6. Linear Spaces

A tropical linear space is the tropicalization of a subspace of the vector space K^n over the Puiseux series field $K = \mathbb{C}\{\{t\}\}$. Algebraically, the subspace is given by an ideal I that is generated by $n-d$ linearly independent linear forms:

$$(1) \qquad I \quad = \quad \langle\, a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \;:\; i = 1,2,\ldots,n-d \,\rangle.$$

Thus a d-dimensional tropical linear space in $\mathbb{R}^n$ is any tropical variety of the form $\mathcal{T}(I)$ where (a_{ij}) is any $(n-d) \times n$-matrix of rank $n-d$ with entries in K. The ideal I is uniquely specified by the vector of Plücker coordinates $p \in G_{d,n} \subset \mathbb{R}^{\binom{n}{d}}$ of the linear subspace of K^n. The Plücker coordinates are

$$(2) \qquad p_{i_1 i_2 \cdots i_d} \quad = \quad (-1)^{i_1+i_2+\cdots+i_d} \cdot \det \begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \cdots & a_{1,j_{n-d}} \\ a_{2,j_1} & a_{2,j_2} & \cdots & a_{2,j_{n-d}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-d,j_1} & a_{n-d,j_2} & \cdots & a_{n-d,j_{n-d}} \end{pmatrix},$$

where $i_1 < \cdots < i_d$, $j_1 < \cdots < j_{n-d}$ and $\{i_1, \ldots, i_d, j_1, \ldots, j_{n-d}\} = \{1, \ldots, n\}$, and the ideal I is expressed in terms of the Plücker coordinates by the formula

$$(3) \qquad I \;=\; \Big\langle \sum_{r=0}^{d} (-1)^r \cdot p_{i_0 i_1 \cdots \widehat{i_r} \cdots i_d} \cdot x_{i_r} \;:\; \text{for all } 1 \le i_0 < i_1 < \cdots < i_r \le n \Big\rangle.$$

In what follows we shall make the technical assumption that all Plücker coordinates (2) are non-zero, so we shall only consider linear subspaces of K^n whose matroid is uniform. This ensures that the tropical Plücker coordinates

$$w_{i_1 i_2 \ldots i_d} \quad = \quad \mathrm{val}(p_{i_1 i_2 \cdots i_d})$$

are well-defined rational numbers. The $\binom{n}{d+1}$ linear forms in (3) are called the *circuits* of the linear ideal I. Their tropicalizations are called *tropical circuits*:

$$(4) \qquad C_{i_0 i_1 \cdots i_r} \quad := \quad \bigoplus_{r=0}^{d} w_{i_0 i_1 \cdots \widehat{i_r} \cdots i_d} \odot x_{i_r}$$

Lemma 6.1. *The circuits in (3) form a tropical basis for the linear ideal I. Hence the tropical d-plane $\mathcal{T}(I)$ equals the intersection in $\mathbb{R}^n$ of the $\binom{n}{d+1}$ tropical hyperplanes $\mathcal{T}(C_{i_0 i_1 \cdots i_d})$ defined by the tropical circuits.*

Proof. This follows from two easy facts about Gröbner bases of linear ideals. First, every reduced Gröbner basis of I consists of circuits, and, second, the linear ideal $\mathrm{in}_w(I)$ contains a monomial if and only if a variable appears in its reduced Gröbner basis. The latter condition is equivalent to $w \notin \mathcal{T}(I)$. ☐

The classical Grassmannian $G_{d,n}$ is the projective variety in $\mathbb{P}^{\binom{n}{d}-1}$ defined by the Plücker ideal $I_{d,n}$. It parametrizes the d-dimensional linear subspaces of $\mathbb{C}^n$. We now derive the analogous result for the tropical Grassmannian $\mathcal{G}_{d,n}$. Here

$\mathcal{G}_{d,n}$ is regarded as a pure fan of dimension $(n-d)d$ in $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\ldots,1) \simeq \mathbb{R}^{\binom{n}{d}-1}$, so the dimensions match correctly with the classical Grassmannian.

