AN INTRODUCTION TO COMBINATORIAL GARSIDE STRUCTURES

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Abstract. The notion of a Garside group was first introduced in a paper of Dehornoy and Paris [14]. Over the past decade they have been used as a tool to better understand the structure of Artin's braid groups [2] and their generalizations. In general, one can use the Garside structure associated with a Garside group to solve the word and conjugacy problems, as well as create a finite dimensional Eilenberg-MacLane space for the group. In general it may take some effort construct a Garside structure for a given group and many groups do not have Garside structures. Recently McCammond [19] has approached this problem from a different angle by combinatorially creating groups with built-in Garside structures.

This paper hopes to serve as an expose on recent developments in the theory of Garside structures, especially McCammond's combinatorial Garside structures, filling in non-trivial details that are often omitted in other expositions.

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1. Introduction and historical background

In this section we will provide a quick historical background to the formation of Garside groups, including E. Artin's description of the braid group in the 1920's, the work of Garside to solve the word and conjugacy problems of the braid group in the late 1960's, and lastly the work of Dehornoy and Paris in the late 1990's to generalize the concepts Garside used to a much broader class of groups.

1.1. Artin's braid groups.

The classical presentation for the braid group on n strands, often denoted B_n, is due to Artin in 1925 [1, 2]. It is named the "braid group" because it describes the different ways in which n strands of string may be braided together by repeatedly twisting adjacent strands.

Definition 1.1 (Braid Group)**.** Given an integer $n \geq 2$, the braid group on n strands, B_n, has a presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (i = 1, \ldots, n-2) \end{array} \right\rangle .$$

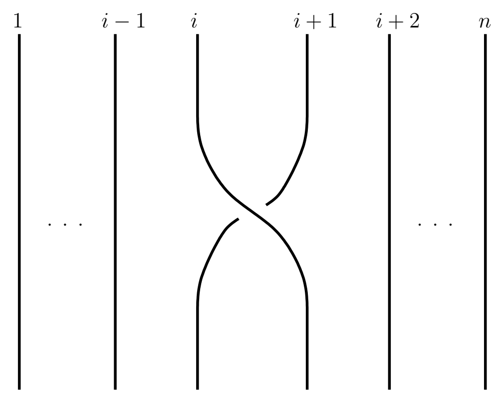


Figure 1. The braid corresponding to the generator σ_i.

Given this presentation of B_n, Artin posed several questions of which he was only able to answer the first:

- **Word Problem:** Given two words in the generators of B_n, does there exist an algorithm that can determine if the two words represent the same element of B_n?
- **Conjugacy Decision Problem:** Given two elements of B_n, does there exist an algorithm to decide whether or not they are conjugate?
- **Conjugacy Search Problem:** Given two elements of B_n which are known to be conjugate, does there exist an algorithm to find a third element of B_n which conjugates the one into the other?

Artin's solution to the word problem of B_n was to study the pure braid group, denoted P_n resulting as the kernel of the natural homomorphism from $B_n \to S_n$ mapping a braid to its permutation of the ends of the strands. The latter two problems remained unsolved for many decades until the work of Garside in 1969 [16] and his new solution to the word problem.

1.2. **Garside's new solution to the word problem.** Garside's solution to the word problem of B_n relied upon several key realizations:

- The presentation of B_n (as in Definition 1.1) defines not only a group but also a monoid ${B_n}^+$.
- The monoid ${B_n}^+$ embeds naturally into the group B_n, in a strong sense that if two positive words represent the same element in B_n they also represent the same element in ${B_n}^+$.
- There is a distinguished element of B_n (and ${B_n}^+$) called the Garside braid (or Garside element) given by

  $$\Delta = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1}).$$

  The Garside element satisfies:
  - σ_i is both a left and right divisor of Δ in ${B_n}^+$.
  - conjugation by Δ maps generators to generators, i.e. $\sigma_i\Delta = \Delta\sigma_{n-i}$.
  - the set of left divisors of Δ (in ${B_n}^+$) and the set of right divisors of Δ are equal.

  In particular these facts allow one to write any element $\omega \in B_n$ in the form $\Delta^r\beta$ for some $r \in \mathbb{Z}$ and $\beta \in {B_n}^+$.
- Of all expressions of a word in the form $\Delta^r\beta$ (with $\beta \in {B_n}^+$) there is a maximum value of r that appears. From now on assume the word is in this form with the maximal value of r achieved.
- Because the relations of B_n are homogeneous (length preserving), there are only a finite number of possible β that could represent the same element. The representative in which the subscripts in β are lexicographically minimal is chosen and the resulting word is said to be in normal form.

This solves the word problem as one wishes to determine if ω_1 and ω_2 represent the same element they merely need to rewrite each in Garside's normal form and check for equivalence.

This new structure also allowed Garside to provide a solution for the conjugacy problems. For detailed description of Garside's solution to the conjugacy problems and recent improvements in the algorithm see [6, 7, 16].

1.3. **Recent developments.** In 1972, Brieskorn and Saito [9] were able to generalize Garside's structural observations of braid groups to all Artin groups of finite type.

In the last decade Dehornoy and Paris [14] have axiomatized Garside's observations and produced examples of groups which contain a similar 'Garside structure' but which are not Artin groups of finite type.

Lastly, McCammond [19] has spent some effort in developing ways to create groups with a specified Garside structure rather than searching for a Garside structure within a group.

2. Preliminaries

2.1. **Posets and lattices.** Before we can properly define the Garside structure of a group we will first give some basic definitions. Readers familiar with the subject may wish to skip to Section 3 for our definition of a Garside group.

Definition 2.1 (Poset)**.** A partially ordered set, or poset, is a pair $(P, \leq)$ where P is a set and "$\leq$" is a binary relation on P satisfying

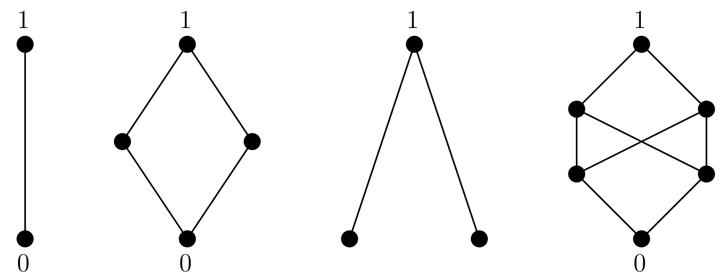


Figure 2. The Hasse diagrams of four posets. The left two posets are bounded lattices. The third is neither bounded nor a lattice. The fourth is a bounded poset, but fails to be a lattice because meets and joins are not well-defined.

- $a \leq a$ for all $a \in P$;
- $a \leq b$ and $b \leq a$ implies $a = b$; and
- $a \leq b$ and $b \leq c$ implies $a \leq c$.

In situations where the binary relation is unambiguous we will often refer to the poset $(P, \leq)$ as just P. Additionally we will write $a < b$ to mean that $a \leq b$ and $a \neq b$.

Definition 2.2 (Chain)**.** A chain C in a poset $(P, \leq)$ is a totally ordered subset of P. In other words, every pair of elements $x, y \in C$ satisfies either $x \leq y$ or $y \leq x$ (or both). If a and b are the minimal and maximal elements of C, we say that C is a chain from a to b.

Definition 2.3 (Finite Height Poset)**.** A poset $(P, \leq)$ is said to have finite height if for all $x \leq y$, there is a bound on the lengths of all chains from x to y.

Definition 2.4 (Covering Relations)**.** In a poset $(P, \leq)$ with $a, b \in P$, we say that b covers a if $a \leq b$, $a \neq b$, and whenever $a \leq c \leq b$ then either $c = a$ or $c = b$. In general we will write $a \prec b$ to mean b covers a.

In a finite height poset, the covering relations alone are sufficient to reconstruct the entire order relation. Hasse diagrams exploit this fact to visually present posets in a relatively clear and concise fashion.

Definition 2.5 (Hasse Diagram)**.** A Hasse diagram for a poset $(P, \leq)$ is a directed graph whose vertices correspond to elements of P and whose edges are drawn from x to y if $x \prec y$.

Because such a graph is acyclic, it is common to not indicate the directionality on the edge itself but rather to rearrange the vertices so that all edges point upwards. The Hasse diagrams for several posets are shown in Figure 2.

Definition 2.6 (Bounded Poset)**.** A poset $(P, \leq)$ is bounded if it has a maximal element (denoted 1 and called the 'top') and a minimal element (denoted 0 and called the 'bottom') so that $0 \leq a \leq 1$ for all $a \in P$.

An element c is said to be an upper bound of the elements a and b if both $a \leq c$ and $b \leq c$. Some posets will admit a least upper bound of the elements a and b, namely an element c which is an upper bound for a and b that happens to be less than all other upper bounds for a and b. If a least upper bound of a and b exists, we will call it the *join* of a and b and denote it as $a \vee b$.

