BMS: Tropical Geometry
Exercises for Wednesday, September 12

(1) Consider the following 2×2-matrix over the rational function field $\mathbf{Q}(t)$:

$$A \quad = \quad \begin{pmatrix} t^2 & t^2 \\ -t^2 & t+t^2+t^3 \end{pmatrix}$$

Compute its determinant, characteristic polynomial, eigenvalues and eigenvectors. The *tropicalization* of the matrix A is the tropical matrix

$$B \quad = \quad \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Define and compute the determinant, characteristic polynomial, eigenvalues and eigenvectors of the tropical matrix B. Compare the results.

(2) Given five general points in $\mathbf{R}^2$, there exists a unique quadratic curve passing through these points. Determine and draw the quadratic curve passing through the points $(0,5)$, $(1,0)$, $(4,2)$, $(7,3)$ and $(9,4)$

(3) A tropical cubic curve in $\mathbf{R}^2$ is *smooth* if it has precisely 9 nodes. Prove that every smooth cubic curve has a unique bounded region, and that this region can has either three, four, five, six, seven, eight, or nine edges. Draw examples for all seven cases.

(4) Draw a quadratic surface in tropical 3-space. Describe the result of intersecting two such tropical quadratic surfaces ?

(5) We have drawn the tropical hypersurface of the 3×3-determinant as a two-dimensional polyhedral complex consisting of triangles and quadrangles. Pick one point each from the relative interior of a triangle and a quadrangle, and find two singular matrices in $\mathbf{C}\{\{t\}\}^{3\times 3}$ that map to your chosen tropical matrices under the valuation map.

(6) The *discriminant* of a quadratic form in three unknowns equals

$$\det \begin{pmatrix} 2x_1 & x_2 & x_3 \\ x_2 & 2x_4 & x_5 \\ x_3 & x_3 & 2x_6 \end{pmatrix}$$

Draw a picture of the tropical hypersurface defined by this polynomial.