Math 239 / Stat 260: Algebraic Statistics
Final Report: An Implicitization Challenge

Maria Angelica Cueto
macueto@math.berkeley.edu

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1 Introduction

In the present work, we discuss the "Implicitization Challenge" proposed as Problem 7.7 in [4], Chapter VI. The problem goes as follows. We are given an undirected graph with 4 observed nodes and two hidden nodes.

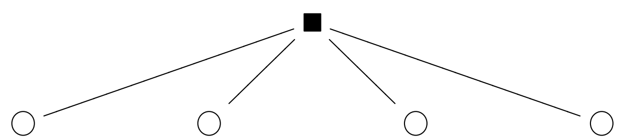

Figure 1: Graphical model for the complete independence model on four nodes given the hidden variable R (root).

Each node represents a binary random variable. Our goal is to describe the algebraic variety corresponding to this model.

Let us discuss the geometry behind the model, a very beautiful one indeed. As a warm-up, consider a graph with four disconnected nodes. We know ([4], Example 1.2.6) that this simplicial complex corresponds to the complete independence model for four identically distributed random variables. Thus, Figure 1 represents a hidden model, where we marginalize over all possible values at the hidden root of the 4-claw tree. We see immediately that this corresponds to the secant variety of the Segre variety $\operatorname{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. From the i.d. assumption we see that this model comes equiped with a natural $\mathbb{S}_4$-action, namely the one obtained by permuting all four observed nodes.

It is worth mentioning that the case of four observed nodes is the first case where the secant variety of the Segre variety $\operatorname{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ doesn't fill in the whole space. In cases $n = 2, 3$ we have $\operatorname{Sec}(\operatorname{Seg}(\mathbb{P}^1 \times \mathbb{P}^1)) = \mathbb{P}^3$ and $\operatorname{Sec}(\operatorname{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = \mathbb{P}^7$. This follows by dimension arguments, since both secant varieties are nondefective. This was studied in full detail in [1], Example 2.1.

The parametrization of the hidden model consists of a toric model corresponding to the fully observed undirected graphical model, composed with the marginalization map over the two hidden nodes. In this case, the marginalization map is given by a map $\mathbb{P}^{63} \to \mathbb{P}^{15}$. According to [4], the Zariski closure of the image is a hypersurface. The goal consists of computing the degree of the equation, or even better, its Newton polytope. Interpolation techniques will allow us to compute the corresponding irreducible homogeneous polynomial equation in 16 unknowns.

As we can imagine, the naive Gröbner basis approach won't give us the desired result in a reasonable amount of time, if it terminates at all. So we need to develop an alternative approach. Tropical Algebraic Geometry will come to the rescue.

The paper is organized as follows. In Section 2 we describe a parametrization of the model and we discuss its symmetric nature. We also describe the ideal defining the model in terms of an elimination task. We also give some insight to motivate the codimension 1 guess for the associated variety, which we'll be discussed further in Section 5.1. In Section 3 we describe the ideal of the secant variety

$\mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ of the Segre variety, following [3] and, more generally, the construction of embedded secant varieties in projective space. In Section 4 we describe the tropical framework to solve the implicitization task. A key-proposition will enable us to translate the tropical implicitization problem to a Minkowski sum of polyhedral fans. We also study the relation between tropical varieties with constant coefficient and base change to the Puiseux series context. We also discuss the notion of multiplicities of maximal cones in tropical varieties, showing that the problem is far from being well behaved. At this stage, `gfan` will come in handy. In addition, the tropical framework will allow us to show that our variety is in fact a hypersurface as predicted. We describe all our computations in Section 5. We closed the section discussing the ray-shooting method to obtain the Newton polytope of our desired equation from it's known tropicalization. In particular, this procedure will allow us to compute the degree of the equation, by computing one vertex of this polytope. We hope to get the right answer in the near future.

