2. Puiseux Series and Tropical Hypersurfaces

Let $K = \mathbb{C}\{\{t\}\}$ denote the field of *Puiseux series* with complex coefficients. The elements of K are the formal power series

$$c(t) \;=\; c_1 t^{a_1} \,+\, c_2 t^{a_2} \,+\, c_3 t^{a_3} \,+\, \cdots \,, \tag{1}$$

where $c_1, c_2, c_3, \ldots$ are non-zero complex numbers and $a_1 < a_2 < a_3 < \cdots$ are rational numbers that have a common denominator. We write K^* for the set of non-zero elements in K. Clearly, K is a field, as it is closed under the usual arithmetic operations of addition, subtraction, multiplication and division. Moreover, we have the following classical result.

Theorem 2.1. (Puiseux's Theorem) *The Puiseux series field $K = \mathbb{C}\{\{t\}\}$ is algebraically closed, i.e. every non-constant polynomial in $K[x]$ has a root in K.*

The algorithmic version of this theorem is fundamental for connecting tropical geometry with classical algebraic geometry. To explain this connection we first introduce some general definitions. The *order* of the Puiseux series $c(t)$ in (1) is the exponent a_1 of the lowest degree term. The map

$$\mathrm{ord} \,:\, K^* \,\rightarrow\, \mathbb{Q}\,,\;\, c(t) \,\mapsto\, a_1 \,=\, \text{the order of } c(t)$$

is a *valuation*, which means that it satisfies

$$\mathrm{ord}(c+d) \,\geq\, \min\{\mathrm{ord}(c), \mathrm{ord}(d)\} \quad \text{and} \quad \mathrm{ord}(c \cdot d) \,=\, \mathrm{ord}(c) + \mathrm{ord}(d).$$

Let $f \in K[x_1, \ldots, x_n]$ be any polynomial in n variables with coefficients in K. Then its *tropicalization*, denoted $\mathrm{trop}(f)$, is the tropical polynomial which gotten from f by replacing each coefficient by its order and by replacing classical arithmetic by tropical arithmetic. Thus the classical polynomial

$$f \,=\, c(t) \cdot x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} \,+\, d(t) \cdot x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \,+\, \cdots$$

has the tropicalization

$$\mathrm{trop}(f) \,=\, \mathrm{ord}(c(t)) \odot x_1^{\odot u_1} \odot \cdot \cdot \odot x_n^{\odot u_n} \,\oplus\, \mathrm{ord}(d(t)) \odot x_1^{\odot v_1} \odot \cdot \cdot \odot x_n^{\odot v_n} \,\oplus\, \cdots$$

Lemma 2.2. *The piecewise-linear concave function $\mathbb{R}^n \rightarrow \mathbb{R}$ given by the tropical polynomial* $\mathrm{trop}(f)$ *is characterized by the following property. For almost all complex numbers $\gamma_1, \ldots, \gamma_n$, we have*

$$(\mathrm{trop}(f))(w_1, \ldots, w_n) \,=\, \mathrm{ord}\big(f(\gamma_1 t^{w_1}, \ldots, \gamma_n t^{w_n})\big) \quad \textit{for all } w_1, \ldots, w_n \in \mathbb{Q}.$$

Proof. Consider any monomial $c(t) x_1^{u_1} \cdots x_n^{u_n}$ appearing in the expansion of f. This contributes the expression $c(t) \cdot \gamma_1^{u_1} \cdots \gamma_n^{u_n} \cdot t^{w_1 u_1 + \cdots + w_n u_n}$ to the expansion of the univariate series $f(\gamma_1 t^{w_1}, \ldots, \gamma_n t^{w_n})$. Since the γ_i are generic, the terms of lowest order in that expansion cannot cancel, and hence the order of $f(\gamma_1 t^{w_1}, \ldots, \gamma_n t^{w_n})$ equals the minimum of $\mathrm{ord}(c(t)) + w_1 u_1 + \cdots + w_n u_n$ over all monomials in f. That minimum is precisely $\mathrm{trop}(f)(w_1, \ldots, w_n)$. $\square$

The previous lemma has the following important corollary.

