A COMBINATORIAL INTRODUCTION TO TROPICAL GEOMETRY

BERND STURMFELS

These are the notes for a short course held September 12-16, 2007, at the Technical University Berlin. The course consists of ten lectures. Originally, I had hoped to write notes for each lecture but that turned out to be too optimistic, and is left for the future. Missing notes are replaced with pointers to the literature, and a fairly substantial bioliography has been compiled.

1. Tropical Arithmetic and Polynomials

Our basic algebraic structure for the geometry in this course is the *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. As a set this is just the real numbers $\mathbb{R}$, together with an extra element ∞ which represents infinity. However, we redefine the arithmetic operations of addition and multiplication of real numbers as follows:

$$x \oplus y \;:=\; \min(x, y) \qquad \text{and} \qquad x \odot y \;:=\; x + y.$$

In words, the *tropical sum* of two numbers is their minimum, and the *tropical product* of two numbers is their sum. Here are some examples of how to do arithmetic in this strange number system. The tropical sum of 4 and 5 is 4. The tropical product of 4 and 5 equals 9. We write this as follows:

$$4 \oplus 5 \;=\; 4 \qquad \text{and} \qquad 4 \odot 5 \;=\; 9.$$

Many of the familar axioms of arithmetic remain valid in tropical mathematics. For instance, both addition and multiplication are *commutative*:

$$x \oplus y \;=\; y \oplus x \qquad \text{and} \qquad x \odot y \;=\; y \odot x.$$

The *distributive law* holds for tropical addition and tropical multiplication:

$$x \odot (y \oplus z) \quad = \quad x \odot y \;\oplus\; x \odot z.$$

Both arithmetic operations have a neutral element. Infinity is the *neutral element* for addition and zero is the *neutral element* for multiplication:

$$x \oplus \infty \;=\; x \qquad \text{and} \qquad x \odot 0 \;=\; 0.$$

Elementary school students tend to prefer tropical arithmetic because the multiplication table is easier to memorize, and even long division becomes easy.

Here are the tropical *addition table* and the tropical *multiplication table*:

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

$\odot$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

Tropical arithmetic can be tricky because there is no subtraction. There is no x which we can call "9 minus 5" because the equation $4 \oplus x = 9$ has no solution x at all. While tropical division does make sense, in these lectures we shall content ourselves with using addition $\oplus$ and multiplication $\odot$ only.

Example 1.1 (Pascal's Triangle)**.** The tropical Pascal's triangle looks like

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 \\
\cdots & \cdots & & \cdots & & \cdots & & \cdots & \cdots
\end{array}
$$

Indeed, the rows of this triangle are the coefficients appearing in the *Binomial Theorem*. For instance, the third row in the triangle represents the identity

$$
\begin{aligned}
(x \oplus y)^3 \quad &= \quad (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\
&= \quad 0 \odot x^3 \ \oplus \ 0 \odot x^2 y \ \oplus \ 0 \odot xy^2 \ \oplus \ 0 \odot y^3.
\end{aligned}
$$

Of course, the zero coefficients can be dropped in this identity:

$$(x \oplus y)^3 \quad = \quad x^3 \ \oplus \ x^2 y \ \oplus \ xy^2 \ \oplus \ y^3.$$

Moreover, the *Freshman's Dream* holds for all powers in tropical arithmetic:

$$(x \oplus y)^3 \quad = \quad x^3 \ \oplus \ y^3.$$

The validity of the three displayed identities is easily verified by noting that the following equations hold in classical arithmetic for all $x, y \in \mathbb{R}$:

$$3 \cdot \min\{x, y\} \quad = \quad \min\{3x, 2x + y, x + 2y, 3y\} \quad = \quad \min\{3x, 3y\}.$$

We must always remember that "$0''$ is the multiplicatively neutral element. $\square$

Let $x_1, x_2, \ldots, x_n$ be variables which represent elements in the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. A *monomial* is any product of these variables, where repetition is allowed. By commutativity, we can sort the product and write monomials in the usual notation, with the variables raised to exponents:

$$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 \quad = \quad x_1^2 x_2^3 x_3^2 x_4.$$

A monomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$. When evaluating this function in classical arithmetic, what we get is a linear function with integer coefficients:

$$x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 \quad = \quad 2x_1 + 3x_2 + 2x_3 + x_4.$$

Throughout this course, we will allow negative integer exponents for our monomials. So, for example, the tropical monomial $x_1^{-17} x_2^{11} x_3^{-8}$ represents the linear function $-17x_1 + 11x_2 - 8x_3$ in classical arithmetic. With this convention, tropical monomials in n variables are precisely the linear functions $\mathbb{Z}^n \to \mathbb{Z}$.