2 The model

In what follows, we describe the parametric representation of the model we wish to study. Recall that all our six random variables are binary, with four observed nodes and two hidden ones. Like any undirected graphical model (see [5] or [4]), the corresponding parametrization is given by

$$p : \mathbb{R}^{32} \to \mathbb{R}^{16} \quad p_{ijkl} = \sum_{s=0}^{1}\sum_{r=0}^{1} a_{si}b_{sj}c_{sk}d_{sl}e_{ri}f_{rj}g_{rk}h_{rl} \qquad \text{for all } (i,j,k,l) \in \{0,1\}^4.$$

Notice that our coordinates are homogeneous of degree 1 in the subset of variables corresponding to each edge of the graph. Therefore, there is a natural interpretation of this model in projective space. On the other hand, by the distributive law we can write down each coordinate as a product of two points in the model corresponding to the 4-claw tree. Namely,

$$p : (\mathbb{P}^1 \times \mathbb{P}^1)^8 \to \mathbb{P}^{15} \quad p_{ijkl} = (\sum_{s=0}^{1} a_{si}b_{sj}c_{sk}d_{sl}) \cdot (\sum_{r=0}^{1} e_{ri}f_{rj}g_{rk}h_{rl}) \qquad \text{for all } (i,j,k,l) \in \{0,1\}^4.$$

From this observation it is natural to define a new operation between projective varieties: the star operation, which we now describe.

Definition 1. *Let $X, Y \subset \mathbb{P}^n$ be two projective varieties. Define a new projective variety as follows:*

$$X * Y = \overline{\{(x_0y_0 : \ldots : x_ny_n) \,|\, x \in X, y \in Y, x * y \neq 0\}} \subset \mathbb{P}^n.$$

where $x * y = (x_0y_0, \ldots, x_ny_n) \in \mathbb{A}^{n+1}$.

Note that this structure is well-defined since each coordinate is bihomogeneous of degree (1,1).

From our construction, it follows immediately that $\overline{\mathrm{im}\, p} = X * X$ where $X = \mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Therefore, we are reduced to study the variety $X * X$.

Notice that the binary nature of our random variables enables us to define a natural $\mathbb{S}_2$-action by permuting the values 0 and 1 on each index in our 4-tuples. Combining this with the $\mathbb{S}_4$-action on the 4-tuples of indices, we see that our model comes equiped with a natural $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4$-action. This group action will be *extremely* helpful for our computations (see Section 5, specially §5.1). We'll describe the semidirect product structure in Section 5.

Why one would expect this variety to be of codimension 1? Dimension arguments will clarify this point. It is well-known that the expected dimension of $\mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ equals $2 \cdot 4 + 1 = 9$. If we are able to show that $X * X$ has dimension 14, then we'll be done. We postpone this discussion to Section 4, where we develope the necessary tropical techniques.

In the remaining of the section, let us describe the ideal associated to $X * Y$, where $X, Y \subset \mathbb{P}^n$. For this we will use the Segre embedding