Corollary 2.3. *Tropicalization of polynomial functions commutes with multiplication, i.e., if f and g are polynomials in $K[x_1,\ldots,x_n]$ then*

$$\mathrm{trop}(f\cdot g) \;=\; \mathrm{trop}(f)\odot\mathrm{trop}(g) \qquad \textit{as functions } \mathbb{R}^n\to\mathbb{R}.$$

Example 2.4. The identity $\mathrm{trop}(f\cdot g) = \mathrm{trop}(f)\odot\mathrm{trop}(g)$ need not hold at the level of polynomials. For instance, if $n=1$, $f = x-t^2$ and $g = x+t^2+t^3$ then $\mathrm{trop}(f\cdot g) = x^2\oplus 3\odot x\oplus 4$ while $\mathrm{trop}(f)\odot\mathrm{trop}(g) = x^2\oplus 2\odot x\oplus 4$. These are different polynomials but they define the same function $\mathbb{R}\to\mathbb{R}$.

We are now prepared to return to the situation of Puiseux's Theorem.

Proposition 2.5. *Let $f(x)$ be a polynomial in one variable with coefficients in the Puiseux series field K and let $u_1,\ldots,u_m$ be the roots of $f(x)$ in K. Their orders* $\mathrm{ord}(u_1),\ldots,\mathrm{ord}(u_m)$ *are the roots of the tropical polynomial* $\mathrm{trop}(f)$.

Proof. We first consider the case $m=1$, when f is a linear polynomial

$$f \;=\; c\cdot x + d \;=\; c\cdot\big(x\,+\,(-d/c)\big).$$

Its tropicalization is the tropical polynomial

$$\mathrm{trop}(f) \;=\; \mathrm{ord}(c)\odot x\,\oplus\,\mathrm{ord}(d) \;=\; \mathrm{ord}(c)\odot\big(x\,\oplus\,(\mathrm{ord}(d)-\mathrm{ord}(c))\big).$$

This proves the proposition for $m=1$ because

$$\mathrm{ord}(-d/c) \;=\; \mathrm{ord}(d/c) \;=\; \mathrm{ord}(d)-\mathrm{ord}(c).$$

For $m\geq 2$ we express f as a product of linear factors by the Fundamental Theorem of Algebra. The result follows directly from Corollary 2.3 and the $m=1$ case. □

We now take a closer look at the geometry of tropical hypersurfaces. To this end we first need to review some notions from polyhedral geometry. A *polyhedron* in $\mathbb{R}^n$ is the intersection of a finite collection of closed halfspaces $\{x : Ax\geq b\}$. If all halfspaces pass through the origin (i.e. $b=0$) then the polyhedron is called a *cone*. Bounded polyhedra are called *polytopes*. If the entries of all the defining matrices A and b are rational numbers then we speak of a *rational polyhedron*, *rational cone* or *rational polytope*. A *face* of a polyhedron P is a subset of P at which a linear form w attains its minimum. We use the following notation for the face of P specified by a linear form w:

$$\mathrm{face}_w(P) \;=\; \big\{\,x\in P \;:\; w\cdot x\,\leq\, w\cdot y \;\text{ for all } y\in P\,\big\}.$$

Here we identify $\mathbb{R}^n$ with its dual vector space, using the standard inner product, and so we simply regard w as a vector in $\mathbb{R}^n$. Given a face F of a polyhedron P we can consider the *normal cone* of P at the face F:

$$N_P(F) \;=\; \{\,w\in\mathbb{R}^n \;:\; \mathrm{face}_w(P)\,\subseteq\, F\,\}.$$

The dimension of this cone is complementary to the dimension of the face:

$$\dim(F)\;+\;\dim(N_P(F)) \quad=\quad n.$$

A *polyhedral complex* is a finite collection $\mathcal{C}$ of polyhedra in $\mathbb{R}^n$ such that

- If P is in $\mathcal{C}$ then every face of P is also in $\mathcal{C}$.
- If P and Q are in $\mathcal{C}$ then $P \cap Q$ is a face of P and is a face of Q.