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1, \ldots, x_n) \quad = \quad a \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \;\oplus\; b \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \;\oplus\; \cdots$$

Here the coefficients $a, b, \ldots$ are real numbers and the exponents $i_1, j_1, \ldots$ are integers. Every tropical polynomial represents a function $\mathbb{R}^n \to \mathbb{R}$. When evaluating this function in classical arithmetic, what we get is the minimum of a finite collection of linear functions (with constant terms), namely,

$$p(x_1, \ldots, x_n) \quad = \quad \min\bigl(a + i_1 x_1 + \cdots + i_n x_n \,,\, b + j_1 x_1 + \cdots + j_n x_n \,,\, \ldots\bigr)$$

This function $p : \mathbb{R}^n \to \mathbb{R}$ has the following three important properties:

(a) p is continuous,
(b) p is piecewise-linear, where the number of pieces is finite, and
(c) p is concave, i.e., $p(\frac{x+y}{2}) \geq \frac{1}{2}(p(x) + p(y))$ for all $x, y \in \mathbb{R}^n$.

Proposition 1.2. *Every function $\mathbb{R}^n \to \mathbb{R}$ which satisfies the three properties (a), (b) and (c) has a representation as the minimum of a finite set of linear functions. Thus, the tropical polynomials in n variables $x_1, \ldots, x_n$ represent the class of piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.*

Proof. Consider a function $p : \mathbb{R}^n \to \mathbb{R}$ which satisfies (a), (b) and (c). Let $\ell_1(x), \ldots, \ell_s(x)$ be the linear functions which appear in the representation of $p(x)$. By continuity, we may assume that for every $i \in \{1, \ldots, s\}$ there is an open subset U_i of $\mathbb{R}^n$ such that $p(x) = \ell_i(x)$ for all $x \in U_i$. Fix any point $z \in \mathbb{R}^n$ and let $k \in \{1, \ldots, s\}$ be an index such that $f(z) = \ell_k(z)$. Consider any other index $i \in \{1, \ldots, s\}$. Then we can find points u and v in U_j such that v lies on the segment between u and z, i.e., there exists $\lambda \in (0, 1)$ such that $v = \lambda z + (1 - \lambda)u$ and hence $z = \frac{1}{\lambda}(v + (\lambda - 1)u)$. Concavity of p implies

$$p(v) \;\geq\; \lambda p(z) + (1 - \lambda) p(u),$$

and hence, using the linearity of ℓ_i,

$$p(z) \;\leq\; \frac{1}{\lambda}\bigl(p(v) + (\lambda - 1)p(u)\bigr) \;=\; \frac{1}{\lambda}\bigl(\ell_i(v) + (\lambda - 1)\ell_i(u)\bigr) \;=\; \ell_i(z).$$

Combining this inequality with the equation $p(z) = \ell_k(z)$, we find that

$$p(z) \;=\; \min\bigl\{\ell_1(z), \ell_2(z), \ldots, \ell_s(z)\bigr\} \quad \text{for all } z \in \mathbb{R}^n.$$

Hence the function $p : \mathbb{R}^n \to \mathbb{R}$ is represented by a tropical polynomial. $\square$

The representation of concave functions by tropical polynomials is not unique.

Example 1.3. Consider the general quadratic polynomial in one variable x:

$$p(x) \quad = \quad a \odot x^2 \ \oplus \ b \odot x \ \oplus \ c, \tag{1}$$

Here a, b, c are arbitrary real numbers. If a and c are fixed then all polynomials $p(x)$ with $b \geq \frac{1}{2}(a+c)$ represent the same function as the complete square

$$\tilde{p}(x) \quad = \quad a \odot \big(x \oplus \frac{c-a}{2}\big) \odot \big(x \oplus \frac{c-a}{2}\big).$$

On the other hand, if $b < \frac{1}{2}(a+c)$ then the quadratic polynomial factors as

$$p(x) \quad = \quad a \odot \big(x \oplus (b-a)\big) \odot \big(x \oplus (c-b)\big).$$

We define the *root* of a linear polynomial $x \oplus C$ to be its constant term C. This leads to the *tropical quadratic formula*: The quadratic polynomial $p(x)$ has two roots $b-a$ and $c-b$ if the *tropical discriminant* $a-2b+c$ is positive and it has only one root $\frac{1}{2}(c-a)$ if the tropical discriminant is negative. □

The following theorem concerns polynomials with non-negative exponents.

