

Math 55, **First Midterm Exam**
SOLUTIONS

(1) The statement $Q(x, y)$ is equivalent to the statement “ $y = 0$ ”. The truth values of this statement are as follows:

- (a) **False** because it is false for $y = 1$.
- (b) **True** because we can choose $y = 0$ regardless of the value of x .
- (c) **False** because it is false for $y = 1$.
- (d) **False** for the same reason as in (c) since “ $y = 0$ ” does not depend on x .
- (e) **True** because we can choose $y = 0$ regardless of the value of x .

(2) From the identities $(-38) \cdot 2 + 1 \cdot 77 = 1$, $(-3)7 + 1 \cdot 22 = 1$ and $(-5) \cdot 11 + 4 \cdot 14 = 1$, we see that $77 \equiv 1 \pmod{2}$, $22 \equiv 1 \pmod{7}$, and 4 is the inverse to 14 modulo 11. By the method used in the proof of the Chinese Remainder Theorem, $n = 0 \cdot 77 + 3 \cdot 22 + 6 \cdot 4 \cdot 14 = \mathbf{402}$ is one solution, and all solutions have the form $n = 402 + 154k$ for some integer k . Taking $k = -2$ gives the smallest positive solution $n = \mathbf{94}$.

(3) In each case we explicitly determine the set to find its cardinality:

- $|A \cap (A - A)| = |A \cap \emptyset| = |\emptyset| = \mathbf{0}$.
- $|A \cup (A \cap A)| = |A \cup A| = |A| = \mathbf{3}$.
- A is disjoint from its power set $P(A)$. Since $|P(A)| = 2^{|A|} = 8$, we get $|A \cup P(A)| = \mathbf{11}$.
- $|P(A) \times P(A)| = |P(A)| \cdot |P(A)| = 8 \cdot 8 = \mathbf{64}$.
- $|P(A \times A)| = 2^{|A \times A|} = 2^{|A| \cdot |A|} = 2^{3 \cdot 3} = 2^9 = \mathbf{512}$.

(4) The sum in (a) can be broken into the sum of $\sum_{i=1}^3 \sum_{j=1}^4 i = 4 \cdot \sum_{i=1}^3 i = 4 \cdot (1+2+3) = 24$ and $\sum_{i=1}^3 \sum_{j=1}^4 2j = 3 \cdot \sum_{j=1}^4 2j = 3 \cdot (2+4+6+8) = 60$, so the answer is **84**.

By the distributive law, the sum in (b) can be factored and evaluated as follows:

$$\left(\sum_{i=1}^3 i\right) \cdot \left(\sum_{j=1}^4 j^2\right) = (1+2+3) \cdot (1+4+9+16) = 6 \cdot 30 = \mathbf{180}.$$

(5) To prove these statements, we let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$. Then $\lfloor x \rfloor = n$. If $\epsilon = 0$ then x is an integer and $-x$ is an integer, and the two assertions are true because we can remove the floor and ceiling symbols when applied to integers. Thus we can assume that $\epsilon > 0$. Under this assumption we have $\lfloor x \rfloor = n$. The negative of the given real number x equals $-x = -n - \epsilon = -n - 1 + (1 - \epsilon)$, where $0 < (1 - \epsilon) < 1$. This implies $\lfloor -x \rfloor = -n - 1 = -(n + 1) = -\lceil x \rceil$, which proves the first statement. Likewise, $\lceil -x \rceil = -n - 1 + 1 = -n = -\lfloor x \rfloor$, which proves the second statement.