Theorem 6.2. *The bijection between the classical Grassmannian $G_{d,n}$ and the d-dimensional subspaces of K^n induces a unique bijection $w \mapsto L_w$ between the tropical Grassmannian $\mathcal{G}'_{d,n}$ and the set of tropical d-planes in n-space.*

Proof. We begin by describing the map which takes a point w in $\mathcal{G}_{d,n}$ to the associated tropical d-plane $L_w \subset \mathbb{R}^n$. Given w, we define L_w as the intersection of the tropical hyperplanes $\mathcal{T}(C_{i_0 i_1 \cdots i_d})$ where $1 \le i_0 < i_1 < \cdots < i_d \le n$. This definition depends only on $w + \mathbb{R}(1,1,\ldots,1)$, as required. By the Extended Kapranov Theorem, we can pick a point $p \in (K^*)^{\binom{n}{d}}$ which is a zero of $I_{d,n}$ and satisfies $w = \mathrm{order}(p)$. Let I be the ideal defined by (3). By Lemma 6.1, we have $L_w = \mathcal{T}(I)$. Hence the map $w \mapsto L_w$ surjects the tropical Grassmannian onto the set of all tropical d-planes, and it is the only such map which is compatible with the classical bijection between $G_{d,n}$ and the d-planes in K^n.

It remains to be shown that the map $w \mapsto L_w$ is injective. We do this by constructing the inverse map. Suppose we are given the linear space L_w as a subset of $\mathbb{R}^n$. We need to reconstruct the coordinates $w_{i_1 \cdots i_d}$ of w up to a global additive constant. Equivalently, for any $(d-1)$-subset I of $[n]$ and any pair $j,k \in [n]\backslash I$, we need to reconstruct the real number $w_{I\cup\{j\}} - w_{I\cup\{k\}}$.

Fix a very large positive rational number M and consider the $(n-d+1)$-dimensional plane defined by $x_i = M$ for $i \in I$. The intersection of this plane with L_w contains at least one point $x \in \mathbb{R}^n$, and this point can be chosen to satisfy $x_j \ll M$ for all $j \in [n]\backslash I$. This can be seen by solving the $d-1$ equations $x_i = t^M$ on any d-plane $V(I) \subset K^n$ which tropicalizes to L_w.

Now consider the tropical circuit C_J as in (4) with $J = I \cup \{j,k\}$. Since x lies $\mathcal{T}(C_J)$, and since $\max(x_j, x_k) \ll M = x_i$ for all $i \in I$, we conclude

$$w_{J\backslash\{k\}} + x_k \quad = \quad w_{J\backslash\{j\}} + x_j.$$

This shows that the desired differences can be read off from the point x:

$$w_{I\cup\{j\}} - w_{I\cup\{k\}} \quad = \quad x_j \; - \; x_k. \tag{5}$$

We thus reconstruct $w \in \mathcal{G}_{d,n}$ by locating $\binom{n}{d-1}$ special points on L_w. □

The above proof offers an (inefficient) algorithm for computing the map $w \mapsto L_w$, namely, by solving all $\binom{n}{d+1}$ tropical circuits. Consider the case $d = 2$. Here the tropical plane $\mathcal{T}(C_{ijk})$ is the solution set to the linear system

$$\begin{array}{rl} & w_{ij} + x_k \;=\; w_{ik} + x_j \;\le\; w_{jk} + x_i \\ \text{or} & w_{ij} + x_k \;=\; w_{jk} + x_i \;\le\; w_{ik} + x_j \\ \text{or} & w_{ij} + x_k \;=\; w_{jk} + x_i \;\le\; w_{ik} + x_j. \end{array}$$

The conjunction of these $\binom{n}{3}$ linear systems is solved efficiently by the Neighbor Joining Algorithm from phylogenetics [61, 66]. If r and $s \in \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ are vertices of this tree connected by an edge e then $r = s + c\sum_{i\in S} e_i$ for some $c > o$ where $S \subset [n]$ is the set of leaves on the s side of e. We regard the tree as a metric space by assigning the length c to edge e. The length of each edge is measured in lattice distance, so we get the tree with metric $-2w$.