In a similar fashion one can describe what it means for an element to be a lower bound or greatest lower bound for two elements of the poset. The greatest lower bound of two elements a and b is called the *meet* of a and b and is denoted $a \wedge b$.

Posets in which meets and joins exist are called lattices.

Definition 2.7 (Lattice)**.** A lattice is a quadruple $(P, \leq, \vee, \wedge)$ so that $(P, \leq)$ is a poset in which meets $(a \vee b)$ and joins $(a \wedge b)$ exist for all pairs of elements $a, b \in P$.

Note that in a lattice we have $a \leq b$ if and only if $a \wedge b = a$, so we will often write a lattice as $(P, \vee, \wedge)$ because the binary relation can be recovered purely from $\wedge$.

2.2. **Monoids.** While the reader is assumed to be familiar with groups, we'll briefly mention the definition of a monoid. In general, a monoid is a group without the requirement that all elements have an inverse.

Definition 2.8 (Monoid)**.** A monoid M is a set together with an associative binary operation $M \times M \to M$ written multiplicatively. In addition, there is an element $e \in M$ called the identity element which satisfies $ae = ea = a$ for all $a \in M$.

If G is a group generated by the elements in some set S, we may write G^+ to indicate the submonoid of G generated by S. That is, G^+ contains all elements of G which can be expressed as a product of non-negative powers of the generators in S. We will often say that G^+ contains all positive words in S or is the 'positive cone' of G generated by S.

Definition 2.9 (Cancellative Monoid)**.** We say that monoid M is *left* [resp. *right*] *cancellative* if whenever $xa = xb$ [resp. $ax = bx$] then $a = b$. We say a monoid M is *cancellative* if it is both left and right cancellative.

Definition 2.10 (Common Multiples)**.** An element $m \in M$ is said to be a *right common multiple* of m_1 and m_2 in M if there exist $x_1, x_2 \in M$ satisfying $m = m_1x_1 = m_2x_2$. Two elements m_1 and m_2 in M are said to *admit a right common multiple* if there exists any $m \in M$ that is a right common multiple of m_1 and m_2.

The atoms of a monoid are the elements $a \in M \setminus \{e\}$ such that whenever $a = bc$ then either $b = e$ or $c = e$. For any $a \in M$, let $\ell(a)$ be the supremum of the lengths of all expressions of a in terms of the atoms of M, setting $\ell(a) = \infty$ if necessary.

Definition 2.11 (Atomic Monoid)**.** A monoid M is said to be atomic if it is generated by its atoms and $\ell(a) < \infty$ for all $a \in M$.

3. Garside groups

Because Garside groups are relatively new, several different, but equivalent, definitions of a Garside group have been published. Some define a Garside monoid and then define a Garside group to be the group of fractions of a Garside monoid. Others start with a Garside group and use a lattice order to define a Garside monoid.

We will use the first approach. The following definition of a Garside monoid appears in [6].

Definition 3.1 (Garside Monoid)**.** A monoid G^+ is a Garside monoid if

(1) G^+ is atomic;

(2) G^+ is cancellative;
(3) every pair of elements in G^+ admit a left and right least common multiple and greatest common divisor; and
(4) there is an element $\Delta \in G^+$ (known as the Garside element) whose left divisors and right divisors coincide, form a lattice, and generate G^+. This set of divisors is traditionally denoted $\mathcal{D}$.

It is a well-known theorem of Ore [21] that any cancellative monoid in which common multiples exist embeds in its group of fractions. For a proof of this theorem see Appendix A.

Definition 3.2 (Garside Group)**.** A group G is a Garside group if it is the group of fractions of a Garside monoid G^+.

Example 3.3. The monoid $\mathbb{Z}^+ = \langle a| \ \rangle$ is a Garside monoid with Garside element $\Delta = a$.

We verify the required conditions:

(1) $\mathbb{Z}^+$ is atomic: the atoms of $\mathbb{Z}^+$ are $\{a\}$ and any element $a^n \in \mathbb{Z}^+$ can be written as a word (in the atoms) of no longer than length n.
(2) $\mathbb{Z}^+$ is cancellative: clear.
(3) greatest common divisors and least common multiples exist: clear.
(4) the left divisors and right divisors of Δ are exactly $\{e, a\}$ which coincide and generate $\mathbb{Z}^+$. Additionally the left (and right) divisors form a simple lattice with two elements.

The group of fractions of $\mathbb{Z}^+$ is the group $\mathbb{Z} = \langle a \mid \ \rangle$, and hence $\mathbb{Z}$ is a Garside group.

For the following examples we will leave the details of the proofs to the reader.

Example 3.4. The monoid $(\mathbb{Z}^n)^+ = \langle x_1, x_2, \ldots, x_n \mid x_i x_j = x_i x_j \text{ (if } i \neq j) \rangle$ is a Garside monoid with Garside element $\Delta = x_1 x_2 \cdots x_n$. Its group of fractions, $\mathbb{Z}^n$, is a Garside group.

Example 3.5. The monoid ${B_3}^+ = \langle a, b \mid aba = bab \rangle$ is a Garside monoid with Garside element $\Delta = aba$. Its group of fractions, B_3, is a Garside group.

Example 3.6. The braid group on three strands has another presentation given by ${B_3}^{\text{BKL}} = \langle a, b, c \mid ab = bc = ca \rangle$ which is due to Birman, Ko, and Lee [8]. Under this presentation the element $\delta = ab$ is a Garside element for the Garside monoid ${B_3}^{\text{BKL}+}$.

Remark 3.7. As the previous two examples demonstrated, a group may have more than one distinct Garside structure associated with it.

Recall (see [19, 20], for example) that a Coxeter group W is completely specified by a finite set of generators S and a mapping $m : S \times S \to \mathbb{N} \cup \{\infty\}$ satisfying

- $m(s, s) = 1$ for all $s \in S$; and
- $m(s, t) = m(t, s) \in \{2, 3, \ldots, \infty\}$ for all $s \neq t$.

The Coxeter group W then has a presentation given by

$$W = \left\langle S \;\middle|\; (st)^{m(s,t)} = 1 \quad (s, t \in S,\, m(s, t) < \infty) \right\rangle$$

and the corresponding Artin group A has a presentation given by

$$A = \left\langle S \;\middle|\; \overbrace{ststst\cdots}^{\text{length } m(s,t)} = \overbrace{tststs\cdots}^{\text{length } m(s,t)} \quad (s,t \in S,\, m(s,t) < \infty) \right\rangle .$$

An Artin group A is said to be of finite type (or spherical type) if the corresponding Coxeter group W is finite. For example B_n is an Artin group with corresponding Coxeter group S_n, the symmetric group on n symbols. In particular B_n is an Artin group of finite type for all n.

Proposition 3.8 (See [7, Example 1]). *Garside structures can be constructed on Artin groups of finite type in two different ways. (Note that in some cases the different constructions actually yield the same Garside structure.)*

This leads to an immediate corollary which one would hope is true considering the original impetus for developing the theory surrounding Garside groups.

Corollary 3.9. *B_n admits a Garside structure for all n.*

In fact B_n admits two distinct Garside structures for all $n \geq 3$, but we will not prove this here. See [4] for example.

4. Combinatorial Garside structures

A combinatorial Garside structure is a labeled poset under certain restrictions which allows it to define a monoid, group, and order complex which behave similarly to a traditional Garside structure. This is an extremely modern concept, first introduced by McCammond [19], with almost no papers published on the topic. The reader is warned that it may take many years for the notation and terminology to standardize.

Combinatorial Garside structures are of some interest because it is a theorem of McCammond that (weakly-graded) Garside groups are in bijection with combinatorial Garside structures [19, Theorem 1.17]. (By a weakly-graded Garside group G, we mean there is a map from the (Garside) generators of G to the positive integers which extends to a homomorphism from $G \to \mathbb{Z}$.)

Before we can form a monoid or group structure from a poset, it is important to provide labels to the order relations in the poset. One way of doing this is by labeling each covering relation in the poset or edge in its Hasse diagram.

Definition 4.1 (Labeled Poset). Let $\mathcal{P} = (P, \leq)$ be a poset and S a set. An edge-labeling on $\mathcal{P}$ is a map from the covering relations on $\mathcal{P}$ to the set S.

We can easily extend the edge-labeling to a labeling on all maximal length chains by concatenation. For example if $x \prec w$ is labeled as a and $w \prec y$ is labeled as b, then we can label the maximal length chain $x \prec w \prec y$ by ab. This can be further extended to what we will call an *interval-labeling* by assigning to each relation (or 'interval') $x \leq y$ a language (i.e. a set of words) consisting of the union of all labels on maximal length chains from x to y.

