3. Computing Tropical Varieties

Given any polynomial $f \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ and a vector $w \in \mathbb{R}^n$, the initial form $\mathrm{in}_w(f)$ is the sum of all terms in f of lowest w-weight. For instance, if $\ell = x_1 + x_2 + x_3 + 1$ then $\mathrm{in}_{(0,0,1)}(\ell) = x_1 + x_2 + 1$ and $\mathrm{in}_{(0,0,-1)}(\ell) = x_3$. Recall that the tropical hypersurface $\mathcal{T}(f)$ is the union of all codimension one cones in the normal fan of the Newton polytope $\mathrm{New}(f)$. In light of the relation

$$\mathrm{face}_w(\mathrm{New}(f)) \;=\; \mathrm{New}(\mathrm{in}_w(f)),$$

the tropical hypersurface can be expressed as follows:

$$\mathcal{T}(f) \quad = \quad \{\, w \in \mathbb{R}^n \,:\, \mathrm{in}_w(f) \;\text{ is not a monomial}\,\}.$$

As before, we represent $\mathcal{T}(f)$ by the polyhedral complex $\mathcal{T}'(f)$ which is obtained by removing the lineality space and intersecting with the unit sphere. A finite intersection of tropical hypersurfaces is called a *tropical prevariety*

For the linear polynomial ℓ above, $\mathcal{T}'(\ell)$ is the complete graph on the 2-sphere having the four nodes $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $-(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. . If we pick the second linear form $\ell' = x_1 + x_2 + 2x_3$ then $\mathcal{T}'(\ell')$ is a graph with two vertices connected by three edges on the 2-sphere, and $\mathcal{T}'(\ell) \cap \mathcal{T}'(\ell')$ consists of three edges of $\mathcal{T}'(\ell)$ which are adjacent to $-(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. This graph is not balanced. The tropical prevariety $\mathcal{T}(\ell) \cap \mathcal{T}(\ell')$ is not a tropical variety.

Every ideal I in the polynomial ring $\mathbb{C}[\mathbf{x}]$ specifies a *tropical variety* $\mathcal{T}(I)$. By definition, $\mathcal{T}(I)$ is the intersection of the tropical hypersurfaces $\mathcal{T}(\mathrm{trop}(f))$, where f runs over all polynomials in the ideal I. The *initial ideal* $\mathrm{in}_w(I)$ is generated by all polynomials $\mathrm{in}_w(f)$ where f runs over I. Using the various initial ideals, the tropical variety of I can be expressed as follows:

$$\mathcal{T}(I) \quad = \quad \{\, w \in \mathbb{R}^n \,:\, \mathrm{in}_w(I) \;\text{ contains no monomial}\,\}.$$

Theorem 3.7 below states that every tropical variety $\mathcal{T}(I)$ is a tropical prevariety, i.e., every ideal I has a finite generating set $\{f_1, f_2, \ldots, f_r\}$ such that

$$\mathcal{T}(I) \quad = \quad \mathcal{T}(f_1) \,\cap\, \mathcal{T}(f_2) \,\cap\, \cdots \,\cap \mathcal{T}(f_r)\,.$$

If this holds then $\{f_1, f_2, \ldots, f_r\}$ is called a *tropical basis* of I. For instance, our ideal $I = \langle \ell, \ell' \rangle$ has the tropical basis $\{\, x_1 + x_2 + 2x_3,\; x_1 + x_2 + 2,\; x_3 - 1\,\}$, and we find that its tropical variety consists of three points on the sphere:

$$\mathcal{T}(I) \;\; = \;\; \big\{(1,0,0),\, (0,1,0),\, -(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\big\}.$$

The aim of this lecture is to discuss a practical algorithm for computing the tropical variety $\mathcal{T}(I)$ from any generating set of its ideal I. The emphasis lies on the geometric and algebraic features of this computation. A key result (Theorem 3.10) states that the tropical variety $\mathcal{T}(I)$ of a prime ideal I is connected in codimension one. This result is the foundation of Algorithm 3.23. We shall also describe procedures for computing tropical bases and tropical prevarieties. Our algorithms have been implemented in the software package `Gfan`, and we shall explain how to use this software for computing $\mathcal{T}(I)$.

A note on the choice of ground field is in order. We here work with varieties defined over $\mathbb{C}$. Our tropical software in `Gfan` requires that the given polynomials have rational coefficients, but the underlying algorithms work verbatim for complex coefficients. In the literature, tropical varieties are usually defined from polynomials with coefficients in a field K with non-archimedean valuation, such as the Puiseux series field $K = \mathbb{C}\{\{t\}\}$. These tropical varieties are polyhedral complexes but usually they are not fans. Our algorithms can be applied to this situation as follows. Consider the field $\mathbb{C}(t)$ of rational functions in the unknown t. Then $\mathbb{C}(t)$ is a subfield of the algebraically closed field K. Suppose we are given an ideal I in $\mathbb{C}(t)[\mathbf{x}]$. The tropical variety of I is the intersection of all tropical hypersurfaces $\mathcal{T}(\mathrm{trop}(f))$ where $f \in I$. To compute $\mathcal{T}(I)$, we consider the polynomial ring $\mathbb{C}[t, \mathbf{x}]$ in $n+1$ variables and its ideal $J = I \cap \mathbb{C}[t, \mathbf{x}]$. Generators of J are computed from generators of I by clearing denominators and saturating with respect to t. The tropical variety $\mathcal{T}(J)$ is a fan in $\mathbb{R}^{n+1}$ which our algorithm will compute. The tropical variety of I is the intersection of the fan $\mathcal{T}(J)$ with the hyperplane $\{x_0 = 1\}$ in $\mathbb{R}^{n+1}$. In symbols

$$\mathcal{T}(I) \quad = \quad \left\{ w \in \mathbb{R}^{n+1} \,:\, (1, w) \in \mathcal{T}(J) \right\} \tag{1}$$

Note that the situation is now analogous to that in the proof of Proposition **??**. The result that $\mathcal{T}(J)$ is a fan implies that $\mathcal{T}(I)$ is (finite) polyhedral complex in $\mathbb{R}^n$ which usually has both bounded and unbounded faces.

A most basic problem in computational tropical geometry is the following:

Problem 3.1. *Given a finite list of polynomials* $f_1, \ldots, f_r \in \mathbb{C}[\mathbf{x}]$ *in* n *unknowns, compute the tropical prevariety* $\mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_r)$ *in* $\mathbb{R}^n$.

The geometry of this problem is best understood by considering the Newton polytopes $\mathrm{New}(f_1), \ldots, \mathrm{New}(f_r)$ of the given polynomials. By definition, $\mathrm{New}(f_i)$ is the convex hull in $\mathbb{R}^n$ of the exponent vectors which appear in f_i. The tropical hypersurface $\mathcal{T}(f_i)$ consists of the $(n-1)$-dimensional cones of the normal fan of the polytope $\mathrm{New}(f_i)$. Our problem is to from a fan by intersecting these hypersurfaces. The resulting tropical prevariety can be a fairly general polyhedral fan. Its maximal cones may have different dimensions.

The tropical variety of an ideal I in $\mathbb{C}[\mathbf{x}]$ is the set $\mathcal{T}(I) := \bigcap_{f \in I} \mathcal{T}(f)$. We first note that it suffices to compute tropical varieties of homogeneous ideals. Let ${}^hI \subset \mathbb{C}[x_0, \mathbf{x}]$ be the homogenization of an ideal I in $\mathbb{C}[\mathbf{x}]$ by a new variable.