Corollary 6.3. *Let w be a point in $\mathcal{G}_{2,n}$ which lies in the cone C_σ for some tree σ. The image of L_w in $\mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is a tree of combinatorial type σ.*

The bijection $w \mapsto L_w$ of Theorem 6.2 is a higher-dimensional generalization of recovering a phylogenetic tree from pairwise distances among n leaves. For instance, for $d = 3$ we can think of w as data giving a proximity measure for any triple among n "leaves". The image of L_w in $\mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is a "phylogenetic surface" which is a geometric representation of such data on triples.

The tropical Grassmannians $\mathcal{G}_{d,n}$ and $\mathcal{G}_{n-d,n}$ are isomorphic because the ideals $I_{d,n}$ and $I_{n-d,n}$ are the same after signed complementation of Plücker coordinates. Theorem 6.2 allows us to define the *dual* $(n-d)$-plane L^* of a tropical d-plane L in $\mathbb{R}^n$. If $L = L_w$ then $L^* = L_{w^*}$ where w^* is the vector whose $([n]\backslash I)$-coordinate is the I-coordinate of w, for all d-subsets I of $[n]$. One can check that a tropical hyperplane $\mathcal{T}(a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n)$ contains the tropical plane L^* if and only if $(a_1,\ldots,a_n) \in L_w$. Furthermore, $(L^*)^* = L$.

The following example, known as the *dual snowflake*, shows that a tropical d-plane in $\mathbb{R}^n$ is generally not the intersection of $n-d$ tropical hyperplanes.

Example 6.4. Let $w = e_{12} + e_{34} + e_{56}$ in $\mathbb{R}^{\binom{6}{2}}$. Then L_w is a tropical 2-plane in $\mathbb{R}^6$. Its image in $\mathbb{R}^6/\mathbb{R}(1,\ldots,1)$ is a *snowflake tree*, i.e. a tree of type $\sigma = \big\{\{12,3456\},\{34,1256\},\{56,1234\}\big\}$. The Plücker vector dual to w is

$$w^* \quad = \quad e_{3456} + e_{1256} + e_{1234} \quad \in \ \mathcal{G}_{4,6} \ \subset \ \mathbb{R}^{\binom{6}{4}}.$$

We shall compute the tropical 4-plane L_{w^*} by applying the algorithm in the proof of Theorem 6.2. There are six tropical circuits C_J as in (4), namely,

$$\begin{array}{lcl}
C_{12345} & = & 0\odot x_1 \ \oplus\ 0\odot x_2 \ \oplus\ 0\odot x_3 \ \oplus\ 0\odot x_4 \ \oplus\ 1\odot x_5 \\
C_{12346} & = & 0\odot x_1 \ \oplus\ 0\odot x_2 \ \oplus\ 0\odot x_3 \ \oplus\ 0\odot x_4 \ \oplus\ 1\odot x_6 \\
C_{12356} & = & 0\odot x_1 \ \oplus\ 0\odot x_2 \ \oplus\ 1\odot x_3 \ \oplus\ 0\odot x_5 \ \oplus\ 0\odot x_6 \\
C_{12456} & = & 0\odot x_1 \ \oplus\ 0\odot x_2 \ \oplus\ 1\odot x_4 \ \oplus\ 0\odot x_5 \ \oplus\ 0\odot x_6 \\
C_{13456} & = & 1\odot x_1 \ \oplus\ 0\odot x_3 \ \oplus\ 0\odot x_4 \ \oplus\ 0\odot x_5 \ \oplus\ 0\odot x_6 \\
C_{23456} & = & 1\odot x_2 \ \oplus\ 0\odot x_3 \ \oplus\ 0\odot x_4 \ \oplus\ 0\odot x_5 \ \oplus\ 0\odot x_6
\end{array}$$