Definition 4.2 (Weakly-Graded Poset). Let $\mathcal{P} = (P, \leq)$ be an edge-labeled poset with edge-labeling set S. $\mathcal{P}$ is said to be weakly-graded if there a map $\phi : S \to \mathbb{Z}_{>0}$ which induces a map $\rho : P \to \mathbb{Z}_{\geq 0}$ so that $\rho(0) = 0$ and whenever $a \prec b$ with edge-label x, then $\rho(b) = \rho(a) + \phi(x)$.

If such a grading can be achieved by setting $\phi(x) = 1$ for all $x \in S$ we say that $(P, \leq)$ is graded.

There are two main conditions that must be satisfied for a bounded, weakly-graded, finite height, edge-labeled lattice to be a combinatorial Garside structure. They are given in the following two definitions.

Definition 4.3 (Group-like)**.** An interval-labeled poset $(P, \leq)$ is said to be group-like if whenever two 3-element chains $x \leq y \leq z$ and $x' \leq y' \leq z'$ have pairs of corresponding labels in common, then the third pair of labels are also equal.

This requirement is put in place to ensures that the labeling will eventually yield a monoid that is left-cancellative, right-cancellative, and multiplicative.

Definition 4.4 (Balanced)**.** Let $(P, \leq)$ be a bounded, finite height, interval-labeled poset and set $\lambda(x, y)$ denote the label assigned to the relation $x \leq y$ under the interval-labeling scheme. Define

$$\begin{aligned} L(P) &= \{\lambda(0, b) \mid b \in P\}\,; \\ C(P) &= \{\lambda(a, b) \mid a, b \in P \text{ and } a \leq b\}\,; \text{ and} \\ R(P) &= \{\lambda(a, 1) \mid a \in P\}\,. \end{aligned}$$

The poset P is called balanced if $L(P) = R(P)$.

Proposition 4.5. *If $(P, \leq)$ is both balanced and group-like, then $L(P) = C(P) = R(P)$.*

Proof. By hypothesis $L(P) = R(P)$ and $L(P) \subseteq C(P)$, so it suffices to show that $C(P) \subseteq L(P)$.

Fix any $\lambda(a, b) \in C(P)$ and consider $\lambda(a, 1) \in R(P)$, which by hypothesis is equal to some $\lambda(0, d) \in L(P)$. Additionally choose any maximal length chain $b = b_0 \prec b_1 \prec \cdots \prec b_n = 1$. We will construct a sequence of elements $d_n, d_{n-1}, \ldots, d_0$ which satisfy $d_0 \prec d_1 \prec \cdots \prec d_n = d$ and $\lambda(a, b_i) = \lambda(0, d_i)$ for all i. In particular, $\lambda(a, b) = \lambda(0, d_0) \in L(P)$ which will complete the proof.

Set $d_n = d$. By induction, assume that we have constructed d_k which satisfies $\lambda(a, b_k) = \lambda(0, d_k)$. We will construct d_{k-1}.

We know b_k covers b_{k-1} and hence $\lambda(b_{k-1}, b_k) = \{x\}$ where x is the edge-label given to the covering relation $b_{k-1} \prec b_k$. Because there is some maximal length chain from a to b_{k-1}, there must be at least one word in one of the languages in $\lambda(a, b_k)$ which ends in x. Because $\lambda(a, b_k) = \lambda(0, d_k)$, this implies that there exists an element $d_{k-1} \in P$ so that the edge-label given to the relation $d_{k-1} \prec d_k$ is also x.

Now we have two chains $a \leq b_{k-1} \leq b_k$ and $0 \leq d_{k-1} \leq d_k$ which satisfy $\lambda(a, b_k) = \lambda(0, d_k)$ and $\lambda(b_{k-1}, b_k) = \lambda(d_{k-1}, d_k)$. Because P is group-like, it follows that $\lambda(a, b_{k-1}) = \lambda(0, d_{k-1})$, completing the inductive step. □

Definition 4.6 (Combinatorial Garside Structure)**.** A poset $(P, \leq)$ is a combinatorial Garside structure if P is a bounded, weakly-graded, finite height, edge-labeled lattice which is also group-like and balanced.

We say that $(P, \leq)$ is Garside-like if it satisfies all of the above conditions except the lattice requirement.

We will postpone giving examples of combinatorial Garside structures until we describe how to create a group and monoid from a combinatorial Garside structure.

5. Group and monoid constructions

To every labeled poset there is a standard construction of a monoid and a group. In the case of a combinatorial Garside structure, the monoid will be a Garside monoid and the group a Garside group.

Definition 5.1. Let $(P, \leq)$ be an edge-labeled poset with S the set of labels. Define $\textsc{Mon}(P)$ [resp. $\textsc{Grp}(P)$] to be the monoid [resp. group] generated by S to which we add relations which, for each $x \leq y$, equate the words of all maximal length chains from x to y.

Note that because we have added relations equating all maximal length chains from x to y, it makes sense to talk about the element of $\textsc{Mon}(P)$ arising from the interval $x \leq y$ in P.

Definition 5.2 (Δ and $\mathcal{D}$ in $\textsc{Mon}(P)$)**.** Let $(P, \leq)$ be a combinatorial Garside structure and $\textsc{Mon}(P)$ the monoid derived from P.

The Garside element Δ is defined to be the element of $\textsc{Mon}(P)$ corresponding to the relation $0 \leq 1$ in P.

The set of simple divisors $\mathcal{D}$ is defined to be set of elements in $\textsc{Mon}(P)$ corresponding to the labeling of all relations in P.

Lemma 5.3. *Let $d \in \textsc{Mon}(P)$ be a left-divisor of Δ, then $d \in \mathcal{D}$.*

Proof. Let $r \in \textsc{Mon}(P)$ satisfy $dr = \Delta$. Then writing d and r as a product of elements of the generating set of $\textsc{Mon}(P)$, we see that dr traces out a chain from 0 to 1 in P. However d arises by truncating this chain at the appropriate length and thus corresponds to some interval $0 \leq a$. Therefore $d \in \mathcal{D}$. □

A similar argument shows that if $d \in \textsc{Mon}(P)$ is a right-divisor of Δ then $d \in \mathcal{D}$.

Lemma 5.4. *Let $d \in \mathcal{D}$ be any element. Then there exist elements $r, s, t \in \mathcal{D}$ so that $rd = \Delta = ds$ and $d\Delta = \Delta t$. In particular d is a left and right divisor of Δ.*

Proof. By construction d consists of a word along a maximal length chain in P starting at 0. However because P is balanced, d must also correspond to the label given to some interval $a \leq 1$. Let $r \in \mathcal{D}$ correspond to the interval $0 \leq a$. Then $rd = \Delta$. Similar reasoning gives the existence of an element $s \in \mathcal{D}$ satisfying $ds = \Delta$.

By repeating this observation there exists $t \in \mathcal{D}$ which satisfies $st = \Delta$. Then $d\Delta = dst = \Delta t$. □

The previous two lemmas show us that the name 'simple divisors' for the set $\mathcal{D}$ is apt, as $\mathcal{D}$ is both the set of left-divisors and the set of right-divisors for Δ.

Lemma 5.5. *Let $m = m_1 \cdots m_k$ be a product of k elements of $\mathcal{D}$. Then there exists an element $r \in \textsc{Mon}(P)$ so that $mr = \Delta^k$. In particular m is a left divisor of Δ^k.*

Proof. Observe: If we fix $k > 0$ and $d \in \mathcal{D}$, repeated application of the previous lemma gives us an element $v \in \mathcal{D}$ which satisfies $d\Delta^k = \Delta^k v$.

The proof is by induction on k. The case $k = 1$ is satisfied by Lemma 5.4. Assume this lemma is true when m is a product of $k-1$ elements of $\mathcal{D}$.

Now let $m = m_1 \cdots m_k$. By Lemma 5.4 there is an element $s \in \mathcal{D}$ so that $m_1 s = \Delta$. By the previous observation, there is an element $v \in \mathcal{D}$ so that $s\Delta^{k-1} =$

$\Delta^{k-1}v$. Lastly, by the induction hypothesis there is an element $r \in \textsc{Mon}(P)$ so that $(m_2 \cdots m_k)r = \Delta^{k-1}$. Putting these results together, we see that

$$\begin{aligned} m(rv) =& m_1(m_2 \cdots m_k r)v \\ =& m_1(\Delta^{k-1}v) \\ =& (m_1 s)\Delta^{k-1} \\ =& \Delta^k \end{aligned}$$

completing the inductive step. □

Lemma 5.6. *Let $(P, \leq)$ be a combinatorial Garside structure and* $\textsc{Mon}(P)$ *the monoid derived from P. Then any two elements $m, n \in \textsc{Mon}(P)$ admit a right common multiple.*

Proof. Because they are elements of $\textsc{Mon}(P)$, each of m and n may be written as a finite product of the generating set and hence as a finite product of elements of $\mathcal{D}$. Suppose m can be written as a product of r elements of $\mathcal{D}$ and n as a product of s elements of $\mathcal{D}$. Then by the previous lemma both m and n are left-divisors of $\Delta^{\max(r,s)}$ and hence m and n admit a right common multiple. □

Theorem 5.7. *Let $(P, \leq)$ be a combinatorial Garside structure. Then* $\textsc{Mon}(P)$ *is a Garside monoid.*

Proof. We must verify the four conditions in the definition of a Garside monoid as given in Definition 3.1.