Lemma 3.2. *Fix an ideal* $I \subset \mathbb{C}[\mathbf{x}]$ *and a vector* $w \in \mathbb{R}^n$. *The initial ideal* $\mathrm{in}_w(I)$ *contains a monomial if and only if* $\mathrm{in}_{(0,w)}({}^hI)$ *contains a monomial.*

Proof. Suppose $\mathbf{x}^u \in \mathrm{in}_w(I)$. Then $\mathbf{x}^u = \mathrm{in}_w(f)$ for some $f \in I$. The $(0, w)$-weight of a term in ${}^hf$ equals the w-weight of the corresponding term in f. Hence $\mathrm{in}_{(0,w)}({}^hf) = x_0^a \mathbf{x}^u \in \mathrm{in}_{(0,w)}({}^hI)$ where a is some non-negative integer.

Conversely, if $\mathbf{x}^u \in \mathrm{in}_{(0,w)}({}^hI)$ then $\mathbf{x}^u = \mathrm{in}_{(0,w)}(f)$ for some $f \in {}^hI$. Substituting $x_0 = 1$ in f gives a polynomial in I. The $(0, w)$-weight of any term in f equals the w-weight of the corresponding term in $f|_{x_0=1}$. Since $\mathrm{in}_{(0,w)}(f)$ is a monomial, only one term in f has minimal $(0, w)$-weight. This term cannot be canceled during the substitution. Hence it lies in $\mathrm{in}_w(I)$. $\square$

Our main goal in this lecture is to solve the following computational task.

Problem 3.3. *Given a finite list of homogeneous polynomials* $f_1, \ldots, f_r \in \mathbb{C}[\mathbf{x}]$*, compute the tropical variety* $\mathcal{T}(I)$ *of their ideal* $I = \langle f_1, \ldots, f_r \rangle$*.*

It is important to note that Problems 3.1 and 3.3 are of a fundamentally different nature. Problem 3.1 is a problem of polyhedral geometry. It involves only polyhedral computations and no algebra. Problem 3.3, on the other hand, combines the polyhedral aspect with an algebraic one. To solve Problem 3.3 we must perform algebraic operations with polynomials (e.g. Gröbner bases).

Our next problem concerns tropical bases. Recall that a finite set $\{f_1, \ldots, f_t\}$ is a tropical basis of I if $\langle f_1, \ldots, f_t \rangle = I$ and $\mathcal{T}(I) = \mathcal{T}(f_1) \cap \cdots \cap \mathcal{T}(f_t)$.

Problem 3.4. *Compute a tropical basis of a given ideal* $I \subset \mathbb{C}[\mathbf{x}]$*.*

A priori, it is not clear that every ideal I has a finite tropical basis, but we shall prove this below. First, here is one case where this is easy:

Example 3.5. If $I = \langle f \rangle$ is a principal ideal, then $\{f\}$ is a tropical basis. $\square$

In [73] it was claimed that any universal Gröbner basis of I is a tropical basis. Unfortunately, this claim is false as the following example shows.

Example 3.6. Let I be the intersection of the three linear ideals $\langle x + y, z \rangle$, $\langle x + z, y \rangle$, and $\langle y + z, x \rangle$ in $\mathbb{C}[x, y, z]$. Then I contains the monomial xyz, so $\mathcal{T}(I)$ is empty. A minimal universal Gröbner basis of I is

$$\mathcal{U} \quad = \quad \{\, x + y + z,\; x^2y + xy^2,\; y^2z + yz^2,\; x^2z + xz^2 \,\},$$

and the intersection of the four corresponding tropical surfaces in $\mathbb{R}^3$ is the line $w_1 = w_2 = w_3$. Thus $\mathcal{U}$ is not a tropical basis of I. $\square$

We now prove that every ideal $I \subset \mathbb{C}[\mathbf{x}]$ has a tropical basis. By Lemma 3.2, one tropical basis of a non-homogeneous ideal I is the dehomogenization of a tropical basis for ${}^hI$. Hence we shall assume that I is a homogeneous ideal.

Tropical bases can be constructed from the *Gröbner fan* of I (see [57], [75]). The Gröbner fan is a complete finite rational polyhedral fan in $\mathbb{R}^n$ whose relatively open cones are in bijection with the distinct initial ideals of I. Two weight vectors $w, w' \in \mathbb{R}^n$ lie in the same relatively open cone of the Gröbner fan of I if and only if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$. The closure of this cell, denoted by $C_w(I)$, is called a *Gröbner cone* of I. The n-dimensional Gröbner cones are in bijection with the reduced Gröbner bases, or equivalently, the monomial initial ideals of I. Every Gröbner cone of I is a face of at least one n-dimensional

Gröbner cone of I. If $\mathrm{in}_w(I)$ is not a monomial ideal, then we can refine w to $\prec_w$ by breaking ties in the partial order induced by w with a fixed term order $\prec$ on $\mathbb{C}[\mathbf{x}]$. Let $G_{\prec_w}(I)$ denote the reduced Gröbner basis of I with respect to $\prec_w$. The Gröbner cone of $G_{\prec_w}(I)$, denoted by $C_{\prec_w}(I)$, is an n-dimensional Gröbner cone that has $C_w(I)$ as a face. The tropical variety $\mathcal{T}(I)$ consists of all Gröbner cones $C_w(I)$ such that $\mathrm{in}_w(I)$ does not contain a monomial. From the description of $\mathcal{T}(I)$ as $\bigcap_{f\in I}\mathcal{T}(f)$ it is clear that $\mathcal{T}(I)$ is supported on a closed subfan of the Gröbner fan. This endows the tropical variety $\mathcal{T}(I)$ with the structure of a polyhedral fan. In this lecture, the tropical variety $\mathcal{T}(I)$ of a homogenous ideal I is always assumed to come with this fan structure.

Theorem 3.7. *Every ideal $I \subset \mathbb{C}[\mathbf{x}]$ has a tropical basis.*

Proof. We may assume that I is a homogeneous ideal. Let $\mathcal{F}$ be any finite generating set of I which is not a tropical basis. Pick a Gröbner cone $C_w(I)$ whose relative interior intersects $\cap_{f\in\mathcal{F}}\mathcal{T}(f)$ non-trivially and whose initial ideal $\mathrm{in}_w(I)$ contains a monomial $\mathbf{x}^{\mathbf{m}}$. Compute the reduced Gröbner basis $G_{\prec_w}(I)$ for a refinement $\prec_w$ of w, and let h be the normal form of $\mathbf{x}^{\mathbf{m}}$ with respect to $G_{\prec_w}(I)$. Let $f := \mathbf{x}^{\mathbf{m}} - h$. Since the normal form of $\mathbf{x}^{\mathbf{m}}$ with respect to $G_{\prec}(\mathrm{in}_w(I)) = \{\mathrm{in}_w(g) : g \in G_{\prec_w}(I)\}$ is 0 and h is the normal form of $\mathbf{x}^{\mathbf{m}}$ with respect to $G_{\prec_w}(I)$, every monomial occurring in h has higher w-weight than $\mathbf{x}^{\mathbf{m}}$. Moreover, h depends only on the reduced Gröbner basis $G_{\prec_w}(I)$ and is independent of the particular choice of w in $C_w(I)$. Hence for any w' in the relative interior of $C_w(I)$, we have $\mathbf{x}^{\mathbf{m}} = \mathrm{in}_{w'}(f)$. This implies that the polynomial $f := \mathbf{x}^{\mathbf{m}} - h$ is a *witness* for the cone $C_w(I)$ not being in the tropical variety $\mathcal{T}(I)$. We now add the witness f to the current basis $\mathcal{F}$ and repeat the process. Since the Gröbner fan has only finitely many cones, this process will terminate after finitely many steps. It eventually removes all cones of the Gröbner fan which violate the condition for $\mathcal{F}$ to be a tropical basis. □

The following lemma is useful for practical computations with Gröbner fans and for finding a low-dimensional representation of the tropical variety $\mathcal{T}(I)$.