The elements of $\mathcal{C}$ are called the *cells* of $\mathcal{C}$. The *support* $|\mathcal{C}|$ of $\mathcal{C}$ is the union of all cells in $\mathcal{C}$. The *dimension* of $\mathcal{C}$ is the maximal dimension of any cell in $\mathcal{C}$. If all maximal cells (with respect to inclusion) have the same dimension then $\mathcal{C}$ is said to be *pure*. A *fan* is a polyhedral complex all of whose cells are cones. An example of a fan is the *normal fan* of a polyhedron P:

$$\mathcal{N}(P) \;=\; \bigl\{\, N_P(F) \;:\; F \text{ face of } P \,\bigr\}.$$

The support $|\mathcal{N}(P)|$ of the normal fan is a cone in $\mathbb{R}^n$. If P is a polytope then $|\mathcal{N}(P)| = \mathbb{R}^n$. A *subdivision* of a polytope P is a polyhedral complex Δ whose support equals P. If all cells in Δ are simplices then Δ is a *triangulation* of P.

Given a tropical polynomial p in n variables, its hypersurface $\mathcal{T}(p)$ was defined as the set of all $x \in \mathbb{R}^n$ at which the function p is not linear. Thus $\mathcal{T}$ is the "corner locus" of the piecewise-linear concave function $p : \mathbb{R}^n \to \mathbb{R}$.

Proposition 2.6. *The tropical hypersurface $\mathcal{T}(p)$ is a pure polyhedral complex of dimension $n-1$. The cells of this complex can be bounded or unbounded.*

Proof. We define a polytope P in $\mathbb{R}^{n+1}$ by taking the convex hull of all points $(c, u_1, \ldots, u_n)$ where $c \odot x_1^{\odot u_1} \odot \cdots \odot x_n^{\odot u_n}$ is a term of the tropical polynomial p. The hypersurface $\mathcal{T}(p)$ consists of all points $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$ such that the linear form given by $(1, w) = (1, w_1, \ldots, w_n)$ attains its minimum at more than one of the points $(c, u_1, \ldots, u_n)$ representing a term in p. Equivalently,

$$\mathcal{T}(p) \;=\; \bigl\{\, w \in \mathbb{R}^n \;:\; \dim\bigl(\mathrm{face}_{(1,w)}(P)\bigr) \;\geq 1 \,\bigr\}. \tag{2}$$

Let $\mathcal{N}(P)_{\leq n}$ denote the set of all non-maximal cones in the normal fan $\mathcal{N}(P)$. This a pure fan of dimension n. If we intersect (the support of) this fan with the hyperplane $\{x_0 = 1\}$ in $\mathbb{R}^{n+1}$ then we get precisely the set $\mathcal{T}(p)$ in (2). Every cone C of $\mathcal{N}(P)_{\leq n}$ emanates from the origin and hence intersects the hyperplane $\{x_0 = 1\}$ transversally, or the intersection is empty. This implies that $\mathcal{T}(p) \;=\; \mathcal{N}(P)_{\leq n} \cap \{x_0 = 1\}$ is a pure polyhedral complex of dimension $n-1$. The cell $C \cap \{x_0 = 1\}$ is bounded if and only if the cone C meets the hyperplane $\{x_0 = 0\}$ just in the origin, and this may or may not happen. $\square$

It is often useful to consider the dual representation of $\mathcal{T}(p)$ as a polyhedral subdivision. The *Newton polytope* $\mathrm{New}(p)$ of the polynomial p is the convex hull in $\mathbb{R}^n$ of all points $(u_1, \ldots, u_n)$ such that $x_1^{\odot u_1} \odot \cdots \odot x_n^{\odot u_n}$ appears with some coefficient in p. Thus $\mathrm{New}(p)$ equals the image of P under the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ onto the last n coordinates. A face F of P is called a *lower face* if it has the form $F = \mathrm{face}_{(1,w)}(P)$ for some $w \in \mathbb{R}^n$. The set of all lower faces of P form a pure polyhedral complex of dimension n. The image of this complex under the map π is a polyhedral subdivision of the Newton polytope

