Math 55 - Discrete Mathematics

UC Berkeley - Spring 2010
Solutions to Homework #2 (due February 3)

§2.1, #5

a. yes
b. no
c. yes
d. no
e. no
f. no

§2.1, #8

a. True
b. True
c. False
d. True
e. True
f. True

§2.1, #21 Recall that $|\mathcal{P}(A)| = 2^{|A|}$. Then:

a. 8
b. 16
c. 2

§2.1, #28

a. $A \times B \times C = \{(a,x,0),(a,x,1),(a,y,0),(a,y,1),(b,x,0),(b,x,1),(b,y,0),(b,y,1),$ $(c,x,0),(c,x,1),(c,y,0),(c,y,1)\}$,
b. $C \times B \times A = \{(0,x,a),(1,x,a),(0,y,a),(1,y,a),(0,x,b),(1,x,b),(0,y,b),(1,y,b),$ $(0,x,c),(1,x,c),(0,y,c),(1,y,c)\}$,
c. $C \times A \times B = \{(0,a,x),(1,a,x),(0,a,y),(1,a,y),(0,b,x),(1,b,x),(0,b,y),(1,b,y),$ $(0,c,x),(1,c,x),(0,c,y),(1,c,y)\}$,
d. $B \times B \times B = \{(x,x,x),(x,y,x),(y,x,x),(y,y,x),(x,x,y),(x,y,y),(y,x,y),(y,y,y)\}$.

§2.1, #38

a. Assume $S \in S$. By definition of the set, this implies that $S \notin S$, a contradiction.
b. Assume $S \notin S$. Followint the definition of the set, this implies that $S \in S$, since it is not an element of the set S. Again, we obtain a contradiction.

§2.2, #4

a. $A \cup B = B = \{a,b,c,d,e,f,g,h\}$
b. $A \cap B = A = \{a,b,c,d,e\}$
c. $A \setminus B = \emptyset$
d. $B \setminus A = \{f,g,h\}$

§2.2, #14 Recall that for any two sets X, Y we have $X = (X \setminus Y) \cup (X \cap Y)$. Thus,

$$A = \{1,5,7,8\} \cup \{3,6,9\} = \{1,3,5,6,7,8,9\},$$

$$B = \{2,10\} \cup \{3,6,9\} = \{2,3,6,9,10\}.$$

§2.2, #24 We want to show that $(A \setminus B) \setminus C) = (A \setminus C) \setminus (B \setminus C)$. By definition of the difference of two sets and conmutativity of the intersection we have

$$(A \setminus B) \setminus C) = (A \cap \overline{B}) \cap \overline{C} = (A \cap \overline{C}) \cap \overline{B}.$$

But,

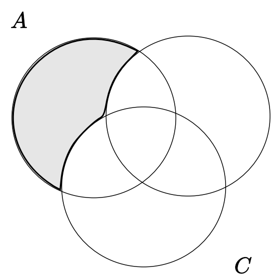
$$(A \cap \overline{C}) \cap \overline{B} = A \cap \overline{C} \cap \overline{B} \cap \overline{C} = (A \cap \overline{C}) \cap (\overline{B} \cap \overline{C}) = (A \setminus C) \cap \overline{(B \cup C)} = (A \setminus C) \setminus (B \cup C),$$

by the first De Morgan law. Now, since $B \cup C = (B \setminus C) \cup (B \cap C)$ we get

$$\begin{aligned}(A \setminus B) \setminus C) &= (A \setminus C) \setminus (B \cup C) = (A \setminus C) \setminus ((B \setminus C) \cup (B \cap C)) \\ &= \big((A \setminus C) \setminus (B \setminus C)\big) \cap \big((A \setminus C) \setminus (B \cap C)\big) \\ &= \big((A \setminus C) \setminus (B \setminus C)\big) \cap \quad (A \setminus C) \\ &= (A \setminus C) \setminus (B \setminus C),\end{aligned}$$

as we wanted to show.

This can also be checked because both sides of the identity have the same Venn diagram:



§2.2, #28
a. See Figure 1.
b. See Figure 2.
c. See Figure 3.

§2.2, #42 Yes. The statement will follow directly from the associativity of $\oplus$:

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C). \tag{1}$$

We know prove (1). By definition, and following De Morgan's rules: $A \oplus (B \oplus C) = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \cup (A \cap B \cap C)$.

The expression in the right-hand side is symmetric with respect to A, B, C and so it also equals $(A \oplus B) \oplus C$. Another way of seeing this is by Venn diagrams, since $(A \oplus B) \oplus C$ and $A \oplus (B \oplus C)$ both have the same Venn diagram (see Figure 4).