Lemma 3.8. *For an ideal $I \subset \mathbb{C}[\mathbf{x}]$ and $w \in \mathbb{R}^n$, the following are equivalent:*

(1) *The ideal I is w-homogeneous;* i.e. *I is generated by a set S of w-homogeneous polynomials, meaning that* $\mathrm{in}_w(f) = f$ *for all $f \in S$.*
(2) *The initial ideal* $\mathrm{in}_w(I)$ *is equal to I.*

Proof. If I has a w-homogeneous generating set then $I \subseteq \mathrm{in}_w(I)$. Any maximal w-homogeneous component of $f \in I$ is in I. In particular $\mathrm{in}_w(f) \in I$. Conversely, the ideal $\mathrm{in}_w(I)$ is generated by w-homogeneous elements by definition so, if $I = \mathrm{in}_w(I)$, then I is generated by w-homogeneous elements. □

The set of w which satisfy conditions (1) and (2) is a linear subspace of $\mathbb{R}^n$. Its dimension is called the *homogeneity* of I and is denoted $\mathrm{homog}(I)$. This space is contained in every cone of the fan $\mathcal{T}(I)$ and can be computed from the Newton polytopes of the polynomials that form any reduced Gröbner basis of I. Passing to the quotient of $\mathbb{R}^n$ modulo that subspace and then to a sphere around the origin, $\mathcal{T}(I)$ can be represented as a polyhedral complex of dimension $n - \mathrm{codim}(I) - \mathrm{homog}(I) - 1 = \dim(I) - \mathrm{homog}(I) - 1$. Here $\mathrm{codim}(I)$ and $\dim(I)$ are the codimension and dimension of I. In what follows, $\mathcal{T}(I)$ is always presented in this way, and every ideal I is presented by a finite list of generators together with the three numbers n, $\dim(I)$ and $\mathrm{homog}(I)$.

Example 3.9. Let I denote the ideal which is generated by the 3×3-minors of a symmetric 4×4-matrix of unknowns. This ideal has $n = 10$, $\dim(I) = 7$ and $\mathrm{homog}(I) = 4$. Hence $\mathcal{T}(I)$ is a two-dimensional polyhedral complex. We regard $\mathcal{T}(I)$ as the tropicalization of the secant variety of the Veronese threefold in $\mathbb{P}^9$, i.e., the variety of symmetric 4×4-matrices of rank ≤ 2, Applying our `Gfan` implementation (see Example 3.28), we find that $\mathcal{T}(I)$ is a simplicial complex consisting of 75 triangles, 75 edges and 20 vertices. □

In our next theorem, we shall assume that I is a homogeneous prime ideal of dimension d in $\mathbb{C}[x_1, \ldots, x_n]$. Its tropical variety $\mathcal{T}(I)$ is called *irreducible*. It is a subfan of the Gröbner fan of I. A theorem due to Bieri and Groves [10] states that $\mathcal{T}(I)$ is pure of dimension d, and we shall strengthen this result. A cone of dimension $d-1$ in $\mathcal{T}(I)$ is called a *ridge*. A *ridge path* is a sequence of d-dimensional cones $F_1, F_2, \ldots, F_k$ such that $F_i \cap F_{i+1}$ is a ridge for all $i \in \{1, 2, \ldots, k-1\}$. The following theorem is crucial for our algorithms.

Theorem 3.10. *The irreducible tropical variety $\mathcal{T}(I)$ is a pure fan of dimension d. This fan is connected in codimension one, which means that two maximal (d-dimensional) cones in $\mathcal{T}(I)$ are connected by a ridge path in $\mathcal{T}(I)$.*

The proof of this theorem will be based on the following important lemma.

Lemma 3.11. (Transverse Intersection Lemma)
Let I and J be ideals in $\mathbb{C}[x_1, \ldots, x_n]$ whose tropical varieties $\mathcal{T}(I)$ and $\mathcal{T}(J)$ meet transversally at a point $w \in \mathbb{R}^n$. Then $w \in \mathcal{T}(I + J)$.

By "meet transversely" we mean that if F and G are the cones of $\mathcal{T}(I)$ and $\mathcal{T}(J)$ which contain w in their relative interior, then $\mathbb{R}F + \mathbb{R}G = \mathbb{R}^n$.

This lemma implies that any transverse intersection of tropical varieties is a tropical variety. In particular, any transverse intersection of tropical hypersurfaces is a tropical variety, and such a tropical variety is defined by an ideal which is a complete intersection in the commutative algebra sense.

Corollary 3.12. *For any two ideals I and J in $\mathbb{C}[x_1, \ldots, x_n]$ we have*

$$\mathcal{T}(I + J) \quad \subseteq \quad \mathcal{T}(I) \cap \mathcal{T}(J).$$

Equality holds if the latter intersection is transverse at every point except the origin and the two fans meet in at least one point other than the origin.

Proof. We have $\mathcal{T}(I)\cap\mathcal{T}(J) = \bigcap_{f\in I}\mathcal{T}(f)\cap\bigcap_{f\in J}\mathcal{T}(f) = \bigcap_{f\in I\cup J}\mathcal{T}(f)$. Clearly, this contains $\mathcal{T}(I+J) = \bigcap_{f\in I+J}\mathcal{T}(f)$. If $\mathcal{T}(I)$ and $\mathcal{T}(J)$ intersect transversally and w is a point of $\mathcal{T}(I)\cap\mathcal{T}(J)$ other than the origin then the preceeding lemma tells us that $w \in \mathcal{T}(I+J)$. Thus $\mathcal{T}(I+J)$ contains every point of $\mathcal{T}(I)\cap\mathcal{T}(J)$ except possibly the origin. In particular, $\mathcal{T}(I+J)$ is not empty. Since every nonempty fan contains the origin, it is in $\mathcal{T}(I+J)$ as well. □

We first derive Theorem 3.10 from Lemma 3.11, which will be proved later. We must at this point address an annoying technical detail. The subset $\mathcal{T}(I) \subset \mathbb{R}^n$ depends only on the ideal $I\mathbb{C}[\mathbf{x}^{\pm 1}]$ generated by I in the Laurent polynomial ring $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}]$. From a theoretical perspective then, it would be better to directly work with ideals in $\mathbb{C}[\mathbf{x}^{\pm 1}]$. One reason is the availability the symmetry group $\mathrm{G}L_n(\mathbb{Z})$ of the multiplicative group of monomials. The action of this group transforms $\mathcal{T}(I)$ by the obvious action on $\mathbb{R}^n$. This symmetry will prove invaluable for simplifying the arguments in this section. Therefore, in this section, we will work with ideals in $\mathbb{C}[\mathbf{x}^{\pm 1}]$. Computationally, however, it is much better to deal with ideals in $\mathbb{C}[\mathbf{x}]$ as it is for such ideals that Gröbner basis techniques have been developed and this is the approach we take here.

