + y^2 \qquad y(0) = 0 \qquad (2)$$

   (a) Find the first four terms in the Picard sequence of approximations for both problems.
   (b) For problem (1), show that the sequence converges uniformly for all x, and say what it converges to.
   (c) For problem (2), show that the sequence converges uniformly for $|x| < 1/\sqrt{2}$.