The tropical 4-plane L_{w^*} is the intersection of these six tropical hyperplanes:

$$\mathcal{T}(C_{12345}) \cap \mathcal{T}(C_{12346}) \cap \mathcal{T}(C_{12356}) \cap \mathcal{T}(C_{12456}) \cap \mathcal{T}(C_{13456}) \cap \mathcal{T}(C_{23456}).$$

We claim that L_{w^*} *is not a complete intersection*, i.e., there do no exist two tropical linear forms F and F' such that $L_{w^*} = \mathcal{T}(F) \cap \mathcal{T}(F')$. A tropical linear form $F = a_1x_1 + \cdots + a_6x_6$ vanishes on the dual 4-plane L_{w^*} if and only if the point $a = (a_1, \ldots, a_6)$ lies in the 2-plane L_w. There are 9 types of such tropical linear forms ℓ, one for each of the 9 edges of the tree L_w. For instance, the bounded edge $\{56, 1234\}$ represents the tropical linear forms

$$\ell \; = \; \alpha \odot (x_1 \oplus x_2 \oplus x_3 \oplus x_4) \;\oplus\; \beta \odot (x_5 \oplus x_6) \quad \text{where } \; 0 < \alpha \leq \beta.$$

By checking all pairs of the 9 edges, we find that any pairwise intersection $\mathcal{T}(\ell) \cap \mathcal{T}(\ell')$ contains a 5-dimensional cone like $\{x_1{+}c = x_2 \ll x_3, x_4, x_5, x_6\}$, $\{x_3{+}c = x_4 \ll x_1, x_2, x_5, x_6\}$ or $\{x_5{+}c = x_6 \ll x_1, x_2, x_3, x_4\}$. □

Our next goal is to give a combinatorial encoding of tropical planes. The basic object in our combinatorial encoding is a d-partition $\{A_1, \ldots, A_d\}$. By a *d-partition* we mean an unordered partition of $[n]$ into d subsets A_i. Let L_w be a tropical d-plane and F a maximal cell of L_w. Thus F is a d-dimensional convex polyhedron in $\mathbb{R}^n$ which is invariant under translation along the line $\mathbb{R}(1, 1, \ldots, 1)$. The affine span of F is a d-dimensional affine space which is defined by equations of the special form

$$x_k \; - \; x_j \quad = \quad w_{J\setminus\{j\}} \; - \; w_{J\setminus\{k\}} \qquad \text{(the right hand side is a constant)}$$

Such a system of equations defines a d-partition $\{A_1, \ldots, A_d\}$, namely, two indices j and k lie in the same block A_i if and only if the difference $x_k - x_j$ is constant on F. The number of blocks clearly equals d, the dimension of F. In examples, we always consider F modulo the line $\mathbb{R}(1, 1, \ldots, 1)$, so it has dimension $d - 1$ and may or may not be bounded.

Remark 6.5. *A maximal face F of L_w is uniquely specified by its d-partition $\{A_1, \ldots, A_d\}$. It is a (bounded) polytope in $\mathbb{R}^n$ if and only if $|A_i| \geq 2$ for all i. Hence a tropical d-plane $L_w \subset \mathbb{R}^n$ has no bounded d-faces if $n \leq 2d - 1$.*

We define the *type* of a tropical d-plane L, denoted $\mathrm{type}(L)$ to be the set of all d-partitions arising from the maximal faces of L. If $d = 2$ and $L = L_w$ with $w \in C_\sigma$ then $\mathrm{type}(L)$ is precisely the set σ together with the pairs $\{\{i\}, [n]\backslash\{i\}\}$ representing the unbounded edges of the tree L. This follows from Corollary 6.3. Thus $\mathrm{type}(L)$ generalizes the representation of a semi-labeled tree [61, Theorem 2.35] by its set of splits to higher-dimensional tropical planes L.