Note that, if $I \subset \mathbb{C}[\mathbf{x}]$ is prime then so is the ideal it generates in the Laurent polynomial ring $\mathbb{C}[\mathbf{x}^{\pm 1}]$. We will signify an application of the $\mathrm{G}L_n(\mathbb{Z})$ symmetry by the phrase "making a multiplicative change of variables". The fan structure on $\mathcal{T}(I)$ induced by the Gröbner fan of I will change under a multiplicative change of variables of $I\mathbb{C}[\mathbf{x}^{\pm}]$ in $\mathbb{C}[\mathbf{x}^{\pm 1}]$, but all of the properties of $\mathcal{T}(I)$ that are of interest to us depend only on the underlying point set.

Proof of Theorem 3.10. As discussed, we replace I by the ideal it generates in $\mathbb{C}[\mathbf{x}^{\pm 1}]$ and, by abuse of notation, continue to denote this ideal as I. The proof is by induction on $d = \dim(\mathcal{T}(I))$. If $d \leq 1$ then the statement is trivially true. We now explain why the result holds for $d = 2$. By a multiplicative change of coordinates, it suffices to check that $\mathcal{T}(I)\cap\{x_n = 1\}$ is connected. Let K be the Puiseux series field over $\mathbb{C}$. Let $I' \subset K[x_1, \ldots, x_{n-1}]$ be the prime ideal generated by I via the inclusion $\mathbb{C}[x_n] \to K$. By Lemma **??**, the tropical variety of I' is $\mathcal{T}(I)\cap\{x_n = 1\}$. The tropical variety of I' is connected since I' defines an irreducible curve. This can be seen by projecting into the plane and using the balancing condition of Proposition **??**. We conclude that $\mathcal{T}(I)\cap\{x_n = 1\}$ is connected, so our result holds for $d = 2$.

We now suppose that $d \geq 3$. Let F and F' be facets of $\mathcal{T}(I)$. We can find

$$H \quad = \quad \big\{\, (u_1, \ldots, u_n) \in \mathbb{R}^n \,:\, a_1 u_1 + \cdots + a_n u_n = 0 \,\big\}$$

such that $a_1, \ldots, a_n$ are relatively prime integers, both $H \cap F$ and $H \cap F'$ are cones of dimension $d-1$, and H intersects every cone of $\mathcal{T}(I)$ except for the origin transversally. To see this, pick rays w and w' in the relative interiors of F and F'. By perturbing w and w' slightly, we may arrange that the span of $\{w, w'\}$ does not meet any ray of $\mathcal{T}(I)$. Here it is important that $d \geq 3$. Now, taking H to be the span of w, w' and a generic $(n-3)$-plane, we get that H also does not contain any ray of $\mathcal{T}(I)$ and hence does not contain any positive-dimensional face of $\mathcal{T}(I)$. So H is transverse to $\mathcal{T}(I)$ everywhere except at the origin. Since $H \cap F$ and $H \cap F'$ are positive-dimensional (as $d \geq 2$), the hyperplane H intersects $\mathcal{T}(I)$ at points other than just the origin. Note that H is the tropical hypersurface of a binomial, namely, $H = \mathcal{T}(\langle f_u \rangle)$, where

$$f_u \quad = \quad \prod_{i:a_i>0} (u_i x_i)^{a_i} \; - \; \prod_{j:a_j<0} (u_j x_j)^{-a_j},$$

and $u = (u_1, u_2, \ldots, u_n)$ is an arbitrary point in the algebraic torus $(\mathbb{C}^*)^n$. Our transversality assumption regarding H and Lemma 3.11 imply that

$$(2) \qquad H \cap \mathcal{T}(I) \quad = \quad \mathcal{T}(\langle f_u \rangle) \cap \mathcal{T}(I) \quad = \quad \mathcal{T}\big(I + \langle f_u \rangle\big).$$

Since I is prime of dimension d, and $f_u \notin I$, the ideal $I + \langle f_u \rangle$ has dimension $d-1$ by Krull's Principal Ideal Theorem [29, Theorem 10.1]. If $I + \langle f_u \rangle$ were a prime ideal then we would be done by induction. Indeed, this would imply that there is a ridge path between the facets $H \cap F$ and $H \cap F'$ in the $(d-1)$-dimensional tropical variety (2). Since $d \geq 3$, the $(d-1)$- and $(d-2)$-dimensional faces of $H \cap \mathcal{T}(I)$ arise uniquely from the intersections of H with d- and $(d-1)$-dimensional faces of $\mathcal{T}(I)$. Hence this path is also a ridge path considered as a path in $\mathcal{T}(I)$.

Let $V(J)$ denote the subvariety of the algebraic torus $(\mathbb{C}^*)^n$ defined by an ideal $J \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The tropical variety in (2) depends only on the subvariety of $(\mathbb{C}^*)^n$ defined by our ideal $I + \langle f_u \rangle$. This subvariety is

$$(3) \qquad V\big(I + \langle f_u \rangle\big) \quad = \quad V(I) \cap V(f_u) \quad = \quad V(I) \cap u^{-1} \cdot V(f_{\mathbf{1}}).$$

Here $\mathbf{1}$ denotes the identity element of $(\mathbb{C}^*)^n$. For generic choices of the group element $u \in (\mathbb{C}^*)^n$, the intersection (3) is an irreducible subvariety of dimension $d-1$ in $(\mathbb{C}^*)^n$. This follows from Kleiman's version of Bertini's Theorem [**?**, Theorem III.10.8], applied to the algebraic group $(\mathbb{C}^*)^n$. Hence (2) is indeed an irreducible tropical variety of dimension $d-1$, defined by the prime ideal $I + \langle f_u \rangle$. This completes the proof by induction. □

Proof of Lemma 3.11: Again, we replace $I \subset \mathbb{C}[\mathbf{x}]$ by the ideal it generates in $\mathbb{C}[\mathbf{x}^{\pm 1}]$. Let F be the cone of $\mathcal{T}(I)$ which contains w in its relative interior and G the cone of $\mathcal{T}(J)$ which contains w in its relative interior. Our hypothesis is that F and G meet transversally at w, that is, $\mathbb{R}F + \mathbb{R}G = \mathbb{R}^n$.

We claim that the ideal $\mathrm{in}_w(I)$ is homogeneous with respect to any weight vector $v \in \mathbb{R}F$ or, equivalently (see Proposition 3.8), that $\mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_w(I)$. According to Proposition 1.13 in [75], for ϵ a sufficiently small positive number, $\mathrm{in}_{w+\epsilon v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$. The vector $w + \epsilon v$ is in the relative interior of F so $\mathrm{in}_{w+\epsilon v}(I) = \mathrm{in}_w(I)$. By the same argument, the ideal $\mathrm{in}_w(J)$ is homogeneous with respect to any weight vector in $\mathbb{R}G$.