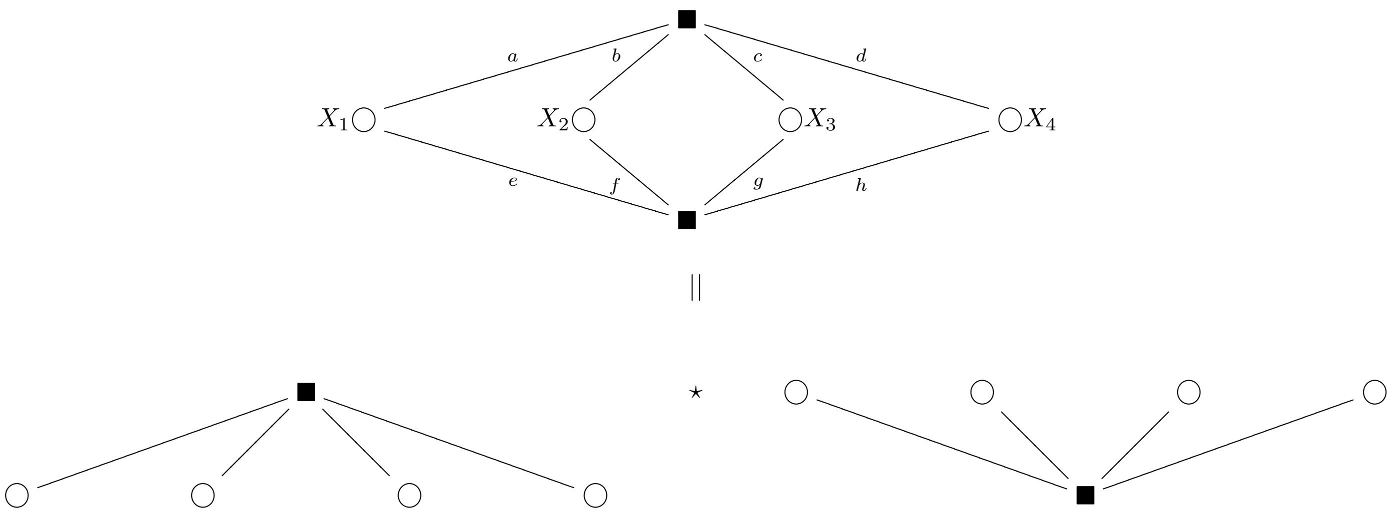

Figure 2: The model corresponds to $X * X$ as a projective variety in $\mathbb{P}^{15}$.

Proposition 1. *[Characterization of $I(X*Y)$]*
Let $J = I(X) = (f_1, \ldots, f_s) \subset \mathbb{C}[x_0, \ldots, x_n]$ and $L = I(Y) = (g_1, \ldots, g_r) \subset \mathbb{C}[y_0, \ldots, y_n]$ where all f_i and g_j are homogeneous polynomials in $n+1$ unknowns. Then

$$I(X*Y) = \underbrace{(z_{ij} - x_i y_j, f_t(\underline{x}), g_l(\underline{y}) : i, j \in [n], t = 1, \ldots, s, l = 1, \ldots, r)}_{\subset \mathbb{C}[\underline{x}, \underline{y}, \underline{z}]} \cap \mathbb{C}[z_{00}, \ldots, z_{nn}]$$

$$= \underbrace{(z_{ii} - x_i y_i, f_t(\underline{x}), g_l(\underline{y}) : i, j \in [n], t = 1, \ldots, s, l = 1, \ldots, r)}_{T \subset \mathbb{C}[\underline{x}, \underline{y}, \underline{z}]} \cap \mathbb{C}[z_{00}, \ldots, z_{nn}].$$

Proof. The proof follows by scheme arguments. Namely, $X*Y$ corresponds to $\phi \circ \iota$, where ϕ is the projection $\mathbb{P}^{n^2+2n}$ to the diagonal $\mathbb{P}^n$ and ι is the inclusion $\iota : X \times Y \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{n^2+2n}$ (the last inclusion is the Segre embedding). □

Note that our initial guess is that the variety $X*X$ is a hypersurface, so we should get only one equation after eliminating all variables except for the diagonal ones.

Remark 1. *In case X and Y are irreducible projective varieties not contained in any coordinate hyperplane, one can see that $X*Y$ is also irreducible. This follows because this variety is the closure of the image of the following map $X \times Y \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{n^2+2n} \to \mathbb{P}^n$, where the last maps are the Segre embedding followed by the projection map to the diagonal, and $X \times Y$ is irreducible (the affine cone $C(X \times Y)$ is irreducible in $\mathbb{A}^{n+1}$ by Exercise 3.15 in [15], Chapter I, so $X \times Y \subset \mathbb{P}^{n^2+2n}$ will also be irreducible by Exercise 2.10 in [15] Chapter I). This condition will be important when dealing with computations, since `gfan` assumes our input to be a prime ideal.*

3 Ideal of flattenings

In this section we describe how to compute the ideal associated to the secant variety $\mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Note that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^{15}$ is irreducible so its ideal is prime. Our task is to compute the ideal of its secant variety.

By definition, the embedded secant variety $\mathrm{Sec}V$ of a projective variety $V \subset \mathbb{P}^n$ corresponds to the closure of the set $\{sx + ty : x, y \in V, (s:t) \in \mathbb{P}^1\} \subset \mathbb{P}^n$ (i.e. the projection of the affine secant variety of the associated affine cone $C(V)$ to projective space.) How to find the ideal of this variety? Pick $I(V) = (f_1, \dots, f_r)$. Call

$$J' = (z_i - sx_i - ty_i, f_j(\underline{x}), f_j(\underline{y}) \,:\, j = 1, \dots, r, i = 0, \dots, n) \subset \mathbb{C}[\underline{x}, \underline{y}, \underline{z}, s, t] \qquad (*)$$