Example 6.6. We present three of the seven types in $\mathcal{G}_{3,6}$. In each case we display $\mathrm{type}(L_w)$ with the 15 obvious tripartitions $\big\{i, j, [6]\backslash\{i, j\}\big\}$ removed. The labeling corresponds to that of the maximal cones in our discussion of $\mathcal{G}_{3,6}$ in the previous lecture. We begin with what is called the *tree type* in [72, §8]:

$$\text{EEFF1}: \quad \begin{array}{c} \big\{\{1,23,456\}, \{1,56,234\}, \{2,13,456\}, \{2,56,134\}, \\ \{3,12,456\}, \{3,56,124\}, \{4,12,356\}, \{4,56,123\}, \\ \{5,12,346\}, \{5,46,123\}, \{6,12,345\}, \{6,45,123\}, \{12,34,56\}\big\} \end{array}$$

The next type is the *bipyramid type.* All three tetrahedra in a bipyramid *FFFGG* have the same type listed below. As the faces of $\mathcal{G}_{3,6}$ contain those

w inducing different initial ideals $\mathrm{in}_w(I_{d,n})$, this example demonstrates that $\mathrm{type}(L_w)$ in general does not determine $\mathrm{in}_w(I_{d,n})$.

$$\text{FFGG}: \quad \begin{array}{c} \big\{\{1,34,256\},\{1,56,234\},\{2,34,156\},\{2,56,134\} \\ \{3,12,456\},\{3,56,124\},\{4,12,356\},\{4,56,123\} \\ \{5,12,346\},\{5,34,126\},\{6,12,345\},\{6,34,125\},\{12,34,56\}\big\} \end{array}$$

For all but one of the seven types in $\mathcal{G}_{3,6}$, the tropical plane L_w has 28 facets. The only exception is the type *EEEE*. Here the tropical plane L_w has only 27 facets, all of them unbounded.

$$\text{EEEE}: \quad \begin{array}{c} \big\{\{1,23,456\},\{1,234,56\},\{2,13,456\},\{2,135,46\} \\ \{3,12,456\},\{3,126,45\},\{4,26,135\},\{4,126,35\} \\ \{5,16,234\},\{5,126,34\},\{6,15,234\},\{6,135,24\}\big\} \end{array}$$

From this analysis, we see that every 2-plane L_w in $\mathbb{R}^6/\mathbb{R}(1,\ldots,1)$ has six vertices, it has ≤ 24 edges (of which ≤ 6 are bounded), and ≤ 28 two-dimensional faces (of which at most one is bounded). □

In order to visualize a tropical 2-plane L_w, we can draw the graph gotten by intersecting L_w with a large sphere. It consists of n trees, each having $n-1$ leaves, interconnected by $\binom{n}{2}$ edges. In general, for $d \geq 3$, each face of a tropical plane L_w is affinely isomorphic to a full-dimensional polytope in $\mathbb{R}^d/\mathbb{R}(1,\ldots,1)$ that is defined by inequalities of the special form $x_i - x_j \leq w_{ij}$ for all $i,j \in \{1,\ldots,d\}$. Such polytopes are the basic building blocks for tropical convexity, as we shall see in the next section, and each of them can appear as a face in a tropical plane in $\mathbb{R}^n$ for some sufficiently large value of n.

The following remarkable theorem about face numbers of tropical linear spaces was recently proved by David Speyer [72, 73]. His proof is rather difficult and involves lots of matroid theory and K-theory of the Grassmannian.