After a multiplicative change of variables in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we may assume that $w = e_1$, $\mathbb{R}\{e_1, e_2, \ldots, e_s\} \subseteq \mathbb{R}F$ and $\mathbb{R}\{e_1, e_{s+1}, \ldots, e_n\} \subseteq \mathbb{R}G$. We change the notation for the variables as follows:

$$t = x_1,\; y = (y_2, \ldots, y_s) = (x_2, \ldots, x_s),\;\; z = (z_{s+1}, \ldots, z_n) = (x_{s+1}, \ldots, x_n).$$

The homogeneity properties of the two initial ideals ensure that we can pick generators $f_1(z), \ldots, f_a(z)$ for $\mathrm{in}_w(I)$ and generators $g_1(y), \ldots, g_b(y)$ for $\mathrm{in}_w(J)$. Since $\mathrm{in}_w(I)$ is not the unit ideal, the Laurent polynomials $f_i(z)$ have a common zero $Z = (Z_{s+1}, \ldots, Z_n) \in (\mathbb{C}^*)^{n-s}$, and likewise the Laurent polynomials $g_j(y)$ have a common zero $Y = (Y_2, \ldots, Y_s) \in (\mathbb{C}^*)^{s-1}$.

Next we consider the following general chain of inclusions of ideals:

$$\mathrm{in}_w(I) \cdot \mathrm{in}_w(J) \subseteq \mathrm{in}_w(I \cdot J) \subseteq \mathrm{in}_w(I \cap J) \subseteq \mathrm{in}_w(I) \cap \mathrm{in}_w(J). \tag{4}$$

The product of two ideals which are generated by (Laurent) polynomials in disjoint sets of variables equals the intersection of the two ideals. Since the set of y-variables is disjoint from the set of z-variables, it follows that the first ideal in (4) equals the last ideal in (4). In particular, we conclude that

$$\mathrm{in}_w(I \cap J) \quad = \quad \mathrm{in}_w(I) \cap \mathrm{in}_w(J). \tag{5}$$

We next claim that

$$\mathrm{in}_w(I + J) \quad = \quad \mathrm{in}_w(I) + \mathrm{in}_w(J). \tag{6}$$

The left hand side is an ideal which contains both $\mathrm{in}_w(I)$ and $\mathrm{in}_w(J)$, so it contains their sum. We must prove that the right hand side contains the left hand side. Consider any element $f + g \in I + J$ where $f \in I$ and $g \in J$. Let $f = f_0(y, z) + t \cdot f_1(t, y, z)$ and $g = g_0(y, z) + t \cdot g_1(t, y, z)$. We have the following representation for some integer $a \geq 0$ and non-zero polynomial h_0:

$$f + g \quad = \quad t^a \cdot h_0(y, z) + t^{a+1} \cdot h_1(t, y, z).$$

If $a = 0$ then we conclude

$$\mathrm{in}_w(f + g) \;=\; h_0(y, z) \;=\; f_0(y, z) + g_0(y, z) \;\in\; \mathrm{in}_w(I) + \mathrm{in}_w(J).$$

If $a \geq 1$ then $f_0 = -g_0$ lies in $\mathrm{in}_w(I) \cap \mathrm{in}_w(J)$. In view of (5), there exists $p \in I \cap J$ with $f_0 = -g_0 = \mathrm{in}_w(p)$. Then $f + g = (f - p) + (g + p)$ and replacing f by $(f - p)/t$ and g by $(g + p)/t$ puts us in the same situation as before, but with a reduced by 1. By induction on a, we conclude that $\mathrm{in}_w(f + g)$ is in $\mathrm{in}_w(I) + \mathrm{in}_w(J)$, and the claim (6) follows.

For any constant $T \in \mathbb{C}^*$, the vector $(T, Y_2, \ldots, Y_s, Z_{s+1}, \ldots, Z_n)$ is a common zero in $(\mathbb{C}^*)^n$ of the ideal (6). We conclude that $\mathrm{in}_w(I + J)$ is not the unit ideal, so it contains no monomial, and hence $w \in \mathcal{T}(I + J)$. $\square$

We are now prepared to describe our solutions for the three computational problems stated earlier on in this lecture. The emphasis is on algorithms for Problem 3.3 for homogeneous prime ideals, taking advantage of Theorem 3.10.

In order to state our algorithms we must first explain how polyhedral cones and polyhedral fans are represented. A polyhedral cone is represented by a canonical minimal set of inequalities and equations. Given arbitrary defining linear inequalities and equations, the task of bringing these to a canonical form involves linear programming. Representing a polyhedral fan requires a little thought. We are rarely interested in all faces of all cones.

Definition 3.13. *A set S of polyhedral cones in $\mathbb{R}^n$ is said to* represent *a fan $\mathcal{F}$ in $\mathbb{R}^n$ if the set of all faces of cones in S is exactly $\mathcal{F}$.*

A representation may contain non-maximal cones, but each cone is represented minimally by its canonical form. A Gröbner cone $C_w(I)$ is represented by the pair $(\mathcal{G}_{\prec_w}(\mathrm{in}_w(I)), \mathcal{G}_{\prec_w}(I))$ of marked reduced Gröbner bases, where $\prec$ is some globally fixed term order. In a *marked* Gröbner basis the initial terms are distinguished. The advantage of using marked Gröbner bases is that the weight vector w need not be stored – we can deduce defining inequalities for its cone from the marked reduced Gröbner bases themselves, see Example 3.25. This is done as follows; see [75, proof of Proposition 2.3]:

Lemma 3.14. *Let $I \subset \mathbb{C}[\mathbf{x}]$ be a homogeneous ideal, $\prec$ a term order and $w \in \mathbb{R}^n$ a vector. For any other vector $w' \in \mathbb{R}^n$:*

$$w' \in C_w(I) \quad \Longleftrightarrow \quad \forall\, f \in \mathcal{G}_{\prec_w}(I) : \, \mathrm{in}_w(\mathrm{in}_{w'}(f)) = \mathrm{in}_w(f).$$

Our first two algorithms perform polyhedral computations, and they solve Problem 3.1. By the *support* of a fan we mean the union of its cones. Recall that, for a polynomial f, the tropical hypersurface $\mathcal{T}(f)$ is the union of the normal cones to the edges of the Newton polytope $\mathrm{New}(f)$.

Algorithm 3.15. *Tropical Hypersurface*
Input: $f \in \mathbb{C}[\mathbf{x}]$.
Output: *A representation S of a polyhedral fan whose support is $\mathcal{T}(f)$.*

```
{
  S := ∅;
  For every vertex v ∈ New(f)
  {
    Compute the normal cone C of v in New(f);
    S := S ∪ {the facets of C};
  }
}
```

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be polyhedral fans in $\mathbb{R}^n$. Their *common refinement* is

$$\mathcal{F}_1 \wedge \mathcal{F}_2 := \{C_1 \cap C_2\}_{(C_1,C_2)\in\mathcal{F}_1\times\mathcal{F}_2}.$$

To compute a common refinement we simply run through all pairs of cones in the fan representations and bring their intersection to canonical form. The canonical form makes it easy to remove duplicates.

Algorithm 3.16. *Common Refinement*
Input: *Representations S_1 and S_2 for polyhedral fans $\mathcal{F}_1$ and $\mathcal{F}_2$.*
Output: *A representation S for the common refinement $\mathcal{F}_1 \wedge \mathcal{F}_2$.*
{
  $S := \emptyset$;
  For every pair $(C_1, C_2) \in S_1 \times S_2$
    $S := S \cup \{C_1 \cap C_2\}$;
}

If refinements of more than two fans are needed, Algorithm 3.16 can be applied successively. Note that the intersection of the support of two fans is the support of the fans' common refinement. Hence Algorithm 3.16 can be used for computing intersections of tropical hypersurfaces. This solves Problem 3.1, but the output may be a highly redundant representation.