In addition, since we have $(s : t) \subset \mathbb{P}^1$, we need to saturate w.r.t. the ideal (s, t). To finish, we need to eliminate all variables except the z-unknowns. Thus

Proposition 2. *Let $V \subset \mathbb{P}^n$ be an irreducible projective variety, and let $I(V) = (f_1, \dots, f_r)$. With the notation (*) we have*

$$I(Sec(V)) = (J' : (s,t)^\infty) \cap \mathbb{C}[z_0, \dots, z_n].$$

Proof. The inclusion ($\supseteq$) follows by construction. To prove the other inclusion, it suffices to show that $(J' : (s,t)^\infty)$ is a radical ideal. If we prove that J' is radical, then we'll be done. Moreover, we claim that J' is a prime ideal. This follows because J is a prime ideal and

$$\mathbb{C}[\underline{x}, \underline{y}, \underline{z}, s, t]/J' = \Big(\big(\underbrace{\mathbb{C}[\underline{x}/I_{\underline{x}}(V)}_{\text{domain}}\big)[\underline{y}]/(I_{\underline{y}}(V))\Big)[s,t].$$

where $I_{\underline{x}}(V) = I(V) \subset \mathbb{C}[\underline{x}]$.

The coefficients ring is a domain since $\{\underline{x}, \underline{y}\}$ are algebraically independent and $I_{\underline{x}}(V), I_{\underline{y}}(V)$ are prime ideals. □

In the case of the variety we're studying, an alternative approach will leads us the a generating set of the associated ideal. As we know, the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{15}$ is determined by the monomial parametrization $p_{ijkl} = u_i \cdot v_j \cdot w_k \cdot x_l$ for $i, j, k, l \in \{0, 1\}$. Its associated prime ideal is generated by the 2×2-minors of all three 4×4-flattenings:

$$F_{(34|12)} := \begin{pmatrix} p_{0000} & p_{0001} & p_{0010} & p_{0011} \\ p_{0100} & p_{0101} & p_{0110} & p_{0111} \\ p_{1000} & p_{1001} & p_{1010} & p_{0111} \\ p_{1100} & p_{1101} & p_{1110} & p_{1111} \end{pmatrix}, F_{(13|24)} := \begin{pmatrix} p_{0000} & p_{0001} & p_{0100} & p_{0101} \\ p_{0010} & p_{0011} & p_{0110} & p_{0111} \\ p_{1000} & p_{1001} & p_{1100} & p_{1101} \\ p_{1010} & p_{1011} & p_{1110} & p_{1111} \end{pmatrix},$$

$$F_{(14|23)} := \begin{pmatrix} p_{0000} & p_{0010} & p_{0100} & p_{0110} \\ p_{0001} & p_{0011} & p_{0101} & p_{0111} \\ p_{1000} & p_{1010} & p_{1100} & p_{1110} \\ p_{1001} & p_{1011} & p_{1101} & p_{1111} \end{pmatrix}.$$

As explained in [3], the three matrices above reflect the bracketings of the parametrization (i.e. associativity formulas), which also correspond to the three topologies for a trivalent tree on four taxa. See Figure 3.

We know that the secant variety $X = \mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{15}$ is the nine-dimensional *irreducible* subvariety consisiting of all $2 \times 2 \times 2 \times 2$-tensors of tensor rank at most 2, which corresponds to the general Markov model for the 4-claw tree, as we mentioned in previous sections.

The prime ideal of X is generated by all the 3×3-minors of the three flattenings. Namely,

$$X = X_{(12|34)} \cap X_{(13|24)} \cap X_{(14|23)},$$

where $X_{(12|34)}$ is the ideal of 3×3 minors of the matrix $F_{(12|34)}$, and similarly for the other two. Notice that $X_{(12|34)}$ is also the ideal corresponding to the general Markov model in the tree corresponding to the quartet (12|34).

4 Tropical land

In this section we discuss some background on tropicalization of varieties over $\mathbb{C}^n$ or $\mathbb{C}\{\{t\}\}^n$ and we prove our main result concerning the interplay between the star operation (Definition 1) and the tropicalization of projective varieties. Tropicalization of algebraic varieties provide a combinatorial shadow of the original varieties and its proven to be a useful tool to study the algebraic variety. Throughout this section we'll define tropicalizations followin the min convention.