Theorem 6.7. (Speyer) *The number of i-dimensional faces of a tropical d-plane in $\mathbb{R}^n$ is at most $\binom{n+1}{d-i}\binom{2n-d-1}{i-1}$, and the number of faces that are bounded modulo* $\mathbb{R}(1,1,\ldots,1)$ *is at most $\binom{n-2i}{d-i}\binom{n-i-1}{i-1}$. These bounds are tight.*

These upper bounds on the face numbers are attained by what Speyer calls *series-parallel planes*. This terminology comes from matroid theory, and it is based on the correspondence between matroids and the following class of tropical planes. Let I be an ideal in $\mathbb{C}[x_1,\ldots,x_n]$ of the form (1), but we now allow some of the Pł"ucker coordinates (2) to be zero. The collection on non-zero Pücker coordinates are the bases of the rank d matroid M which is associated with I. Each tropical Plücker coordinate $w_{i_1 i_2 \cdots i_d}$ is either 0 or ∞, so the knowledge of w is equivalent to the knowledge of the matroid M. The tropical plane $\mathcal{T}(I)$ is a pure d-dimensional fan in $\mathbb{R}^n$ which depends only on the matroid M, and we denote it by $\mathcal{T}(M)$. The lineality space of $\mathcal{T}(M)$

is the one-dimensional space spanned by $(1, 1, \ldots, 1)$, so we can consider the $(d-2)$-dimensional polyhedral complex $\mathcal{T}(M)'$. The fan $\mathcal{T}(M)$ is called the *Bergman fan* and the complex $\mathcal{T}(M)'$ is called the *Bergman complex* of the matroid M.

Given any $w \in \mathbb{R}^n$, we call $w_{i_1} + \cdots + w_{i_d}$ the *w-weight* of a basis $\{i_1, \ldots, i_d\}$ of M. The set of all bases of M that have maximal w-weight is itself the set of bases of a new matroid M_w. We call M_w the *initial matroid* of M with respect to w. This is precisely the matroid associated with the linear ideal $\mathrm{in}_w(I)$. A *circuit* of the matroid M is a subset γ of $\{1, \ldots, n\}$ which is not contained in any basis and is minimal with this property. With this convention, the circuits (4) are the tropical linear forms $C = \bigoplus_{i \in \gamma} x_i$ where γ is any circuit of M.

Lemma 6.8. *For $w \in \mathbb{R}^n$ the following are equivalent:*

(1) *The vector w lies in the Bergman fan $\mathcal{T}(M)$.*
(2) *The initial ideal $\mathrm{in}_w(I)$ contains no variable x_i.*
(3) *The initial matroid M_w has no loop (i.e. a circuit that is a singleton).*
(4) *For each circuit γ of M the minimum of $\{x_i : i \in \gamma\}$ is attained at least twice.*

Proof. The equivalence of (1) and (2) is the definition of $\mathcal{T}(M) = \mathcal{T}(I)$. Statements (2) and (3) are equivalent because M_w is the matroid of $\mathrm{in}_w(I)$, while (1) and (4) are equivalent by (the argument in the proof of) Lemma 6.1. $\square$

A subset F of $[n]$ is a *flat* of the matroid M if there is no circuit C such that $C \backslash F$ has precisely one element. The incidence vector of a subset $F \subseteq [n]$ is denoted by $e_F = \sum_{i \in F} e_i$. We are interested in the negative of that vector.

Remark 6.9. *The vector $-e_F$ lies in $\mathcal{T}(M)$ if and only if F is a flat of M.*

The collection of all flats of M is partially ordered by inclusion, with $\emptyset$ as the smallest flat and $[n]$ as the largest flat. This poset is denoted by $\mathcal{L}_M$ and it is the *geometric lattice* associated with M. Each maximal chain $\mathcal{F}$ in $\mathcal{L}_M$ has length d, so it can be written as

$$\mathcal{F} \;:\; \emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{d-1} \subset F_d = [n].$$

The 0-1 vectors $e_{F_1}, e_{F_2}, \ldots, e_{F_d}$ arising from such a chain $\mathcal{F}$ are linearly independent, so they span a simplicial cone $C_{\mathcal{F}}$ of dimension d in $\mathbb{R}^n$.