Recall (from the proof of Theorem 3.7) that a witness $f \in I$ is a polynomial which certifies $\mathcal{T}(f) \cap \mathrm{rel\ int}(C_w(I)) = \emptyset$. Computing witnesses is essential for solving Problems 3.3 and 3.4. The first step of constructing a witness is to check if the ideal $\mathrm{in}_w(I)$ contains monomials, and, if so, compute one such monomial. The check for monomial containment can be implemented by saturating the ideal with respect to the product of the variables (cf. [75, Lemma 12.1]). Knowing that the ideal contains a monomial, a simple way of finding one is to repeatedly reduce powers of the product of the variables by applying the division algorithm until the remainder is 0.

Algorithm 3.17. *Monomial in Ideal*
Input: *A set of generators for an ideal $I \subset \mathbb{C}[\mathbf{x}]$.*
Output: *A monomial $m \in I$ if one exists,* **no** *otherwise.*
{
  If $((I : x_1 \cdots x_n^\infty) \neq \langle 1 \rangle)$ *return* **no**;
  $m := x_1 \cdots x_n$;
  While $(m \notin I)$ $m := m \cdot x_1 \cdots x_n$;
  Return m;
}

Remark 3.18. *To pick the smallest monomial in I with respect to a term order, we first compute the largest monomial ideal contained in I using* [**?**, Algorithm 4.2.2] *and then pick the smallest monomial generator of this ideal.*

Constructing a witness from a monomial was already explained in the proof of Theorem 3.7. We only state the input and output of this algorithm.

Algorithm 3.19. *Witness*
Input: *A set of generators for an ideal $I \subset \mathbb{C}[\mathbf{x}]$ and a vector $w \in \mathbb{R}^n$ with $\mathrm{in}_w(I)$ containing a monomial.*
Output: *A polynomial $f \in I$ such that the tropical hypersurface $\mathcal{T}(f)$ and the relative interior of $C_w(I)$ have empty intersection.*

Combining Algorithm 3.17 and Algorithm 3.19 with known methods (e.g. [75, Algorithm 3.6]) for computing Gröbner fans, we can now compute the tropical variety $\mathcal{T}(I)$ and a tropical basis of I. This solves Problem 3.3 and Problem 3.4. but this approach is too slow to be useful in practise.

We next present a practical algorithm for computing $\mathcal{T}(I)$ when I is prime. An ideal $I \subset \mathbb{C}[\mathbf{x}]$ is said to define a *tropical curve* if $\dim(I) = 1 + \text{homog}(I)$. Our problems are easier in this case because a tropical curve consists of only finitely many rays and the origin modulo the homogeneity space.

Algorithm 3.20. *Tropical Basis of a Curve*
Input: *A set of generators $\mathcal{G}$ for an ideal I defining a tropical curve.*
Output: *A tropical basis $\mathcal{G}'$ of I.*

{
  Compute a representation S of $\bigwedge_{g \in \mathcal{G}} \mathcal{T}(g)$;
  For every $C \in S$
  {
    Let w be a generic relative interior point in C;
    If ($\text{in}_w(I)$ contains a monomial)
      then add a witness to $\mathcal{G}$ and restart the algorithm;
  }
  $\mathcal{G}' := \mathcal{G}$;
}

Proof of correctness. The algorithm terminates because I has only finitely many initial ideals and at least one is excluded in every iteration. If a vector w passes the monomial test (which verifies $w \in \mathcal{T}(I)$) then C has dimension 0 or 1 modulo the homogeneity space since we are looking at a curve and w is generic in C. Any other relative interior point of C would also have passed the monomial test. (This property fails in higher dimensions, when $\mathcal{T}(I)$ is no longer a tropical curve). Hence, when we terminate only points in the tropical variety are covered by S. Thus $\mathcal{G}'$ is a tropical basis. $\square$

In the curve case, combining Algorithms 3.15 and 3.16 with Algorithm 3.20 we get a reasonable method for solving Problem 3.3. This method is used as a subroutine in Algorithm 3.22 below. In the remainder of this section we concentrate on providing a better algorithm for Problem 3.3 in the case of a prime ideal. The idea is to use connectivity to traverse the tropical variety.

The next algorithm is an important subroutine for us. We only specify the input and output. This algorithm is one step in the Gröbner walk [18].

Algorithm 3.21. *Lift*
Input: *Marked reduced Gröbner bases $\mathcal{G}_{\prec'}(I)$ and $\mathcal{G}_{\prec_w}(\text{in}_w(I))$ where $w \in C_{\prec'}(I)$ is an unspecified vector and $\prec$ and $\prec'$ are unspecified term orders.*
Output: *The marked reduced Gröbner basis $\mathcal{G}_{\prec_w}(I)$.*

We now suppose that I is a monomial-free prime ideal with $d = \dim(I)$, and $\prec$ is a globally fixed term order. We first describe the local computations needed for a traversal of the d-dimensional Gröbner cones contained in $\mathcal{T}(I)$.

Algorithm 3.22. *Neighbors*
Input: *A pair* $(\mathcal{G}_{\prec_w}(\mathrm{in}_w(I)), \mathcal{G}_{\prec_w}(I))$ *such that* $\mathrm{in}_w(I)$ *is monomial-free and* $C_w(I)$ *has dimension* d*.*
Output: *The collection* N *of pairs of the form* $(\mathcal{G}_{\prec_{w'}}(\mathrm{in}_{w'}(I)), \mathcal{G}_{\prec_{w'}}(I))$ *where one* w' *is taken from the relative interior of each* d*-dimensional Gröbner cone contained in* $\mathcal{T}(I)$ *that has a facet in common with* $C_w(I)$*.*
{
  $N := \emptyset$;
  Compute the set $\mathcal{F}$ *of facets of* $C_w(I)$;
  For each facet $F \in \mathcal{F}$
  {
    Compute the initial ideal $J := \mathrm{in}_u(I)$
      where u *is a relative interior point in* F;
    Use Algorithm 3.20 and Algorithm 3.16 to produce a relative
      interior point v *of each ray in the curve* $\mathcal{T}(J)$;
    For each such v
    {
      Compute $(\mathcal{G}_{\prec_v}(\mathrm{in}_v(J)), \mathcal{G}_{\prec_v}(J)) = (\mathcal{G}_{\prec_{v_u}}(\mathrm{in}_v(J)), \mathcal{G}_{\prec_{v_u}}(J))$;
      Apply Algorithm 3.21 to $\mathcal{G}_{\prec_w}(I)$ *and* $\mathcal{G}_{\prec_{v_u}}(J)$ *to get* $\mathcal{G}_{\prec_{v_u}}(I)$;
      $N := N \cup \{(\mathcal{G}_{\prec_{v_u}}(\mathrm{in}_v(J)), \mathcal{G}_{\prec_{v_u}}(I))\}$;
    }
  }
}

Proof of correctness. Facets and relative interior points are computed using linear programming. The initial ideal $\mathrm{in}_u(I)$ is homogeneous with respect to the span of F. Hence $\mathrm{homog}(I) = d - 1$. The Krull dimension of $\mathbb{C}[\mathbf{x}]/\mathrm{in}_u(I)$ is d. Hence $\mathrm{in}_u(I)$ defines a curve and $\mathcal{T}(\mathrm{in}_u(I))$ can be computed using Algorithm 3.20. The identity $\mathrm{in}_v(\mathrm{in}_u(I)) = \mathrm{in}_{u+\varepsilon v}(I)$ for small $\varepsilon > 0$, see [75, Proposition 1.13], implies that we run through all the desired $\mathrm{in}_{w'}(I)$ where $w' = u + \varepsilon v$ for small $\varepsilon > 0$. The lifting step can be carried out since $u \in C_{\prec_w}(I)$. $\square$