For simplicity call K any of the previous two fields. We define

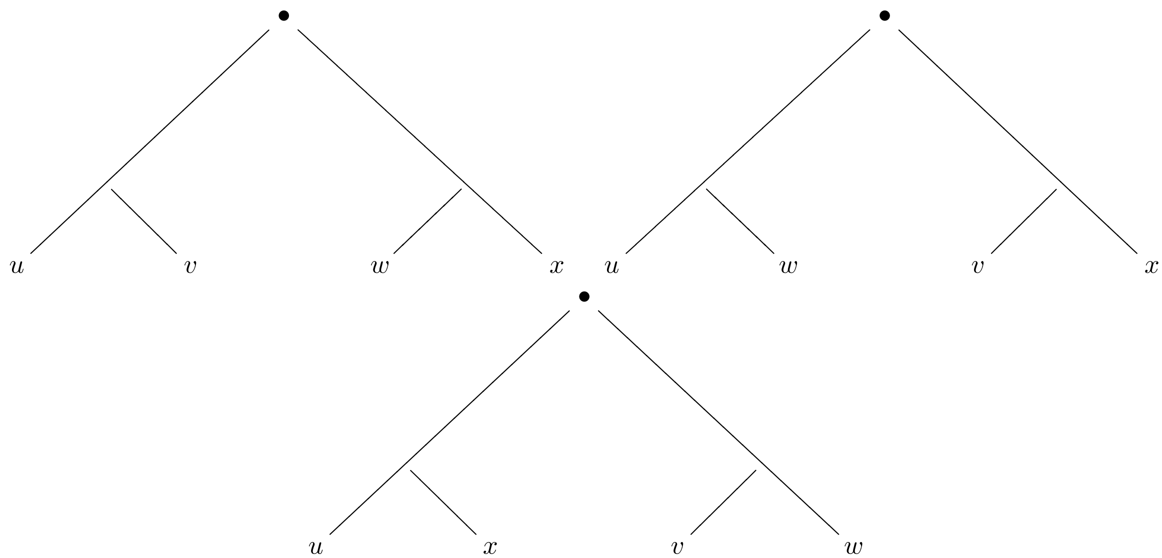

Figure 3: Three topologies for trivalent trees on four taxa corresponding to the bracketing of the Segre embedding $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

Definition 2. *Given $X \subset K^n$ an algebraic variety, with defining ideal $I = I(X) \subset K[x_1, \ldots, x_n]$, we define the* tropicalization *of X as:*

$$\mathscr{T}X = \{w \in \mathbb{R}^{n+1} \mid in_w(I) \text{ contains no monomial}\}.$$

were $in_w(I) = (in_w(f) : f \in I)$. Likewise, we can define the same tropicalization construction for ideals in the Laurent polynomial ring. All the relevant information regarding the tropical variety will be encoded in the tropicalization of $X' = X \cap (K^*)^n \subset (K^*)^n$, in case X' is non-empty.

Depending on the nature of the field K, we'll define the initial term of a polynomial w.r.t. w in two different ways. If $K = \mathbb{C}$ (known as the constant coefficient case), we define $in_w(f)$ as the sum of all nonzero terms of $f = \sum_\alpha c_\alpha x^\alpha$ where $\alpha \cdot w$ is minimum *over all α's with $c_\alpha \neq 0$. On the other hand, if $K = \mathbb{C}\{\{t\}\}$ (the non-constant coefficient case) with its standard valuation, we define $in_w(f)$ as follows. For $f = \sum_\alpha c_\alpha(t) x^\alpha$, we set $W = \min\{val(c_\alpha) + \alpha \cdot w : c_\alpha \neq 0\}$, and so*

$$in_w(f) = \overline{t^W \sum_{\alpha \in \mathbb{Z}^n} c_\alpha(t) t^{w \cdot \alpha} x^\alpha} \subset \mathbb{C}[\underline{x}^{\pm 1}].$$

From the definition in the constant coefficient case, we see that $\mathscr{T}X$ is a cone. Moreover, it is a pure polyhedral fan and it is connected in codimension one. In addition, in case X is a projective variety there will be a linear space inside the tropical variety, called the lineality space. It corresponds to the set $\{w \in \mathscr{T}X | \mathrm{in}_w I(X) = I(X)\}$. Thus, to analyze the tropical variety we'll often mod out by the lineality space, since it provided no interesting combinatorial information.