Theorem 6.10. *The cones $C_{\mathcal{F}}$ and their faces form a fan in $\mathbb{R}^n$ whose support is precisely the Berman fan $\mathcal{T}(M)$. Hence the Bergman complex $\mathcal{T}(M)'$ equals the simplicial complex of chains in the geometric lattice $\mathcal{L}(M)$. This simplicial complex is shellable, and its homotopy type is that of a wedge of $\mu(M)$ spheres of dimension $d-2$, where $\mu(M)$ is the Möbius invariant of the matroid M.*

The first two statements in this theorem are due to Ardila and Kilvans [7]. The third sentence is a well-known result in topological combinatorics [14].

The Bergman fans of matroids are important for the study of arbitrary tropical linear spaces because they appear both as the links of their faces and as their "asymptotes". Another way to think about tropical linear spaces is as dual complexes to matroid subdivisions of the second hypersimplex. This point of view has been developed by Speyer [72, 73], and it also plays an important role in the connection of tropical linear spaces to affine buildings [27, 47] and to moduli problems in algebraic geometry [40, 50].

We close our discussion of tropical linear equations with a few remarks about their defining equations. We already saw in Example 6.4 that the number of tropical hyperplanes needed to cut out a tropical d-plane in $\mathbb{R}^n$ can be larger than $n-d$. The following result from [12] shows that this number can actually be very close to the upper bound $\binom{n}{d+1}$ which we saw in Lemma 6.1.

Proposition 6.11. *For any $1 \leq d \leq n$, there is a linear ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ such that any tropical basis of linear forms in I has size at least $\frac{1}{n-d+1}\binom{n}{d}$.*

Proof. Suppose that all $d \times d$-minors of the coefficient matrix (a_{ij}) are nonzero. Equivalently, the matroid of I is uniform. There are $\binom{n}{n-d+1}$ circuits in I, each supported on a different $(n-d+1)$-subset of $\{x_1, \ldots, x_n\}$. Since the circuits form a tropical basis of I and each circuit has support of size $n-d+1$, the tropical variety $\mathcal{T}(I)$ consists of all vectors $w \in \mathbb{R}^n$ whose smallest $d+1$ components are equal. The latter condition is necessary and sufficient to ensure that no single variable in a circuit becomes the initial form of the circuit with respect to w. Consider any vector $w \in \mathbb{R}^n$ satisfying

$$w_{i_1} = w_{i_2} = \cdots = w_{i_d} \; < \; \min\bigl(w_j \, : \, j \in \{1, \ldots, n\} \backslash \{i_1, i_2, \ldots, i_d\}\bigr).$$

Since $w \notin \mathcal{T}(I)$, any tropical basis of linear forms in I contains an f such that $\mathrm{in}_w(f) \in \{x_{i_1}, \ldots, x_{i_d}\}$. This implies that f is one of the d circuits whose support contains the $n-d$ variables x_j with $j \notin \{i_1, \ldots, i_d\}$. The support of each circuit has size $n-d+1$, hence contains $n-d+1$ distinct $(n-d)$-subsets. There are $\binom{n}{d}$ $(n-d)$-subsets of $\{x_1, \ldots, x_n\}$ to be covered. Hence any tropical basis consisting of linear forms has size at least $\frac{1}{n-d+1}\binom{n}{d}$. $\square$

Example 6.12. Let $d = 3, n = 5$. The Bergman fan $\mathcal{T}(I)$ corresponds to the line in tropical projective 4-space which consists of the five rays in the coordinate directions. We have $\frac{1}{n-d+1}\binom{n}{d} = 10/3$. Hence this line is not a complete intersection of three tropical hyperplanes, but it requires four. $\square$

Proposition 6.11 raises the question whether, for special classes of matroids M, the Bergman fan has a smaller tropical basis which has a nice combinatorial characterization. This question was studied by Yu and Yuster [89] who obtained a range of interesting results, including a characterization of tropical bases for graphic and co-graphic matroids.