Algorithm 3.23. *Traversal of an Irreducible Tropical Variety*
Input: *A pair* $(\mathcal{G}_{\prec_w}(\mathrm{in}_w(I)), \mathcal{G}_{\prec_w}(I))$ *such that* $\mathrm{in}_w(I)$ *is monomial free and* $C_w(I)$ *has dimension* d*.*
Output: *The collection* T *of pairs of the form* $(\mathcal{G}_{\prec_{w'}}(\mathrm{in}_{w'}(I)), \mathcal{G}_{\prec_{w'}}(I))$ *where one* w' *is taken from the relative interior of each* d*-dimensional Gröbner cone contained in* $\mathcal{T}(I)$*. The union of all the* $C_{w'}(I)$ *is* $\mathcal{T}(I)$*.*
{
  $T := \{(\mathcal{G}_{\prec_w}(\mathrm{in}_w(I)), \mathcal{G}_{\prec_w}(I))\}$;

Old $:= \emptyset$*;*
While $(T \neq Old)$
{
Old $:= T$*;*
$T := T \cup \text{Neighbors}(T)$*;*
}
}

Proof of correctness. By Neighbors(T) we mean the union of all the output of Algorithm 3.22 applied to all pairs in T. The algorithm computes the connected component of the starting pair. Since I is a prime ideal, Theorem 3.10 implies that the union of all the computed $C_{w'}(I)$ is $\mathcal{T}(I)$. □

To use Algorithm 3.23 we must know a starting d-dimensional Gröbner cone contained in the tropical variety. One inefficient method for finding one would be to compute the entire Gröbner fan. Instead we currently use heuristics, which are based on the following probabilistic recursive algorithm:

Algorithm 3.24. *Starting Cone*
Input: *A marked reduced Gröbner basis* $\mathcal{G}$ *for an ideal* I *whose tropical variety is pure of dimension* $d = \dim(I)$*. A term order* $\prec$ *for tie-breaking.*
Output: *Two marked reduced Gröbner bases:*

- *One for an initial ideal* $\text{in}_{w'}(I)$ *without monomials, where the homogeneity space of* $\text{in}_{w'}(I)$ *has dimension* d*. The term order is* $\prec_{w'}$*.*
- *A marked reduced Gröbner basis for* I *with respect to* $\prec_{w'}$*.*

{
If $(\dim(I) = \text{homog}(I))$
Return $(\mathcal{G}_{\prec}(I), \mathcal{G}_{\prec}(I))$*;*
If not
{
Repeat
{
Compute a random reduced Gröbner basis of I*;*
Compute a random extreme ray w *of its Gröbner cone;*
}
Until $(\text{in}_w(I)$ *is monomial free);*
Compute $\mathcal{G}_{\prec_w}(I)$*;*
$(\mathcal{G}_{Init}, \mathcal{G}_{Full})$*:= Starting Cone(*$\mathcal{G}_{\prec_w}(\text{in}_w(I))$*);*
Apply Algorithm 3.21 to $\mathcal{G}_{\prec_w}(I)$ *and* $\mathcal{G}_{Full}$
to get a marked reduced Gröbner basis $\mathcal{G}'$ *for* I*;*
Return $(\mathcal{G}_{Init}, \mathcal{G}')$*;*
}
}

The above algorithms have been implemented in the software package `Gfan` [?]. In what follows we illustrate the use of `Gfan` in computing various tropical varieties.

Example 3.25. *We consider the prime ideal* $I \subset \mathbb{C}[a,b,c,d,e,f,g]$ *which is generated by the* 3×3 *minors of the generic* Hankel matrix *of size* 4×4*:*

$$\begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{pmatrix}.$$

Its tropical variety is a 4*-dimensional fan in* $\mathbb{R}^7$ *with* 2*-dimensional homogeneity space. Its combinatorics is given by the graph in Figure 1. To compute* $\mathcal{T}(I)$ *in* ***Gfan****, we write the ideal generators on a file* `hankel.in`*:*

```
% more hankel.in
{-c^3+2*b*c*d-a*d^2-b^2*e+a*c*e,-c^2*d+b*d^2+b*c*e-a*d*e-b^2*f+a*c*f,
-c*d^2+c^2*e+b*d*e-a*e^2-b*c*f+a*d*f,-d^3+2*c*d*e-b*e^2-c^2*f+b*d*f,
-c^2*d+b*d^2+b*c*e-a*d*e-b^2*f+a*c*f,-c*d^2+2*b*d*e-a*e^2-b^2*g+a*c*g,
-d^3+c*d*e+b*d*f-a*e*f-b*c*g+a*d*g,-d^2*e+c*e^2+c*d*f-b*e*f-c^2*g+b*d*g,
-c*d^2+c^2*e+b*d*e-a*e^2-b*c*f+a*d*f,-d^3+c*d*e+b*d*f-a*e*f-b*c*g+a*d*g,
-d^2*e+2*c*d*f-a*f^2-c^2*g+a*e*g,-d*e^2+d^2*f+c*e*f-b*f^2-c*d*g+b*e*g,
-d^3+2*c*d*e-b*e^2-c^2*f+b*d*f,-d^2*e+c*e^2+c*d*f-b*e*f-c^2*g+b*d*g,
-d*e^2+d^2*f+c*e*f-b*f^2-c*d*g+b*e*g,-e^3+2*d*e*f-c*f^2-d^2*g+c*e*g}
```

We then run the command

```
gfan_tropicalstartingcone < hankel.in > hankel.start
```

which applies Algorithm 3.24 to produce a pair of marked Gröbner bases. This represents a maximal cone in $\mathcal{T}(I)$*, as explained prior to Lemma 3.14.*

```
% more hankel.start
{
c*f^2-c*e*g,
b*f^2-b*e*g,
b*e*f+c^2*g,
b*e^2+c^2*f,
b^2*g-a*c*g,
b^2*f-a*c*f,
b^2*e-a*c*e,
a*f^2-a*e*g,
a*e*f+b*c*g,
a*e^2+b*c*f}
{
c*f^2+e^3-2d*e*f+d^2*g-c*e*g,
b*f^2+d*e^2-d^2*f-c*e*f+c*d*g-b*e*g,
b*e*f+d^2*e-c*e^2-c*d*f+c^2*g-b*d*g,
b*e^2+d^3-2c*d*e+c^2*f-b*d*f,
b^2*g+c^2*e-b*d*e-b*c*f+a*d*f-a*c*g,
b^2*f+c^2*d-b*d^2-b*c*e+a*d*e-a*c*f,
b^2*e+c^3-2b*c*d+a*d^2-a*c*e,
a*f^2+d^2*e-2c*d*f+c^2*g-a*e*g,
a*e*f+d^3-c*d*e-b*d*f+b*c*g-a*d*g,
a*e^2+c*d^2-c^2*e-b*d*e+b*c*f-a*d*f}
```