Remark 2. *In case X is an irreducible projective hypersurface in $\mathbb{P}^n$ not contained in any coordinate hyperplane, say $X = (f = 0)$ where f is irreducible, we have $\mathscr{T}X = \{w \in \mathbb{R}^{n+1} \mid in_w(f) \text{ is not a monomial}\}$ and the lineality space L is $L = \{w \in \mathbb{R}^{n+1} \mid in_w(f) = f\}$. The space L will be closely related to the support of the Newton polytope of f. Moreover, the tropical variety $\mathscr{T}X$ will allow us to recover $NP(f)$. We'll discuss this method in Section 5.2.*

In the non-constant coefficient case, the tropicalization $\mathscr{T}X$ have bounded components as well as unbounded components, and it is a pure polyhedral complex. In addition, there won't be any lineality space (unless our tropical variety comes from an ideal with constant coefficients).

It will be important for us to relate $\mathscr{T}X$ and $\mathscr{T}\tilde{X}$ where $\tilde{X}$ is the variety associated to the extended ideal $I(X)\mathbb{C}\{\{t\}\}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We do this in Lemma 1.

Another important feature from tropical varieties is the so called "balancing condition". In fact, tropical varieties will be *weighted* polyhedral fans with this extra balancing condition satisfied. Our interest in the problem we're studying will rely on the weights or *multiplicities* of all maximal cones in our tropical variety. This notion will be closely related to multiplicities of minimal primes of initial ideals. More precisely,

Definition 3. *Let $K = \mathbb{C}$ or $\mathbb{C}\{\{t\}\}$, and $X \subset (K^*)^n$ an algebraic variety. Pick $\sigma \subset \mathscr{T}X$ a maximal polyhedron and let w be a vector in the relative interior of σ. We define the* multiplicity *of σ in $\mathscr{T}X$ as follows:*

$$m_{\sigma,\mathscr{T}X} = \sum_{\substack{P \subset Spec\ S \\ P\ minimal \\ over\ in_w I}} mult(P, S/in_w I),$$

where $mult(P, S/in_w I)$ is the multiplicity of P as an associated minimal prime to $in_w I \subset S$.

In the homogeneous case, we'll define $m_{\sigma,\mathscr{T}C}$ as the some of the multiplicities of all monomial-free associated minimal primes to $in_w I \subset S$.

How to define the multiplicity of P? In general, for a noetherian ring R, a finitely generated R-module, and a minimal P- primary component of $(0) \subset M$ we wish to define mult(P, R). As we will show, it will be describe as a length of certain artinian R_P-module. We will follow the exposition in [13], Section 3.6.

For this we need to define a set $H^0_I(M)$ for an ideal I:

$$H^0_I(M) := \{m \in M \mid I^n m = 0 \text{ for } n \gg 0\}.$$

An important result concerning this set (which is a submodule of M and it only depends on the radical of I) is the following:

Fact (Proposition 3.13 in [13])**.** *Let $0 = \bigcap_i M_i$ be a primary decomposition of $(0) \subset M$, with M_i P_i-primary. Then the submodule $H^0_I(M)$ is the intersection of those M_i s.t. $P_i \notin \{P \in Ass\,M | P \supset I\}$. In particular, this intersection is independent of the primary decomposition chosen.*

In case $I = P$ is a prime ideal, then $H^0_P(M) \subset M_P$ is the unique largest submodule of finite length and we define:

Definition 4. *$mult(P, M) = length_{R_P}(H^0_P(M))_P$.*

In Lemma 1 we discuss the relation between multiplicities of maximal cones in $\mathscr{T}X$ and in $\mathscr{T}\tilde{X}$. Multiplicities will be extremely important for our implicitization problem. See Section 5.2 and [2].

Proposition 3. *Given $X, Y \subset \mathbb{P}^n$ two projective irreducible varieties none of which is contained in a proper coordinate hyperplane, we can consider the associated proejctive variety $X * Y \subset \mathbb{P}^n$. Then as* sets*:*

$$\mathscr{T}(X * Y) = \mathscr{T}(X) + \mathscr{T}(Y)$$

where the sum on the (RHS) is the Minkowski sum of the corresponding polyhedral fans in $\mathbb{C}^{n+1}$.