Theorem 1.4. (Tropical Fundamental Theorem of Algebra) *Every tropical polynomial function $p(x)$in one variable x can be written uniquely as a product*

$$p(x) \quad = \quad u \odot (x \oplus v_1) \odot (x \oplus v_2) \odot \cdots \odot (x \oplus v_m). \tag{2}$$

The rational numbers $v_1, v_2, \ldots, v_m$ are the roots *of $\tilde{p}(x)$ but u is a real number.*

Proof. The tropical polynomial function can be written uniquely as

$$p(x) \quad = \quad d_1 \odot x^{\odot c_1} \ \oplus \ d_2 \odot x^{\odot c_2} \ \oplus \cdots \oplus d_r \odot x^{\odot c_r},$$

where $c_1 > c_2 > \cdots > c_r$ are non-negative integers, $d_1, \ldots, d_r$ are arbitrary real numbers, and none of the terms $d_i \odot x^{\odot c_i}$ are redundant in representing the function $p : \mathbb{R} \to RR$. Then the tropical polynomial $p(x)$ equals

$$d_1 \odot \big(x^{c_1 - c_2} \oplus (d_2 - d_1)\big) \odot \big(x^{c_2 - c_3} \oplus (d_3 - d_2)\big) \odot \cdots \odot \big(x^{c_{r-1} - c_r} \oplus (d_r - d_{r-1})\big) \odot x^{c_r}.$$

Each binomial factor can be factored further into a product of linear forms:

$$d_1 \odot \big(x \oplus \frac{d_2 - d_1}{c_1 - c_2}\big)^{\odot c_1 - c_2} \odot \big(x \oplus \frac{d_2 - d_3}{c_2 - c_3}\big)^{\odot c_2 - c_3} \odot \cdots \odot \big(x \oplus \frac{d_r - d_{r-1}}{c_{r-1} - c_r}\big)^{\odot c_{r-1} - c_r} \odot x^{c_r}$$

This is the desired representation of $p(x)$ as a product of linear terms.

To see that the factorization (2) is unique, we argue as follows. We can recover the constants u and m because $p(x) = mx + u$ for $x \ll 0$. The roots v_i are the places where the graph of $p(x)$ has a breakpoint, and the multiplicity of a root is the difference of the slope to the left minus the slope to the right. □

Unique factorization of polynomials no longer holds in two or more variables, because the decomposition of polytopes as Minkowski sums is not unique.

Example 1.5. The following example shows that the factorization of multivariate tropical polynomials into irreducible tropical polynomials is not unique:

$$\begin{aligned} &(0 \odot x \oplus 0) \;\odot\; (0 \odot y \oplus 0) \;\odot\; (0 \odot x \odot y \oplus 0) \\ = \;\; &(0 \odot x \odot y \oplus 0 \odot x \oplus 0) \odot (0 \odot x \odot y \oplus 0 \odot y \oplus 0). \end{aligned}$$

All five factors are irreducible, in the sense that they do not admit any non-trivial factorization. Note that all coefficients of the polynomials are zero. □

A tropical polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ is given as the minimum of a finite set of linear functions. We define the *hypersurface* $\mathcal{T}(p)$ to be the set of all points $x \in \mathbb{R}^n$ at which this minimum is attained at least twice. Equivalently, a point $x \in \mathbb{R}^n$ lies in $\mathcal{T}(p)$ if and only if p is not linear at x. For example, if $n = 1$ and p is the polynomial in (2) then $\mathcal{T}(p) = \{ v_1, v_2, \ldots, v_m \}$. Thus the hypersurface $\mathcal{T}(p)$ is the set of roots of the polynomial $p(x)$.

In this section we consider the case of a polynomial in two variables:

$$p(x,y) \quad = \quad \bigoplus_{(i,j)} c_{ij} \odot x^i \odot y^j .$$

Proposition 1.6. *The tropical curve $\mathcal{T}(p)$ is a finite graph which is embedded in the plane $\mathbb{R}^2$. It has both bounded and unbounded edges, all edge directions are rational, and this graph satisfies a* zero tension condition *around each node.*

The zero tension condition is the following geometric condition. Consider any node (x,y) of the graph and suppose it is the origin, i.e., $(x,y) = (0,0)$. Then the edges adjacent to this node lie on lines with rational slopes. On each such ray emanating from the origin consider the first non-zero lattice vector. *Zero tension* at (x,y) means that the sum of these vectors is zero.