New(p). We denote this subdivision by $\Delta(p)$ and we call it the *subdivided Newton polytope* of the tropical polynomial p. Each i-dimensional cell of $\Delta(p)$ corresponds to an i-dimensional lower face of P and, by way of the formula (2), to an $(n-1-i)$-dimensional cell of $\mathcal{T}(p)$. We conclude the following:

Corollary 2.7. *The tropical hypersurface $\mathcal{T}(p)$ is dual to the subdivided Newton polytope $\Delta(p)$. There is an order-reversing bijection between the cells of $\mathcal{T}(p)$ and the positive-dimensional cells of $\Delta(p)$. Unbounded cells of $\mathcal{T}(p)$ correspond to cells of $\Delta(p)$ that lie in the boundary of the Newton polytope $|\Delta(p)|$.*

In the previous section we have seen several examples of tropical curves in the plane, the case $n = 2$. For $n = 3$, a tropical hypersurface is a piecewise-linear surface in $\mathbb{R}^3$ which is dual to a subdivided 3-dimensional polytope.

Example 2.8. The simplest example of a surface $\mathcal{T}(p)$ is a tropical plane in 3-space. Here the tropical polynomial p has degree one, so we can write:

$$p(x,y,z) \quad = \quad a \odot x \;\oplus\; b \odot y \;\oplus\; c \odot z \;\oplus\; d.$$

The tropical plane $\mathcal{T}(p)$ is a two-dimensional fan with apex $(d-a, d-b, d-c)$. To this apex we attach the rays with directions $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$, and we add the six cones spanned by any two of these rays. Thus a tropical plane in $\mathbb{R}^3$ has one vertex, four unbounded edges and six unbounded 2-cells. Now consider a second plane $\mathcal{T}(p')$ whose apex does not lie on any of the rays of the first plane, and vice versa. Then their intersection $L = \mathcal{T}(p) \cap \mathcal{T}(p')$ is a *tropical line*. A tropical line in $\mathbb{R}^3$ has two vertices, one bounded edge and four unbounded edges. The bounded edge connects the two vertices and the four rays point in the four distinguished directions. □

Example 2.9. Consider a tropical polynomial of degree 3 in three variables:

$$p(x,y,z) \quad = \bigoplus_{i+j+k\leq 3} c_{ijk} \odot x^{\odot i} y^{\odot j} z^{\odot k}$$

We assume that all twenty terms are present with a coefficient $c_{ijk} \in \mathbb{R}$. The Newton polytope of p is the standard tetrahedron scaled by a factor of three:

$$\text{New}(p) \;=\; \text{conv}\big\{(0,0,0),\,(0,0,3),\,(0,3,0),\,(3,0,0)\big\}.$$

If the c_{ijk} are sufficiently general then the subdivision $\Delta(p)$ is a triangulation of the tetrahedron. We are particularly interested in the case when this triangulation is *unimodular*, i.e., it consists of 27 tetrahedra each having volume 1/6. The corresponding tropical cubic surface $\mathcal{T}(p)$ is the dual surface which which has 27 vertices. We refer to the recent paper by Vigeland [87] for several picture and for a careful analysis of tropical cubic surfaces that contain 27 tropical lines. There are still many open problems in this subject [87, §7.2]. □

We now take a step towards algebraic geometry over the Puiseux series field K. As our ambient space we take the *algebraic torus* $(K^*)^n = (K\backslash\{0\})^n$. The valuation map of the field extends to the torus by coordinate-wise application

$$\text{ord} \,:\, (K^*)^n \,\rightarrow\, \mathbb{Q}^n \,, \;\; u = (u_1, \ldots, u_n) \,\mapsto\, \big(\text{ord}(u_1), \ldots, \text{ord}(u_n)\big).$$

The *hypersurface* defined by a polynomial $f \in K[x_1, \ldots, x_n]$ is the subvariety

$$V(f) \quad := \quad \{\, u \in (K^*)^n \,:\, f(u) = 0 \,\}$$

Tropical geometry mirrors algebraic geometry by way of the following result.