At the end of this section we'll study the relationship between the multiplicities of maximal cones on each set. As we will see, there is no general behavior of multiplicities in $\mathscr{T}(X * Y)$ compared to multiplicities in $\mathscr{T}X$ and $\mathscr{T}Y$.

As one can easily imagine, this set-theoretic result is motivated by Kapranov's theorem and the fact that valuations turn products into sums. However, we are dealing with the constant coefficient case, so we need to enlarge our variety to include it in the affine ring with coefficients in the Puiseux series ring. This will be done by the following lemma.

Lemma 1. *[Base change lemma] Let $X \subset \mathbb{P}^n$ an irreducible variety not contained in a coordinate hyperplane, and consider the fiber product $X' = X \times_{Spec\,\mathbb{C}} Spec\,\mathbb{C}\{\{t\}\} \subset \mathbb{P}^n_{\mathbb{C}\{\{t\}\}}$. Let $\tilde{X} = C(X') \subset T^{n+1}$, where $T^{n+1} = \mathbb{C}\{\{t\}\}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$. Then*

$$\mathscr{T}X = \mathscr{T}\tilde{X},$$

both as sets and with the corresponding multiplicities, i.e. they have the same structure as weighted polyhedral fans.

Proof. First, we prove the equality as sets. Consider $I = I(X) = (f_1, \ldots, f_r)$ the homogeneous ideal defining X. Then $\tilde{X}$ is defined by

$$J = I\mathbb{C}\{\{t\}\}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}].$$

By definition, $\mathscr{T}\tilde{X} = \{w \in \mathbb{R}^{n+1} \mid \text{in}_w(J) \text{ contains no monomial}\}$. Note that we can write $g = \sum_i h_i f_i$ as $\sum_{j \geq l} t^{j/N} \sum_{i=0}^{r} h_{ij} f_i$, where h_{ij} is the coefficient of h in $\left(\mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]\right)((t^{1/N}))$ associated to the j-th. power of $t^{1/N}$ for some N. It is clear that $\mathscr{T}\tilde{X} \subset \mathscr{T}X$. For the converse we need to make one useful remark. We claim that $\text{in}_w(\sum_i h_i f_i)$ equals in_w of a truncation of the series in t with coefficients in $\mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. This follows because g involves only finitely many monomials in $\underline{x}$, and so the minimum value $\{val(c_\alpha) + w \cdot \alpha : c_\alpha \neq 0\}$ will be attained at only finitely many values of $t^{j/N}$. Therefore, we can truncate the series. Hence, $\text{in}_w(g) = \text{in}_w(\sum_{j=l}^{M} (\sum_i g_{ij} f_i) t^{j/N}$. Since $w \in \mathscr{T}X$ we have $\text{in}_w(\sum_i g_{ij} f_i)$ is not a monomial for all $\sum_i (g_{ij} f_i)$, and there will. In particular for any two choices of j, say $j < j'$ there will be no cancellations because different values of j will give different values of exponents α. Thus, $\text{in}_w(g)$ won't be a monomial, meaning $w \in \mathscr{T}\tilde{X}$.

Concerning the multiplicities, pick σ a d-dimensional polyhedron in $\mathscr{T}X = \mathscr{T}\tilde{X}$ and fix w in the relative interior of σ. Let $S = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We need to show that

$$m_{\sigma, \mathscr{T}X} = \sum_{\substack{P \subset \text{Spec } S \\ P \text{ minimal} \\ \text{over in}_w I}} \text{mult}(P, S/\text{in}_w I) = \sum_{\substack{Q \subset \text{Spec } S \\ Q \text{ minimal} \\ \text{over in}_w J}} \text{mult}(Q, S/\text{in}_w J) = m_{\sigma, \mathscr{T}\tilde{X}}.$$

where the multiplicities $\text{mult}(P, S/\text{in}_w I)$ denote the multiplicity of P as an associated minimal prime to $\text{in}_w I \subset S$, and likewise for Q and $\text{in}_w J$ (see Definition 4.)