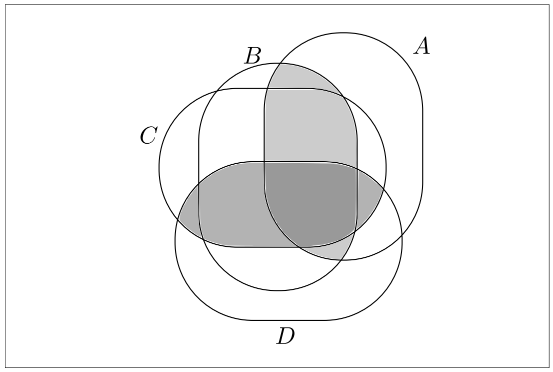


Figure 1: Venn diagram of the set $(A \cap B) \cup (C \cap D)$.

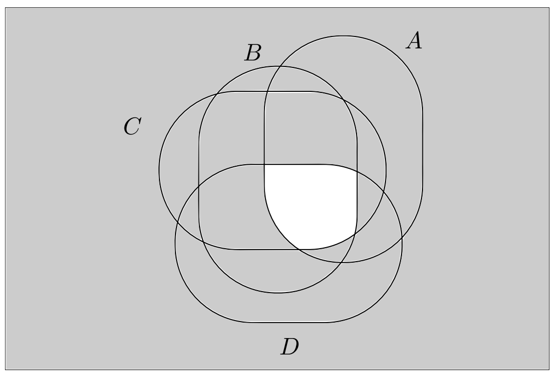


Figure 2: Venn diagram of the set $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$.

We now show that $\oplus$ is symmetric:

$$A \oplus B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \oplus A,$$

since the union is commutative.

Using the associativity and commutativity of $\oplus$, we have

$$\begin{aligned}(A \oplus B) \oplus (C \oplus D) &= A \oplus (B \oplus (C \oplus D)) = A \oplus (B \oplus (D \oplus C)) \\ &= A \oplus ((B \oplus D) \oplus C)) = A \oplus ((D \oplus B) \oplus C)) \\ &= A \oplus (D \oplus (B \oplus C)) \\ &= (A \oplus D) \oplus (B \oplus C),\end{aligned}$$

as we wanted to show.

§2.3, #7 We call A the domain of the function and B the range of the function.

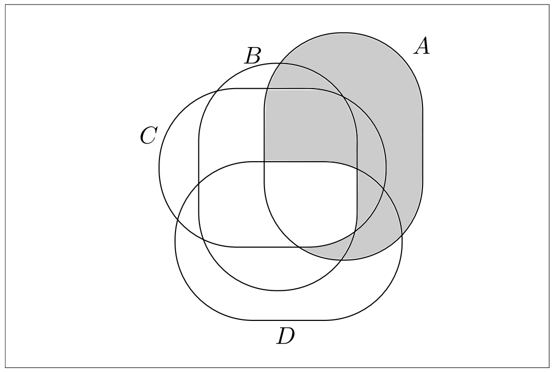


Figure 3: Ven diagram of the set $A \setminus (B \cap C \cap D)$.

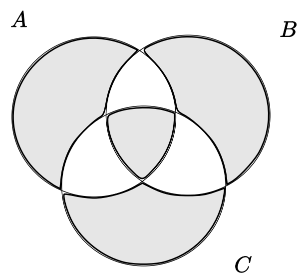


Figure 4: The Venn diagram of $(A \oplus B) \oplus C$ and $A \oplus (B \oplus C)$.

a. $A = \mathbb{N} \times \mathbb{N}$, $B = \mathbb{N}$.

b. $A = \mathbb{N}$, $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c. $A = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$, $B = \mathbb{N} \cup \{0\}$.

d. $A = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$, $B = \mathbb{N}$.

§2.3, #9

a. 1

b. 0

c. 0

d. -1

e. 3

f. -1

g. $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$

h. $\lfloor \frac{1}{2} \cdot 2 \rfloor = \lfloor 1 \rfloor = 1$

§2.3, #12

a. It is one-to-one.

b. Not one-to-one: $f(-1) = 2f(1)$.

c. It is one-to-one.

d. Not one-to-one: $f(1) = 1 = f(2)$.