(1) $\textsc{Mon}(P)$ is atomic:
The atoms of $\textsc{Mon}(P)$ are exactly the set S of edge-labels of P. Clearly the atoms generate $\textsc{Mon}(P)$. Also by hypothesis P is weakly-graded, so there exists a map $\phi : S \to \mathbb{Z}_{>0}$. We claim that ϕ extends to a homomorphism $\textsc{Mon}(P) \to \mathbb{Z}_{\geq 0}$, as for every relation $r_1 = r_2$ in the presentation of $\textsc{Mon}(P)$ we will have $\phi(r_1) = \phi(r_2)$. Now any element $m \in \textsc{Mon}(P)$ cannot be written as a product of more than $\phi(m)$ atoms, and hence $\textsc{Mon}(P)$ is atomic.

(2) $\textsc{Mon}(P)$ is cancellative:
The proof is technical and relies upon the lattice structure of P. See [19, Remark 1.15] for a sketch of the proof.

(3) Every pair of elements in $\textsc{Mon}(P)$ admit a left and right least common multiple and greatest common divisor:
Again the proof is omitted and the reader is referred to [19, Theorem 1.7 and 1.17].

(4) The set of left and right divisors of Δ coincide, form a lattice, and generate $\textsc{Mon}(P)$:
In Lemmas 5.3 and 5.4 we showed that $\mathcal{D}$ is exactly equal to the set of left and right divisors of Δ. Note as well that elements in $\mathcal{D}$ are in bijection with the elements of P and hence form a lattice. Lastly $\mathcal{D}$ contains all edge-labels of P and therefore generates $\textsc{Mon}(P)$. □

Lemma 5.8. *Let $(P, \leq)$ be a combinatorial Garside structure. Then* $\textsc{Grp}(P)$ *is the group of right fractions of* $\textsc{Mon}(P)$*.*

Proof. We have shown that $\textsc{Mon}(P)$ admits right common multiples and is cancellative. By the theorem of Ore (see Appendix A), $\textsc{Mon}(P)$ embeds in its group of right fractions G. Because $\textsc{Mon}(P)$ and $\textsc{Grp}(P)$ share a presentation, it must be that $G \cong \textsc{Grp}(P)$. □

Corollary 5.9. *Let* $(P, \leq)$ *be a combinatorial Garside structure. Then* $\textsc{Grp}(P)$ *is a Garside group.*

6. Examples

Now that we have set up much of the theory behind combinatorial Garside structures, we will cover some basic examples to show the utility of the method. Because Garside structures occur very naturally in Artin groups of finite type, we will give examples of posets which give rise to two such groups, B_3 and $\mathbb{Z}^3$.

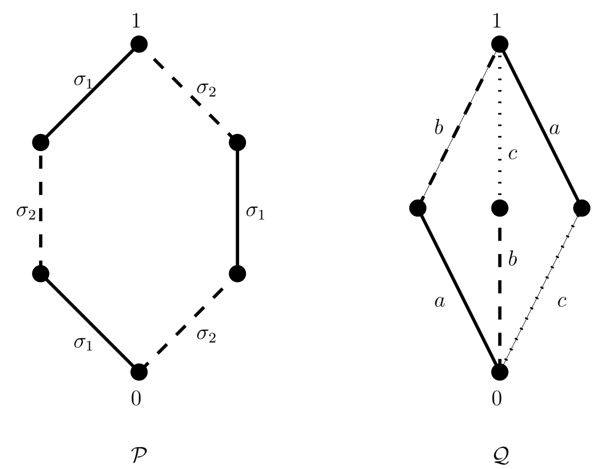


Figure 3. Two combinatorial Garside structures $\mathcal{P}$ and $\mathcal{Q}$.

Example 6.1 (Combinatorial Garside Structures for B_3)**.** Figure 3 shows the Hasse diagrams of two edge-labeled posets, $\mathcal{P}$ and $\mathcal{Q}$. We verify the required conditions for each poset to be a combinatorial Garside structure:

(1) Bounded: yes.
(2) Weakly-graded: yes, in fact each poset is graded.
(3) Finite height: yes.
(4) Edge-labeled: yes.
(5) Lattice: yes.
(6) Group-like: yes, this is easy to check for $\mathcal{P}$ and vacuous for $\mathcal{Q}$.
(7) Balanced:

For $\mathcal{P}$, the set of languages that appear starting at the bottom element are

$$L(\mathcal{P}) = \big\{\{\}, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_1\sigma_2\}, \{\sigma_2\sigma_1\}, \{\sigma_1\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2\}\big\}$$

and those ending at the top element are

$$R(\mathcal{P}) = \big\{\{\}, \{\sigma_1\}, \{\sigma_2\}, \{\sigma_1\sigma_2\}, \{\sigma_2\sigma_1\}, \{\sigma_1\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2\}\big\},$$

hence $L(\mathcal{P}) = R(\mathcal{P})$.

For $\mathcal{Q}$, one can easily check that

$$L(\mathcal{Q}) = R(\mathcal{Q}) = \big\{\{\}, \{a\}, \{b\}, \{c\}, \{ab, bc, ca\}\big\}.$$

Therefore we can conclude that $\mathcal{P}$ and $\mathcal{Q}$ are both combinatorial Garside structures.

The monoid and group associated to $\mathcal{P}$ both have the presentation

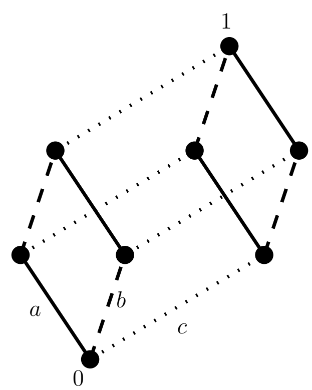
$$\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

which we recognize as $\textsc{Mon}(\mathcal{P}) = {B_3}^{+}$ and $\textsc{Grp}(\mathcal{P}) = B_3$.

The monoid and group associated to $\mathcal{Q}$ both have the presentation

$$\langle a, b, c \mid ab = bc = ca \rangle$$

which we recognize as $\textsc{Mon}(\mathcal{Q}) = {B_3}^{\mathrm{BKL}+}$ and $\textsc{Grp}(\mathcal{Q}) = B_3$, where the alternative presentation of B_3 is due to Birman, Ko, and Lee [8].



$\mathcal{R}$

Figure 4. A combinatorial Garside structure $\mathcal{R}$. Note that we have omitted the labels on many of the edges as to avoid confusion. All solid edges should be labeled a, all long-dashed edges b, and all short-dashed edges c.

Example 6.2 (Combinatorial Garside Structure for $\mathbb{Z}^3$)**.** Figure 4 shows the Hasse diagram of an edge-labeled poset, $\mathcal{R}$. We verify the required conditions for this poset to be a combinatorial Garside structure:

(1) Bounded: yes.
(2) Weakly-graded: yes, in fact the poset is graded.
(3) Finite height: yes.
(4) Edge-labeled: yes.
(5) Lattice: yes, one can tediously verify this by hand.
(6) Group-like: yes.
(7) Balanced: yes, one can check that

$$L(\mathcal{R}) = R(\mathcal{R}) = \big\{\{\}, \{a\}, \{b\}, \{c\}, \{ab, ba\}, \{ac, ca\}, \{bc, cb\}, \\ \{abc, acb, bac, bca, cab, cba\}\big\}$$

The monoid and group associated to $\mathcal{R}$ both have the presentation

$$\langle a, b, c \mid ab = ba,\ ac = ca,\ bc = cb,\ abc = acb = bac = bca = cab = cba\rangle$$

although the last set of relations are easily seen to be superfluous, so we can simplify the presentation to

$$\langle a, b, c \mid ab = ba,\ ac = ca,\ bc = cb\rangle.$$

We easily identify $\textsc{Mon}(\mathcal{R}) = (\mathbb{Z}^3)^+$ and $\textsc{Grp}(\mathcal{R}) = \mathbb{Z}^3$.

Remark 6.3. Replacing the lattice $\mathcal{R}$ in Figure 4 with an appropriately labeled n-dimensional cube will provide a combinatorial Garside structure for $\mathbb{Z}^n$.

7. Two lattice operations that preserve lattice structures

Because the hardest or most tedious condition to check in the definition of a combinatorial Garside structure is often the lattice condition, we will give two methods of creating a larger combinatorial Garside structure from a pair of smaller combinatorial Garside structures. In terms of the resulting group structures, the first operation will correspond to the direct product of two Garside groups and the second will correspond to the free product of two Garside groups amalgamating the infinite cyclic subgroups generated by the Garside elements.