Theorem 2.10. (Kapranov's Theorem) *The image of $V(f)$ under the valuation map is the set of all $\mathbb{Q}$-rational points in the tropical hypersurface, i.e.*

$$\text{ord}(V(f)) \;\;=\;\; \mathcal{T}\big(\text{trop}(f)\big) \cap \mathbb{Q}^n.$$

Proof. Set $f = \sum c_a x^a$ and $p = \text{trop}(f) = \bigoplus \text{ord}(c_a) \odot x^{\odot a}$. Here a runs over a finite subset of $\mathbb{N}^n$. We first show that $\text{ord}(V(f))$ is contained in $\mathcal{T}(p)$. Let $u \in (K^*)^n$ be any point with $f(u) = \sum c_a u^a = 0$. Consider the minimum among the numbers $\text{ord}(c_a u^a) = \text{ord}(c_a) \odot u^{\odot a}$. This minimum is attained at least twice because the summands of lowest order in $\sum c_a u^a$ must cancel. This means that u is in the tropical hypersurface $\mathcal{T}(p)$. Hence $\text{ord}(V(f)) \subset \mathcal{T}(p)$.

We next show $\mathcal{T}(p) \cap \mathbb{Q}^n \subseteq \text{ord}(V(f))$. Consider any point $v \in \mathbb{Q}^n$ which lies in $\mathcal{T}(p)$. We must construct $u \in V(f)$ such that $\text{ord}(u) = v$. Consider the univariate polynomial $g(x) = f(t^{v_1}x, t^{v_2}x, \ldots, t^{v_n}x)$. Then $\text{trop}(g)(z) = p(v_1 \odot z, v_2 \odot z, \ldots, v_n \odot z)$ and $z = 0$ is a root of this univariate tropical polynomial. By Theorem 2.1, there exists $s \in K^*$ with $g(s) = 0$ and $\text{ord}(s) = 0$. The point $u = (st^{v_1}, st^{v_2}, \ldots, st^{v_n})$ satisfies $f(u) = 0$ and $\text{ord}(u) = v$. □

Frequently, we shall be interested in tropicalizing polynomials f whose coefficients do not depend on t at all but are just scalars in $\mathbb{C}$. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be such a polynomial. Then $\text{trop}(f)$ is a tropical polynomial all of whose coefficients are zero. For example, if $f = 23x^3 - 17xy + \sqrt{-1}y^2 - 59$ then $\text{trop}(f) = 0 \odot x^{\odot 3} \oplus 0 \odot x \odot y \oplus 0 \odot y^{\odot 2} \oplus 0$. The tropical hypersurfaces of such polynomials are special in that each of their cells is a cone with apex 0.

Corollary 2.11. *Let p be a tropical polynomial all of whose coefficients are zero. Then the tropical hypersurface $\mathcal{T}(p)$ is a pure fan of codimension* 1.

Proof. The polytope P in the proof of Proposition 2.6 lies in the hyperplane $\{x_0 = 0\}$ and can be identified with the Newton polytope $\text{New}(p)$. Under this identification, we have $\text{face}_{0,w}(P) = \text{face}_w(\text{New}(p))$. Thus, (2) translates into

$$\mathcal{T}(p) \;=\; \big\{\, w \in \mathbb{R}^n \,:\, \dim\big(\text{face}_w(\text{New}(p))\big) \,\geq 1 \,\big\}. \tag{3}$$

Thus $\mathcal{T}(p)$ is the union of all non-maximal cones in the normal fan $\mathcal{N}(\text{New}(p))$. This is a pure fan of codimension 1 in $\mathbb{R}^n$. □