§2.3, #16 Many examples are possible for each one of these.

a. $f \colon \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$. It is one-to-one but not onto ($1 \notin$ image of f).

b. $f \colon \mathbb{N} \to \mathbb{N}$ defined as

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 2, \\ n - 1 & \text{otherwise.} \end{cases}$$

It is onto but not one-to-one.

c. $f\colon \mathbb{N} \to \mathbb{N}$ defined as

$$f(n) = \begin{cases} 2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ n & \text{otherwise.} \end{cases}$$

f is a bijection, so it is one-to-one and onto.

d. $f\colon \mathbb{N} \to \mathbb{N}$, $f(n) = 1$ for all n. f is not onto nor one-to-one.

§2.3, #27

a. $f(S) = \{0, 1, 3\}$

b. $f(S) = \{0, 1, 3, 5, 8\}$

c. $f(S) = \{0, 8, 16, 40\}$

d. $f(S) = \{1, 12, 33, 98\}$

§2.3, #41 We want to show that $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$. We follow by definition:

$$f^{-1}(\overline{S}) = \{x \in A : f(a) \in \overline{S}\} = \{x \in A : f(a) \notin S\} = \overline{f^{-1}(S)}.$$

§2.4, #9

a. We construct the sequence as follows: at each instance we put a string of size n of all ones follows by a string of size n of all zeros. So $f(n) = \{1\}^n, \{0\}^n$. So the three terms following are: 1, 1, 1.

b. $a_n = \begin{cases} 2k & \text{if } n = 3k \\ 2k+1 & \text{if } n = 3k+1 \\ 2k+2 & \text{if } n = 3k+2. \end{cases}$ The next three elements are: 9, 10, 10.

c. $a_n = \begin{cases} 2^{n-1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$. The next three elements are: 32, 0, 64.

d. $a_n = 3 \cdot 2^{n-1}$. The next three elements are: 384, 768, 1536.

e. $a_n = 22 - 7 \cdot n$ The next three elements are: -34, -41, -48.

f. $a_n = 2 + n + \frac{n(n-1)}{2}$ The next three elements are: 57, 68, 80.

g. $a_n = 2n^3$. The next three elements are: 1024, 1458, 2000.

h. $a_n = n! + 1$. The next three elements are: 362881,3628801, 39916801.

§2.4, #16

a. $\displaystyle\sum_{j=0}^{8} (1 + (-1)^j) = 2 + 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2 = 2 \cdot 5 = 10.$

b. $\displaystyle\sum_{j=0}^{8} (3^j - 2^j) = \sum_{j=0}^{8} 3^j - \sum_{j=0}^{8} 2^j = \frac{3^9 - 1}{3 - 1} - \frac{2^9 - 1}{2 - 1} = 9841 - 511 = 9330.$

c. $\displaystyle\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j) = 2 \sum_{j=0}^{8} 3^j + 3 \sum_{j=0}^{8} 2^j) = 2\, \frac{3^9 - 1}{3 - 1} + 3\, \frac{2^9 - 1}{2 - 1} = 21215.$

d. $\sum_{j=0}^{8}(2^{j+1} - 2^j) = \sum_{j=1}^{9} 2^j - \sum_{j=0}^{8} 2^j = 2^9 - 1 = 511.$

§2.4, #18(*)

a.

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j) &= \sum_{i=1}^{3}(i-1)+(i-2)\\ &= (1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2)\\ &= 3.\end{aligned}$$

b.

$$\begin{aligned}\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j) &= \sum_{i=0}^{3}\sum_{j=0}^{2}3i + \sum_{i=0}^{3}\sum_{j=0}^{2}2j\\ &= 3\sum_{i=0}^{3}3i + 4\sum_{j=0}^{2}2j\\ &= 9\cdot 6 + 8\cdot 3 = 78.\end{aligned}$$

c.

$$\sum_{i=1}^{3}\sum_{j=0}^{2} j = 3\sum_{j=0}^{2} j = 3\cdot 3 = 9.$$

d.

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3} i^2 j^3 &= \Big(\sum_{i=0}^{2} i^2\Big)\cdot\Big(\sum_{j=0}^{3} j^3\Big)\\ &= (0+1+4)\cdot(0+1+8+27) = 5\cdot 36 = 180.\end{aligned}$$

§2.4, #31

a. Countable. A bijection is given by $f\colon \mathbb{N} \to \mathbb{Z}_{<0}$, $f(n) = -n$.

b. Countable. A bijection is given by $f\colon \mathbb{N} \to \{\text{even integers}\}$

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd.} \end{cases}$$

c. Uncountable.

d. Countable. A bijection is given by $f\colon \mathbb{N} \to \{\text{integers multiples of } 7\}$,

$$f(n) = \begin{cases} 7k & \text{if } n = 2k \\ -7k & \text{if } n = 2k+1. \end{cases}$$