7.1. **Direct products.**

Definition 7.1 (Direct Product of Lattices)**.** Let $\mathcal{P} = (P, \vee_P, \wedge_P)$ and $\mathcal{Q} = (Q, \vee_Q, \wedge_Q)$ be two lattices. Then the direct product of $\mathcal{P}$ and $\mathcal{Q}$ is

$$\mathcal{P} \times \mathcal{Q} = (P \times Q, \vee, \wedge)$$

where we define the meet and join component wise, i.e.

$$(p_1, q_1) \vee (p_2, q_2) = (p_1 \vee_P p_2, q_1 \vee_Q q_2)$$
$$(p_1, q_1) \wedge (p_2, q_2) = (p_1 \wedge_P p_2, q_1 \wedge_Q q_2).$$

See Figure 5 for an example of the direct product of two lattices.

Remark 7.2. The covering relations in $\mathcal{P} \times \mathcal{Q}$ are exactly those of the form $(p_1, q) \prec (p_2, q)$ where $p_1 \prec_P p_2$ or $(p, q_1) \prec (p, q_2)$ where $q_1 \prec_Q q_2$.

With this observation, $\mathcal{P} \times \mathcal{Q}$ inherits a natural edge-labeling from the edge-labeling of $\mathcal{P}$ and $\mathcal{Q}$ (where we have implicitly assumed that the set of edge-labels of $\mathcal{P}$ and $\mathcal{Q}$ are disjoint).

Proposition 7.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two combinatorial Garside structures. Then $\mathcal{P} \times \mathcal{Q}$ is a combinatorial Garside structure.*

Proof. We verify the conditions required for $\mathcal{P} \times \mathcal{Q}$ to be a combinatorial Garside structure:

(1) Bounded: yes.
 The minimal element of $\mathcal{P} \times \mathcal{Q}$ is $(0_P, 0_Q)$ and the maximal element is $(1_P, 1_Q)$.
(2) Weakly-graded: yes.
 Let $\phi_{\mathcal{P}} : S_{\mathcal{P}} \to \mathbb{Z}_{>0}$, $\rho_{\mathcal{P}} : P \to \mathbb{Z}_{\geq 0}$, $\phi_{\mathcal{Q}} : S_{\mathcal{Q}} \to \mathbb{Z}_{>0}$, and $\rho_{\mathcal{Q}} : Q \to \mathbb{Z}_{\geq 0}$ be the maps which satisfy the requirements to make $\mathcal{P}$ and $\mathcal{Q}$ weakly-graded as in Definition 4.2.

Then $\phi : S \to \mathbb{Z}_{>0}$ defined by

$$\phi(s) = \begin{cases} \phi_{\mathcal{P}}(s) & \text{if } s \text{ is an edge label of } P \\ \phi_{\mathcal{Q}}(s) & \text{if } s \text{ is an edge label of } Q \end{cases}$$

extends to a map $\phi : (P \times Q) \to \mathbb{Z}_{\geq 0}$ as $\phi((p,q)) = \phi_{\mathcal{P}}(p) + \phi_{\mathcal{Q}}(q)$ satisfying all the required conditions.

(3) Finite height: yes.
The maximal length of a chain from (p_1, q_1) to (p_2, q_2) is the sum of the maximal length of a chain from p_1 to p_2 in $\mathcal{P}$ and the maximal length of a chain from q_1 to q_2 in $\mathcal{Q}$.

(4) Edge-labeled: yes.
See the previous remark.

(5) Lattice: yes.
We have constructed $\vee$ and $\wedge$ as required.

(6) Group-like: yes.
Because all chains from (p_1, q_1) to (p_2, q_2) arise as a product of a chain from p_1 to p_2 in $\mathcal{P}$ with a chain from q_1 to q_2 in $\mathcal{Q}$. The language that the interval $(p_1, q_1) \leq (p_2, q_2)$ then arises naturally from the intervals $p_1 \leq p_2$ and $q_1 \leq q_2$. The fact that $\mathcal{P}$ and $\mathcal{Q}$ are group-like then trivially extends to $\mathcal{P} \times \mathcal{Q}$.

(7) Balanced: yes.
Again because the labels on all intervals $(0, 0)$ to (p_1, q_1) arise in a predictable fashion from the labels on $0 \leq p$ and $0 \leq q$, and similarly for the labels on all intervals (p_2, q_2) to $(1, 1)$. The balanced condition on $\mathcal{P} \times \mathcal{Q}$ is satisfied because it is satisfied on $\mathcal{P}$ and $\mathcal{Q}$ separately.

Therefore $\mathcal{P} \times \mathcal{Q}$ is a combinatorial Garside structure. □

Proposition 7.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two combinatorial Garside structures. Then* $\text{GRP}(\mathcal{P} \times \mathcal{Q}) = \text{GRP}(\mathcal{P}) \times \text{GRP}(\mathcal{Q})$.

Proof. We investigate the presentation for $\mathcal{P} \times \mathcal{Q}$. The generators for $\text{GRP}(\mathcal{P} \times \mathcal{Q})$ are exactly the edge-labels that appear in either $\mathcal{P}$ or in $\mathcal{Q}$. Recall that the relations in $\text{GRP}(\mathcal{P} \times \mathcal{Q})$ arise from maximal length chains corresponding to all intervals. Intervals in $\mathcal{P} \times \mathcal{Q}$ are of the form:

- $(p_1, q) \leq (p_2, q)$: This will give us a relation among only the generators of $\text{GRP}(\mathcal{P})$. It is easy to see that all such relations in $\text{GRP}(\mathcal{P})$ will be obtained from chains of this type.
- $(p, q_1) \leq (p, q_2)$: This will give us a relation among only the generators of $\text{GRP}(\mathcal{Q})$. It is easy to see that all such relations in $\text{GRP}(\mathcal{Q})$ will be obtained from chains of this type.
- $(p_1, q_1) \leq (p_2, q_2)$:
  - If it is the case that $p_1 \prec p_2$ and $q_1 \prec q_2$, we will have two maximal length chains

$$(p_1, q_1) \prec (p_2, q_1) \prec (p_2, q_2)$$
$$(p_1, q_1) \prec (p_1, q_2) \prec (p_2, q_2).$$

However the relation that arises in this fashion will be $xy = yx$ where x is some edge-label in $\mathcal{P}$ and y is some edge-label in $\mathcal{Q}$. In fact we will

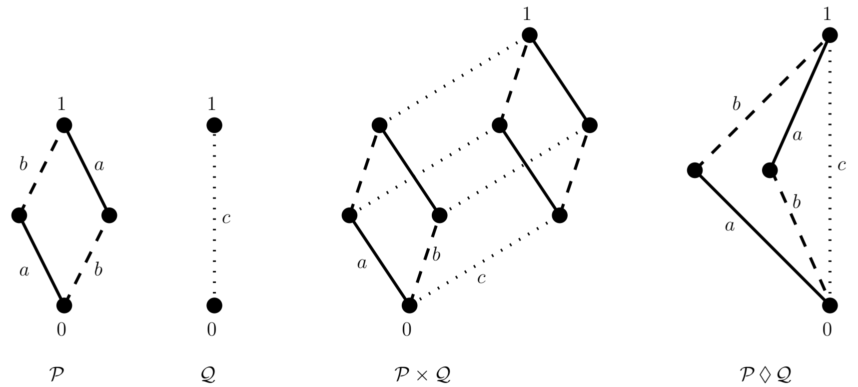


FIGURE 5. The Hasse diagrams of two edge-labeled lattices $\mathcal{P}$ and $\mathcal{Q}$, shown with their direct product $\mathcal{P} \times \mathcal{Q}$ and amalgamated union $\mathcal{P} \lozenge \mathcal{Q}$. Many of the labels on the direct product $\mathcal{P} \times \mathcal{Q}$ are omitted for clarity, but edges with the same shading should be assumed to have the same label.

get that any generator of $\mathrm{GRP}(\mathcal{P})$ will commute with any generator of $\mathrm{GRP}(\mathcal{Q})$.
 – Otherwise, we will get more complicated relation among generators of $\mathrm{GRP}(\mathcal{P})$ and $\mathrm{GRP}(\mathcal{Q})$. By the previous commutativity remark, this will split into the product of two known relations among generators of $\mathrm{GRP}(\mathcal{P})$ and generators of $\mathrm{GRP}(\mathcal{Q})$, yielding no relation that could not have been derived from the others.

Therefore, we have categorized the relations of $\mathrm{GRP}(\mathcal{P} \times \mathcal{Q})$ as:
- all relations appearing in $\mathrm{GRP}(\mathcal{P})$,
- all relations appearing in $\mathrm{GRP}(\mathcal{Q})$,
- relations ensuring that generators of $\mathrm{GRP}(\mathcal{P})$ commute with generators of $\mathrm{GRP}(\mathcal{Q})$,
- and no other relations that could not be derived from those previously mentioned.