Our first example is a *line* in the plane. It is defined by a polynomial:

$$p(x,y) \quad = \quad a \odot x \;\oplus\; b \odot y \;\oplus\; c \qquad\quad \text{where } a,b,c \in \mathbb{R}.$$

The curve $\mathcal{T}(p)$ consists of all points (x,y) where the function

$$p \;:\; \mathbb{R}^2 \to \mathbb{R}\,, \qquad (x,y) \;\mapsto\; \min\big(a+x,\, b+y,\, c \big)$$

is not linear. It consists of three half-rays emanating from the point $(x,y) = (c-a, c-b)$ into northern, eastern and southwestern direction.

Here is a general method for drawing a tropical curve $\mathcal{T}(p)$ in the plane. Consider any term $\gamma \odot x^i \odot y^j$ appearing in the polynomial p. We represent this term by the point (γ, i, j) in $\mathbb{R}^3$, and we compute the convex hull of these points in $\mathbb{R}^3$. Now project the lower envelope of that convex hull into the plane under the map $\mathbb{R}^3 \to \mathbb{R}^2$, $(\gamma, i, j) \mapsto (i,j)$. The image is a planar convex polygon together with a distinguished subdivision Δ into smaller polygons. The tropical curve $\mathcal{T}(p)$ is the dual graph to this subdivision.

As an example we consider the general quadratic polynomial

$$p(x,y) \quad = \quad a \odot x^2 \;\oplus\; b \odot xy \;\oplus\; c \odot y^2 \;\oplus\; d \odot x \;\oplus\; e \odot y \;\oplus\; f.$$

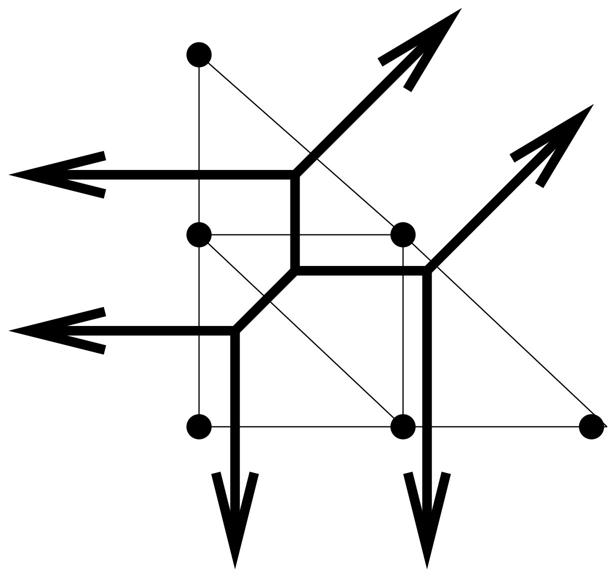

FIGURE 1. The subdivision Δ and the tropical curve

Then Δ is a subdivision of the triangle with vertices $(0,0)$, $(0,2)$ and $(2,0)$. The lattice points $(0,1)$, $(1,0)$, $(1,1)$ are allowed to be used as vertices in these subdivisions. Assuming that $a, b, c, d, e, f \in \mathbb{R}$ are general solutions of

$$2b \leq a + c\,,\;\; 2d \leq a + f\,,\;\; 2e \leq c + f,$$

the subdivision Δ consists of four triangles, three interior edges and six boundary edges. The curve $\mathcal{T}(p)$ has four vertices, three bounded edges and six half-rays (two northern, two eastern and two southwestern). In Figure 2, $\mathcal{T}(p)$ is shown in bold, and the subdivision Δ is shown in thin lines.

Proposition 1.7. *Tropical curves intersect and interpolate like algebraic curves:*

(1) *Two general lines meet in one point, a line and a quadric meet in two points, two quadrics meet in four points, etc....*
(2) *Two general points lie on a unique line, five general points lie on a unique quadric, etc...*

For a general discussion of *Bézout's Theorem* in tropical algebraic geometry, and for many pictures illustrating Fact 4, we refer to the article [65].