The *lineality space* of a fan is the largest linear subspace of the ambient vector space which is contained in all cones of the fan. A fan is called *pointed* if its lineality space is $\{0\}$. If $\mathcal{F}$ is a fan with positive-dimensional lineality space L then it can be represented by the quotient fan $\mathcal{F}/L$. This is a pointed fan of dimension $\dim(\mathcal{F}) - \dim(L)$ which lives in the quotient space $\mathbb{R}^n/L$. We can reduce the dimension further by intersecting this pointed fan with a sphere $\mathbb{S}$ around the origin in $\mathbb{R}^n/L$. The result is a polyhedral complex of $\dim(\mathcal{F}) - \dim(L) - 1$, which retains all the geometric and combinatorial information about $\mathcal{F}$. If p is a tropical polynomial with all coefficients zero then this construction is used to represent its tropical hypersurface $\mathcal{F} = \mathcal{T}(p)$ by the lower-dimensional polyhedral complex $\mathcal{T}'(p) := \mathcal{T}(p)/L \cap \mathbb{S}$.

Example 2.12. Consider a tropical linear form in four variables:

$$p \;=\; 0 \odot x_1 \;\oplus\; 0 \odot x_2 \;\oplus\; 0 \odot x_3 \;\oplus\; 0 \odot x_4.$$

Then $\mathcal{T}(p)$ is a three-dimensional fan with a one-dimensional lineality space L. The one-dimensional complex $\mathcal{T}'(p)$ is the complete graph on four nodes K_4. The six edges of $\mathcal{T}'(p) = K_4$ correspond to the six three-dimensional cones of $\mathcal{T}(p)$. Thus this example is combinatorially isomorphic to Example 2.8. □

Example 2.13. *(The Tropical Determinant)* The determinant of an $m \times m$-matrix is a polynomial *det* of degree m in m^2 unknowns having $m!$ terms. The *tropical determinant* is the tropicalization of that polynomial. It is denoted by tropdet := trop(*det*). A real $m \times m$-matrix is called *tropically singular* if it lies in the tropical hypersurface $\mathcal{T}(\text{tropdet}) \subset \mathbb{R}^{m^2}$ defined by the determinant.

For instance, if $m = 3$ then the tropical determinant equals

$$\text{tropdet}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \;=\; \begin{matrix} & x_{11} \odot x_{22} \odot x_{33} & \oplus & x_{11} \odot x_{23} \odot x_{32} \\ \oplus & x_{12} \odot x_{21} \odot x_{33} & \oplus & x_{12} \odot x_{23} \odot x_{32} \\ \oplus & x_{13} \odot x_{21} \odot x_{32} & \oplus & x_{13} \odot x_{22} \odot x_{31}. \end{matrix}$$

This is a tropical polynomial in $n = 9$ variables. The linealiy space L of its hypersurface consists of all matrices of the form $(u_i + v_j)_{1 \leq i,j \leq 3}$, so L is 5-dimensional. Thus $\mathcal{T}(\text{tropdet})/L$ is a 3-dimensional fan in a 4-dimensional space. Intersecting with a sphere we get the 2-dimensional polyhedral complex $\mathcal{T}'(\text{tropdet})$. This representation of the tropical 3×3-determinant is shown in [61, Figure 3.5, page 114]. The complex $\mathcal{T}'(\text{tropdet})$ has 9 vertices, 18 edges and 15 two-dimensional cells (9 squares and 6 triangles).

The tropical $m \times m$-determinant has a nice interpretation in the context of combinatorial optimization. We regard the $m \times m$-matrix (x_{ij}) as an instance for the following *assignment problem.* A company has m workers and m jobs, and it seeks to assign the jobs to the workers. The matrix entry x_{ij} is the cost if job j is performed by worker i. An assignment of jobs to workers is a permutation π of $\{1, 2, \ldots, m\}$ and the total cost of that assignment equals

$$x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{m\pi(m)} \quad = \quad x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{m\pi(m)}.$$

The company wishes to find the minimum over these $m!$ expressions. This is precisely the problem of evaluating the tropical determinant. A well-known method, called the *Hungarian Algorithm*, performs this task in polynomial time. So, there is no need to inspect all $m!$ tropical summands in order to compute the tropical determinant of the cost matrix (x_{ij}). We conclude:

Remark 2.14. *The tropical hypersurface* $\mathcal{T}(\text{tropdet})$ *is the set of all instances* (x_{ij}) *of the assignment problem for which the optimal solution is not unique.*