Hence $\mathrm{GRP}(\mathcal{P} \times \mathcal{Q}) = \mathrm{GRP}(\mathcal{P}) \times \mathrm{GRP}(\mathcal{Q})$. □

7.2. **Amalgamated free products.**

Definition 7.5 (Amalgamated Union of Bounded Lattices)**.** Let $\mathcal{P} = (P, \vee_P, \wedge_P)$ and $\mathcal{Q} = (Q, \vee_Q, \wedge_Q)$ be two bounded lattices, with minimal elements 0_P and 0_Q and maximal elements 1_P and 1_Q. Then the amalgamated union of $\mathcal{P}$ and $\mathcal{Q}$ is a lattice

$$\mathcal{P} \lozenge \mathcal{Q} = \big((P \mathbin{\dot\cup} Q)/(0_P \sim 0_Q, 1_P \sim 1_Q), \vee, \wedge\big)$$

whose set is the disjoint union of P and Q where we have identified the bottom element and top element of each lattice and will refer to them as 0 and 1. We define $\vee$ and $\wedge$ as

$$a \vee b = \begin{cases} a \vee_P b & \text{if } a, b \in P, \\ a \vee_Q b & \text{if } a, b \in Q, \\ 1 & \text{otherwise;} \end{cases} \qquad \text{and} \qquad a \wedge b = \begin{cases} a \wedge_P b & \text{if } a, b \in P, \\ a \wedge_Q b & \text{if } a, b \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Pictorially, the Hasse diagram of $\mathcal{P} \Diamond \mathcal{Q}$ is the diagram formed by placing the Hasse diagrams for $\mathcal{P}$ and $\mathcal{Q}$ side by side and identifying the minimal element of $\mathcal{P}$ with the minimal element of $\mathcal{Q}$ and likewise for the maximal elements. See Figure 5 for an example.

Remark 7.6. Covering relations of $\mathcal{P} \Diamond \mathcal{Q}$ are in one-to-one correspondence with covering relations of $\mathcal{P}$ and covering relations of $\mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{Q}$ are both edge-labeled then $\mathcal{P} \Diamond \mathcal{Q}$ naturally inherits an edge-labeling.

Proposition 7.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two combinatorial Garside structures. Then $\mathcal{P} \Diamond \mathcal{Q}$ is a combinatorial Garside structure.*

Proof. We verify the conditions required for $\mathcal{P} \Diamond \mathcal{Q}$ to be a combinatorial Garside structure:

(1) Bounded: yes.
The minimal element of $\mathcal{P} \Diamond \mathcal{Q}$ is 0 and the maximal element is 1.

(2) Weakly-graded: yes.
Let $\phi_{\mathcal{P}} : S_{\mathcal{P}} \to \mathbb{Z}_{>0}$, $\rho_{\mathcal{P}} : P \to \mathbb{Z}_{\geq 0}$, $\phi_{\mathcal{Q}} : S_{\mathcal{Q}} \to \mathbb{Z}_{>0}$, and $\rho_{\mathcal{Q}} : Q \to \mathbb{Z}_{\geq 0}$ be the maps which satisfy the requirements to make $\mathcal{P}$ and $\mathcal{Q}$ weakly-graded as in Definition 4.2.
Then $\phi : S \to \mathbb{Z}_{>0}$ defined by

$$\phi(s) = \begin{cases} \phi_{\mathcal{P}}(s)\rho_{\mathcal{Q}}(1_Q) & \text{if } s \text{ is an edge label of } P \\ \phi_{\mathcal{Q}}(s)\rho_{\mathcal{P}}(1_P) & \text{if } s \text{ is an edge label of } Q \end{cases}$$

extends to a map $\rho : (P \Diamond Q) \to \mathbb{Z}_{\geq 0}$ as

$$\rho(a) = \begin{cases} \rho_{\mathcal{P}}(a)\rho_{\mathcal{Q}}(1_Q) & \text{if } a \in P \\ \rho_{\mathcal{P}}(1_P)\rho_{\mathcal{Q}}(a) & \text{if } a \in Q \end{cases}$$

satisfying all the required conditions.

(3) Finite height: yes.
Any interval in $\mathcal{P} \Diamond \mathcal{Q}$ corresponds to either an interval in $\mathcal{P}$ or $\mathcal{Q}$ (or both). The maximal length of any chain along that interval then follows directly from the maximum length in $\mathcal{P}$ or $\mathcal{Q}$.

(4) Edge-labeled: yes.
Yes, see the previous remark.

(5) Lattice: yes.
We have constructed $\vee$ and $\wedge$ as required.

(6) Group-like: yes.
Suppose that $x \leq y \leq z$ and $x' \leq y' \leq z'$ share labeling on two corresponding intervals. Then y and y' must both either belong in $\mathcal{P}$ or in $\mathcal{Q}$, and hence the group-like condition follows from the fact that $\mathcal{P}$ and $\mathcal{Q}$ are both group-like.

(7) Balanced: yes.
We examine $L(\mathcal{P} \Diamond \mathcal{Q})$.
Any interval $0 \leq a$ in $\mathcal{P} \Diamond \mathcal{Q}$ is of the form:
- $0 \leq a$ for $a \in \mathcal{P}$. Then there is an element $b \in \mathcal{P}$ so that $b \leq 1$ has the same label as $0 \leq a$.
- $0 \leq a$ for $a \in \mathcal{Q}$. Then there is an element $b \in \mathcal{Q}$ so that $b \leq 1$ has the same label as $0 \leq a$.
- $0 \leq 1$. This is trivially in $R(\mathcal{P} \Diamond \mathcal{Q})$.

Therefore $L(\mathcal{P} \lozenge \mathcal{Q}) \subseteq R(\mathcal{P} \lozenge \mathcal{Q})$. A similar argument shows the reverse inclusion and thus $\mathcal{P} \lozenge \mathcal{Q}$ is balanced.

Therefore $\mathcal{P} \lozenge \mathcal{Q}$ is a combinatorial Garside structure. □

Proposition 7.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two combinatorial Garside structures. Let $\Delta_P \in \mathrm{Grp}(\mathcal{P})$ and $\Delta_Q \in \mathrm{Grp}(\mathcal{Q})$ be the Garside elements of each group. Then $\mathrm{Grp}(\mathcal{P} \lozenge \mathcal{Q}) = \mathrm{Grp}(\mathcal{P}) *_{\langle\Delta\rangle} \mathrm{Grp}(\mathcal{Q})$, the free product where we have amalgamated the infinite cyclic subgroups generated by Δ_P and Δ_Q.*

Proof. We examine the presentation of $\mathrm{Grp}(\mathcal{P} \lozenge \mathcal{Q})$. The generators of the group are exactly the generators of $\mathrm{Grp}(\mathcal{P})$ and the generators of $\mathrm{Grp}(\mathcal{Q})$. The relations of the group arise from maximal chains along intervals of the form:

- $a \le b$ where either $a \ne 0$ or $b \ne 1$ (or both): This gives us a relation among the edge-labels of either $\mathcal{P}$ or $\mathcal{Q}$.
- $0 \le 1$: This gives us the relations that arise from $0 \le 1$ in $\mathcal{P}$ and $0 \le 1$ in $\mathcal{Q}$, and equates those sets of relations. Recall however that the elements Δ_P and Δ_Q correspond to the elements of the groups equal to those relations.

Therefore the relations in $\mathrm{Grp}(\mathcal{P} \lozenge \mathcal{Q})$ are exactly those in $\mathrm{Grp}(\mathcal{P})$ and $\mathrm{Grp}(\mathcal{Q})$ with one additional relation giving $\Delta_P = \Delta_Q$

Thus we see $\mathrm{Grp}(\mathcal{P} \lozenge \mathcal{Q}) = \mathrm{Grp}(\mathcal{P}) *_{\langle\Delta\rangle} \mathrm{Grp}(\mathcal{Q})$ as desired. □

8. Eilenberg-MacLane spaces

Definition 8.1 (Eilenberg-MacLane Space)**.** Let G be a group and n an integer. A pointed, connected topological space X is called an Eilenberg-MacLane space of type $K(G, n)$ if all homotopy groups $\pi_k(X)$ are trivial except $\pi_n(X)$ which is isomorphic to G.

In general one can construct a CW complex which is an Eilenberg-MacLane space of type $K(G, n)$ for all groups G when $n = 1$ and for all abelian groups G when $n \ge 2$.

If we restrict our attention to only consider CW complexes (and spaces homotopic to CW complexes), it turns out that there is a unique space of type $K(G, n)$ up to homotopy equivalence. So with a slight abuse of notation we will often refer to this space itself as $K(G, n)$. This is a rather technical point that is a consequence of Whitehead's Theorem [17, Theorem 4.5].