Using Lemma 3.14 we can easily read off the canonical equations and equalities for the corresponding Gröbner cone $C_w(I)$*. For example, the polynomials* cf^2-

ceg and $cf^2 + e^3 - 2def + d^2g - ceg$ *represent the equation*

$$w_c + 2w_f \;=\; w_c + w_e + w_g$$

and the inequalities

$$w_c + 2w_f \;\leq\; \min\big\{3w_e, w_d + w_e + w_f, 2w_d + w_g, w_c + w_e + w_g\big\}.$$

At this point, we could run Algorithm 3.23 using the following command:

```
gfan_tropicaltraverse < hankel.start > hankel.out
```

However, we can save computing time and get a better idea of the structure of $\mathcal{T}(I)$ *by instructing* ***Gfan*** *to take advantage of symmetries of* I *as it produces cones. The only symmetries that can be used in* ***Gfan*** *are those that simply permute variables. The output will show which cones of* $\mathcal{T}(I)$ *lie in the same orbit under the action of the symmetry group we provide.*

Our ideal I *is invariant under reflecting the* 4×4*-matrix along the anti-diagonal. This reverses the variables* $a, b, \ldots, g$*. To specify this permutation, we add the following line to the bottom of the file* ***hankel.start****:*

```
{(6,5,4,3,2,1,0)}
```

We can add more symmetries by listing them one after another, separated by commas, inside the curly braces. ***Gfan*** *will compute and use the group generated by the set of permutations we provide, and it will return an error if we input any permutation which does not keep the ideal invariant.*

After adding the symmetries, we run the command

```
gfan_tropicaltraverse --symmetry < hankel.start > hankel.out
```

to compute the tropical variety. We show the output with some annotations:

```
% more hankel.out

Ambient dimension: 7
Dimension of homogeneity space: 2
Dimension of tropical variety: 4
Simplicial: true
Order of input symmetry group: 2
F-vector: (16,28)
```

A short list of basic data: the dimensions of the ambient space, of $\mathcal{T}(I)$*, and of its homogeneity space, and also the face numbers (*f*-vector) of* $\mathcal{T}(I)$ *and the order of symmetry group specified in the input.*

```
Modulo the homogeneity space:
{(6,5,4,3,2,-1,0),
 (5,4,3,2,1,0,-1)}
```

A basis for the homogeneity space. The rays are considered in the quotient of $\mathbb{R}^7$ *modulo this 2-dimensional subspace.*

```
Rays:
{0: (-1,0,0,0,0,0,0),
 1: (-5,-4,-3,-2,-1,0,0),
 2: (1,0,0,0,0,0,0),
 3: (5,4,3,2,1,0,0),
 4: (2,1,0,0,0,0,0),
 5: (4,3,2,1,0,0,0),
 6: (0,-1,0,0,0,0,0),
 7: (6,5,4,3,2,0,0),
 8: (3,2,1,0,0,0,0),
 9: (0,0,-1,0,0,0,0),
 10: (0,0,0,0,-1,0,0),
 11: (0,0,0,-1,0,0,0),
 12: (-6,-4,-3,-3,-1,0,0),
 13: (-3,-2,-2,-1,-1,0,0),
 14: (3,2,2,1,1,0,0),
 15: (3,2,2,0,1,0,0)}
```

The direction vectors of the tropical rays. Since the homogeneity space is positive-dimensional, the directions are not uniquely specified. For instance, the vectors $(-5,-4,-3,-2,-1,0,0)$ *and* $(0,0,0,0,0,0,-1)$ *represent the same ray. Note that* ***Gfan*** *uses negated weight vectors.*

```
Rays incident to each
dimension 2 cone:
{{2,6}, {3,7},
{2,4}, {3,5},
{4,9}, {5,10},
{4,8}, {5,8},
{8,11},
{0,12}, {1,12},
{0,1},
{1,6}, {0,7},
{1,9}, {0,10},
{0,13}, {1,13},
{6,14}, {7,14},
{9,13}, {10,13},
{6,10}, {7,9},
{6,7},
{11,12},
{11,15},
{14,15}}
```

The cones in $\mathcal{T}(I)$ *are listed from highest to lowest dimension. Each cone is named by the set of rays on it. There are 28 two-dimensional cones, broken down into 11 orbits of size 2 and 6 orbits of size 1.*

The further output, which is not displayed here, shows that the 16 rays break down into 5 orbits of size 2 and 6 orbits of size 1.

Using the same procedure, we now compute several more examples.

Example 3.26. *Let* I *be the ideal generated by the* 3×3 *minors of the generic* 5×5 *Hankel matrix. We again use the symmetry group* $\mathbb{Z}/2$. *The tropical variety is a graph with vertex degrees ranging from 2 to 7.*

```
Ambient dimension: 9
Dimension of homogeneity space: 2
Dimension of tropical variety: 4
Simplicial: true
F-vector: (28,53)
```

Example 3.27. *Let* I *be the ideal generated by the* 3×3 *minors of a generic* 3×5 *matrix. We use the symmetry group* $S_5 \times S_3$*, where* S_5 *acts by permuting the columns and* S_3 *by permuting the rows.*

```
Ambient dimension: 15
Dimension of homogeneity space: 7
Dimension of tropical variety: 12
Simplicial: true
F-vector: (45,315,930,1260,630)
```

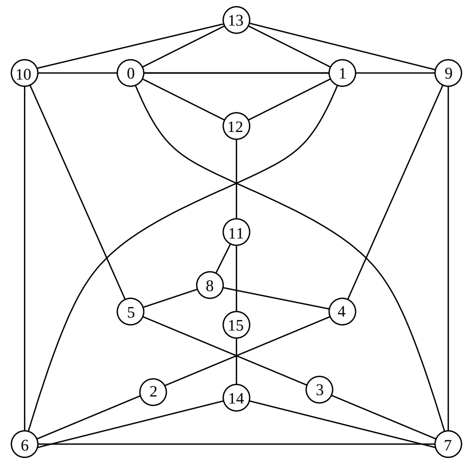

FIGURE 1. The tropical variety of the ideal generated by the 3×3 minors of the generic 4×4 Hankel matrix.

Example 3.28. *Let I be the ideal generated by the 3×3 minors of a generic 4×4 symmetric matrix. We use the symmetry group S_4 which acts by simultaneously permuting the rows and the columns.*

```
Ambient dimension: 10
Dimension of homogeneity space: 4
Dimension of tropical variety: 7
Simplicial: true
F-vector: (20,75,75)
```

If we take the 3×3 minors of a generic 5×5 symmetric matrix then we get

```
Ambient dimension: 15
Dimension of homogeneity space: 5
Dimension of tropical variety: 9
Simplicial: true
F-vector:    (75, 495, 1155, 855)
```

Example 3.29. *Let I be the prime ideal of a pair of commuting 2×2 matrices. That is, $I \subset \mathbb{C}[a, b, \ldots, h]$ is defined by the matrix equation*

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} - \begin{pmatrix} e & g \\ f & h \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 0.$$

*The tropical variety is the graph K_4, which **Gfan** reports as follows:*

```
Ambient dimension: 8
Dimension of homeogeneity space: 4
Dimension of tropical variety: 6
Simplicial: true
F-vector: (4,6)
```

If I is the ideal of 3×3 commuting symmetric matrices then we get:

```
Ambient dimension: 12
Dimension of homeogeneity space: 2
Dimension of tropical variety: 9
Simplicial: false
F-vector: (66,705,3246,7932,10888,8184,2745)
```