It is well known that the (group) cohomology of G with coefficients in M is the same as the (topological) cohomology of $K(G, 1)$ with coefficients in M. Hence the cohomological dimension of G is equal to the cohomological dimension of $K(G, 1)$. As we will provide an explicit construction of $K(G, 1)$ for our constructed Garside groups as a finite dimensional CW complex, it follows that the cohomological dimension of G is finite and hence G is necessarily torsion free.

Definition 8.2 (Order Complex)**.** Let $(P, \le)$ be a poset. The order complex of P, denoted $\Delta(P)$ is the simplicial complex whose n-simplices correspond to chains $x_1 < x_2 < \cdots < x_n$.

Lemma 8.3. *Let P be a poset with a minimal element denoted 0. Then $\Delta(P)$ is a cone over the simplicial complex $\Delta(P - \{0\})$. In particular $\Delta(P)$ is contractible.*

Proof. The order complex $\Delta(P)$ has simplices of three types:

- 0,

- $0 < \sigma$ for some $\sigma \in \Delta(P - \{0\})$, and
- σ for some $\sigma \in \Delta(P - \{0\})$.

This is exactly the construction of a cone over $\Delta(P - \{0\})$. Because cones over any space are contractible it follows that $\Delta(P)$ is contractible. □

Proposition 8.4 (Garside structures admit $K(G, 1)$ of finite dimension)**.** *Let $\mathcal{P}$ be a combinatorial Garside structure. Then there exists a CW complex $\mathrm{K}(\mathcal{P})$ of finite dimension with fundamental group isomorphic to $\mathrm{Grp}(\mathcal{P})$ and contractible universal cover. In particular $\mathrm{K}(\mathcal{P})$ is a finite dimensional $K(\mathrm{Grp}(\mathcal{P}), 1)$ space.*

Proof. Consider the order complex $\Delta(P)$ in which every one-simplex $x < y$ has a label and orientation induced from the labeling of P.

Additionally, we can think of every n-simplex can as having a labeling and orientation induced from its 1-skeleton. In the case of 0-simplices, we will treat these as all having the same (empty) label.

We form a quotient space of $\Delta(P)$ by: if σ and τ are two n-simplices and $f : \sigma \to \tau$ is a label and orientation preserving isometry we then identity σ and τ using f. At most one such f can exist for any pair of simplices σ and τ as the orientation given to the 1-skeleton of any subsimplex of σ and τ is necessarily acyclic by the existence of a minimal element in the chain corresponding to the subsimplex.

Note as well that if σ and τ are both 0-simplices then the map $f : \sigma \mapsto \tau$ is an orientation and label preserving isometry and hence all 0-simplices are identified in this construction.

Let $\mathrm{K}(\mathcal{P})$ be the CW complex arising from this construction. We will show that $\pi_1(\mathrm{K}(\mathcal{P})) \cong \mathrm{Grp}(\mathcal{P})$. We omit listing the base point as $\mathrm{K}(\mathcal{P})$ contains only a single 0-cell which we will assume to be the base point.

It is a well known fact that the fundamental group of any CW complex is completely determined by its 2-skeleton (see [17, Section 1.2] for example). In general one calculates the fundamental group by:

- Choose a base-point among the 0-cells. In this case we have only one 0-cell.
- Calculate the fundamental group of the 1-skeleton: i.e. the set of all words which can be traced around following paths which start and end at the base-point. Because we have only one 0-cell, this will be the free group on the labels which appear on 1-cells.
- For each 2-cell which is glued onto the 1-skeleton, add a relation to the previous group given by $x(a_1 \cdots a_n)x^{-1} = 1$ where $a_1 \cdots a_n$ is the path traced by the boundary of the cell and x is a path from the base-point to the start of the boundary of the 2-cell. Because we have only one 0-cell, we are free to choose x to be the empty word.

The 2-cells that we glue in will either be of the form that ensures the group is multiplicative (i.e. tracing around 'a' and then 'b' is the same as tracing around 'ab') or be of the form that will equate words along maximal length chains in $\mathcal{P}$. This is exactly the construction for the presentation of $\mathrm{Grp}(\mathcal{P})$ and thus $\pi_1(\mathrm{K}(\mathcal{P})) \cong \mathrm{Grp}(\mathcal{P})$.

To see that $\mathrm{K}(\mathcal{P})$ is an Eilenberg-MacLane space one can show that the universal cover of $\mathrm{K}(\mathcal{P})$ is contractible. As it turns out the universal cover of $\mathrm{K}(\mathcal{P})$ is tiled by copies of the contractible order complex $\Delta(P)$ (see Lemma 8.3) attached in a fashion that ensures that the final space remains contractible. However this argument is quite technical and beyond the scope of this paper. The reader is referred to the work of Charney, Meier and Whittlesey [12, Theorem 3.6] and to the work of McCammond [19] for a more complete discussion.

Lastly because P is bounded and of finite height, $\Delta(P)$ is of finite dimension and hence $\mathrm{K}(\mathcal{P})$ is a finite dimensional CW complex. □

Corollary 8.5. *Let $\mathcal{P}$ be a combinatorial Garside structure. Then* $\textsc{Grp}(\mathcal{P})$ *is torsion free.*

Proof. The cohomology of $\textsc{Grp}(\mathcal{P})$ is equal to that of $\mathrm{K}(\mathcal{P})$, thus the cohomological dimension of $\textsc{Grp}(\mathcal{P})$ is bounded by the overall dimension of $\mathrm{K}(\mathcal{P})$ which is finite. Therefore $\textsc{Grp}(\mathcal{P})$ is torsion free. Please refer to [17, Proposition 2.45] for a proof that all groups of finite cohomological dimension are torsion free. □

Example 8.6 (Eilenberg-MacLane space for $\mathbb{Z}$)**.** Consider the lattice $\mathcal{Q}$ in Figure 5, and recall that $\textsc{Grp}(\mathcal{Q}) = \langle c \rangle = \mathbb{Z}$.

The order complex of $\mathcal{Q}$ has two 0-simplices and one 1-simplex and is exactly the Hasse diagram of $\mathcal{Q}$. We then construct $\mathrm{K}(\mathcal{Q})$ by joining all simplices with the same labels. In this case, the two 0-simplices have the same (empty) label, and thus we identify them. This leaves us with one 0-cell and a 1-cell attached to the 0-cell in a loop, i.e. a circle. Therefore $\mathrm{K}(\mathcal{Q})$ is homotopic to S^1.

It is well known that $\pi_1(S^1) = \mathbb{Z}$ and S^1 is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 1)$ as we would expect.

Example 8.7 (Eilenberg-MacLane space for $\mathbb{Z}^n$)**.** As mentioned in Remark 6.3 an appropriately labeled n-dimensional cube can provide a combinatorial Garside structure $\mathcal{P}$ with $\textsc{Grp}(\mathcal{P}) = \mathbb{Z}^n$. The maximal length of any chain from 0 to 1 in this poset is then n.

From this we may construct an Eilenberg-MacLane space of type $K(\mathbb{Z}^n, 1)$ of dimension no more than n. In fact, one can see that the space created will be homotopic to the n-dimensional torus

$$\mathbb{T}^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}.$$

9. Open questions

As mentioned before, Garside structures are a relatively new invention and with many avenues for future research.

Question 9.1. A large class of problems stem from better understanding the relationship between traditional Garside structures and McCammond's combinatorial Garside structures. While every combinatorial Garside structure gives rise to a group with a traditional Garside structure, the converse is not always true as the lattice of simple divisors $\mathcal{D}$ is not always weakly-graded.

This leads to the following question, paraphrased from [19]: What is the precise condition that must be placed on the lattice of combinatorial Garside structures to have a complete bijection between combinatorial Garside structures and traditional Garside structures?

Question 9.2. Recently S. Lee has been able to show that if G and H are two Garside groups with Garside elements Δ_G and Δ_H, and if $\rho : H \to \mathrm{Aut}(G^+)$ is a homomorphism then $G^+ \rtimes_\rho H^+$ is a Garside monoid and $G \rtimes_\rho H$ is its group of fractions (where with a slight abuse of notation have allowed an automorphism of G^+ to induce an automorpism of G in the natural fashion). See [18, Section 4].

One can ask if the notion of a semi-direct product of groups can be reinterpreted as a construction on combinatorial Garside structures. What does it mean to have an automorphism of a labeled lattice? What is a homomorphism from a labeled lattice to the automorphism group of another labeled lattice? How can one construct the 'semi-direct product' of two labeled lattices?

Question 9.3. We were able to show that the amalgamated union is a natural operation that preserves combinatorial Garside structures and leads to an amalgamated free product of the respective Garside groups. Is this true for all traditional Garside groups?

Our work is only sufficient to show that the amalgamated free product of two Garside groups is a Garside group when the starting groups are both weakly-graded.

Question 9.4. Another area of tremendous interest is in determining if the braid group analogues for complex reflection groups have Garside structures. In some cases this has been established, most trivially being the real reflection groups treated as complex reflection groups. To date no universal theory has been able to establish the existence of Garside structures for the braid group analogues in general. For a longer introduction to this topic see Appendix B.

Question 9.5. Does the notion of a Garside structure transfer over to category theory? Relatively few papers have been written on this topic. The reader is referred to the work of Bessis [5] and Digne and Michel [15] for an introduction.

Appendix A. Ore condition

Theorem A.1 (Ore)**.** *Let M be a cancellative monoid and suppose that any two elements of M admit a common right multiple. Then M embeds in its group of right fractions, i.e. there exists a group G in which M embeds and every element of G can be written in the form xy^{-1} for some $x, y \in M$.*

The following proof is adapted from O. Ore [21] in his work on non-commutative rings, but an alternative proof due to D. Rees (see [22] or [13, Theorem 1.23]) is commonly cited. Additionally we use the fact that M is a monoid to simplify the proof slightly, but in general the statement still holds when we relax the hypothesis to only require M to be a semigroup (i.e. a monoid without the assumption of an identity element).

Proof. Let M be a cancellative monoid in which right common multiples exist. We will construct a group G in which M embeds. The method of construction will be similar to the standard constructions of $\mathbb{Z}$ from $\mathbb{N}$ or $\mathbb{Q}$ from $\mathbb{Z}$.

Elements of G will be defined by putting an equivalence relation on the set $M \times M$. In general it will aid the reader to think of $(a, b) \in M \times M$ as loosely representing "ab^{-1}".

Define an equivalence relation $\sim$ on $M \times M$ by

$$(a, b) \sim (c, d) \qquad \Leftrightarrow \qquad \exists\, x, y \in M \text{ such that } ax = cy \text{ and } bx = dy.$$

It is routine to verify that $\sim$ is indeed an equivalence relation on $M \times M$:

- Reflexivity: $(a, b) \sim (a, b)$ is true by choosing x, y to be anything.

- Symmetry: The condition for $(a,b) \sim (c,d)$ is equivalent to that of $(c,d) \sim (a,b)$.
- Transitivity: Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (f,g)$ so there exist $x,y,z,w \in M$ which satisfy $ax = cy$, $bx = dy$, $cz = fw$, and $dz = gw$. By hypothesis y and z admit a right common multiple, i.e. elements $s,t \in M$ which satisfy $ys = zt$. Then
$$axs = cys = czt = fwt$$
and
$$bxs = dys = dzt = gwt$$
and hence $(a,b) \sim (f,g)$.

Let $G = (M \times M)/\sim$ and let $\{(a,b)\}$ represent the equivalence class of (a,b) in G. We define a multiplication operation in G as

$$\{(a,b)\} \cdot \{(c,d)\} := \{(ax,dy)\} \text{ for any } x,y \in M \text{ which satisfy } bx = cy.$$

Note that this definition of multiplication is always defined as b and c admit at least one right common multiple by hypothesis.

We must verify that that this multiplication operation is well-defined, by ensuring that the definition is independent of the choice of x and y, as well as the choice of representative for each element of G:

- Independent of the choice of $x,y \in M$:
  Suppose that $bx_1 = cy_1$ and $bx_2 = cy_2$. We must show that $(ax_1, dy_1) \sim (ax_2, dy_2)$. By hypothesis x_1 and x_2 admit a right common multiple, namely elements $s,t \in M$ which satisfy $x_1 s = x_2 t$. This implies that $cy_1 s = bx_1 s = bx_2 t = cy_2 t$ and hence $y_1 s = y_2 t$ by cancellativity. Thus $(ax_1, dy_1) \sim (ax_2, dy_2)$ as $ax_1 s = ax_2 t$ and $dy_1 s = dy_2 t$.
- Independence of the choice of representative of $\{(a,b)\}$:
  Suppose $\{(a_1,b_1)\} = \{(a_2,b_2)\}$ so there exist s,t so that $a_1 s = a_2 t$ and $b_1 s = b_2 t$. We will show that $\{(a_1,b_1)\} \cdot \{(c,d)\} = \{(a_2,b_2)\} \cdot \{(c,d)\}$, i.e. $(a_1 x_1, dy_1) \sim (a_2 x_2, dy_2)$ for some choice of $x_1, y_1, x_2, y_2 \in M$ satisfying $b_1 x_1 = cy_1$ and $b_2 x_2 = cy_2$.
  Because y_1 and y_2 admit a right common multiple there exist $u,v \in M$ which satisfy $y_1 u = y_2 v$ b. Note that $b_1 x_1 u = cy_1 u = cy_2 v = b_2 x_2 v$. Additionally, s and $x_1 u$ admit a right common multiple, so there are $m,n \in M$ which satisfy $sm = x_1 un$. Finally, one can verify that $b_2 x_2 vn = b_1 x_1 un = b_1 sm = b_2 tm$ and hence $tm = x_2 vn$ by cancellativity.
  Together these results give $a_1 x_1 un = a_1 sm = a_2 tm = a_2 x_2 vn$ and $dy_1 un = dy_2 vn$, hence $(a_1 x_1, dy_1) \sim (a_2 x_2, dy_2)$ as desired.
- Independence of the choice of representative of $\{(c,d)\}$:
  This can be verified in a method similar to the previous check, but is omitted for brevity.

Next we verify that the multiplication we have defined on G induces a group structure on G:

- Multiplication is associative:
  It can easily be shown that
$$\Big(\{(a,b)\} \cdot \{(c,d)\}\Big) \cdot \{(f,g)\} = \{(a,b)\} \cdot \Big(\{(c,d)\} \cdot \{(f,g)\}\Big)$$
  however we will leave the proof to the reader.

- Identity element:
  The identity element of G is $\{(e,e)\}$ where e is the identity element of M. It is almost immediate to see that
  $$\{(e,e)\}\cdot\{(a,b)\} = \{(a,b)\}\cdot\{(e,e)\} = \{(a,b)\}.$$
- Inverses:
  Define the inverse of any element as $\{(a,b)\}^{-1} = \{(b,a)\}$. One can easily verify that $\{(a,b)\}\cdot\{(a,b)\}^{-1} = \{(a,b)\}^{-1}\cdot\{(a,b)\} = \{(e,e)\}$.

Therefore G is a group. We check that M embeds in G. The map $\phi : M \to G$ defined by $\phi(m) = \{(m,e)\}$ is easily seen to be a homomorphism, as
$$\phi(mn) = \{(mn,e)\} = \{(mn,n)\}\cdot\{(n,e)\} = \{(m,e)\}\cdot\{(n,e)\} = \phi(m)\phi(n).$$
Additionally ϕ is one-to-one, as if $\phi(m) = \{(m,e)\}$ is equal to $\phi(n) = \{(n,e)\}$ then there exist $x,y \in M$ which satisfy $mx = ny$ and $ex = ey$ and hence $m = n$ by cancellation.

Finally, note that the description of G as the group of fractions of M is apt, as any $\{(a,b)\} \in G$ can be written as
$$\phi(a)\phi(b)^{-1} = \{(a,b)\}.$$
□

Appendix B. Complex reflection groups

A *complex reflection* is an invertible linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ of finite order so that all eigenvalues are 1 except possibly one eigenvalue that is a k-th root of unity. A *complex reflection group* is a group generated by complex reflections in $\mathbb{C}^n$. Shephard and Todd [23] provided a classification of all finite irreducible complex reflection groups in 1954. They are:

(1) $G(m,p,n)$ for some positive integers m, p, and n with p dividing m; and
(2) one of 34 exceptional cases.

In general $G(m,p,n)$ is the group of $n \times n$ complex matrices so that all entries are either 0 or m-th roots of unity, there is exactly one non-zero entry in each row and column, and the (m/p)-th power of the product of the non-zero entries is 1. (See for example [3, Appendix B.2].)

All finite real reflection groups (i.e. finite Coxeter groups) can be made into a complex reflection group by extension of scalars from $\mathbb{R}$ to $\mathbb{C}$. This immediately raises the question:

Question B.1. What is the proper way to generalize the transition from Coxeter groups (real reflection groups) to Artin groups (generalized braid groups) to the case of complex reflection groups?

This question has been mostly answered by Broué, Malle, and Rouquier [10, 11], in which they give presentations for the braid group analogue for all irreducible finite complex reflection groups in all but six cases. However many aspects of the theory remain unstudied.

Question B.2. Do the braid group analogues for (finite) complex reflection groups admit a Garside structure?

The answer is known in certain cases, most trivially being the braid group analogues associated with real reflection groups (acting on complex space). To date the question has not been answered in more generality than on a case-by-case analysis.

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