The key-fact is that in this case $\text{in}_w I = \text{in}_w J$. This follows by similar arguments as the ones used above. It is clear that $\text{in}_w I \subset \text{in}_w J$. For the converse, pick $g \in J$. By definition, we'll have $g = \sum_j h_j t^{j/N}$ for some $h_j \in I$. By definition of in_w, we see that $\text{in}_w(g)$ coincides with the initial of a truncation of this series. So we can assume g is a polynomial in t. And we see that the maximum $W = \max\{j + \alpha \cdot w : c_\alpha \neq 0\}$ at each summand $h_j t^{j/N}$ will be realized at indices (j, α), where α is a monomial in $\text{in}_w(h_j)$. By construction, we won't get any cancellations. Hence $\text{in}_w(g) \subset \text{in}_w(I)$, as we wanted to show.

□

Let's go back to the main result in this section. Before proving it, we recall Kapranov's theorem adapted to our setting (see [11] Theorem 2.3 for a general statement):

Theorem 1 (Kapranov's Theorem). *Let $K = \mathbb{C}\{\{t\}\}$ be the ring of Puiseux series with its associated valuation, and let X be a variety $X \subset T_K^m$ with radical ideal $I \subset K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Then the following subsets of $\mathbb{R}^m$ coincide:*

(i) $\mathscr{T}(X)$;

(ii) $\{w \in \mathbb{R}^m : init_w I \neq (1)\}$;

(iii) The closure in $\mathbb{R}^m$ of $\{val(\underline{x}) := (val(x_1), \ldots, val(x_m)) \in \mathbb{R}^m : \underline{x} = (x_1, \ldots, x_m) \in X\}$.

Proof. (Proposition 3) Pick $x \in \tilde{X}$ and $y \in \tilde{Y}$, following the notation in Lemma 1. By definition, $val(x * y) = val(x) + val(y)$. Since the sum in $\mathbb{R}^n$ is a continuous function w.r.t. the product topology, the result follows by Kapranov's theorem. □

Therefore, the same proof gives us:

Proposition 4. *Let $\mathbb{R} \subset K = \mathbb{C}\{\{t\}\}$ be the ring of Puiseux series with its associated nontrivial valuation, and let $X, Y \subset T_K^m$ with radical ideals $I, J \subset K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Consider the variety $X * Y \subset T_K^m$ as the closure of the image of the coordinatewise multiplication map $X \times Y \to T_K^m$, $(\underline{x}, \underline{y}) \mapsto (x_i y_i)_{i=1}^m$. Then as* sets*:*

$$\mathscr{T}(X * Y) = \mathscr{T}X + \mathscr{T}Y.$$

Note that we're **not** claiming that they have the same fan structure. In general, it might happen that maximal cones in (RHS) get subdivided to give maximal cones in (LHS) or, moreover, the union of several cones in (RHS) gives a maximal cone in the (LHS). The following example illustrates the latter. In general, for the constant coefficient case, a fan structure can be determined by means of the Gröbner fan of the ideal $I(X * Y)$.

Example 1. Consider the variety $X = Y = V(y^{10} - xzw^8, x^4 + y^4 + z^4 + w^4) \subset \mathbb{C}^4$, which we know is irreducible, with ideal given by these two polynomials (we check this by doing a primary decomposition). We compute it's fan structure with `gfan`

```
gfan_buchberger  <input.gfan.n4 | gfan_tropicalstartingcone | gfan_tropicaltraverse
&>output.gfan.n4
Q[x,y,z,w]
{y^10-x*z*w^8, x^4+y^4+z^4+w^4}

_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
2

LINEALITY_DIM
1

RAYS
-10 -1 0 0 # 0
5 1 5 0 # 1
0 -1 -10 0 # 2

N_RAYS
3

LINEALITY_SPACE
1 1 1 1

ORTH_LINEALITY_SPACE
0 0 1 -1
0 1 0 -1
1 0 0 -1

F_VECTOR
1 3

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}

MAXIMAL_CONES
{0} # Dimension 2
{1}
```

```
{2}

PURE
1

MULTIPLICITIES
4 # Dimension 2
8
4
```

As we see, the fan structure consists of 3 rays with multiplicities 4, 8 and 4. After moding out by the lineality space, we can draw the Minkowski sum. And in this case, the union of the three 2-dimensional cones give a 2-dimensional vector space. So we have a 3-dimensional vector space in $\mathbb{R}^4$ corresponding to the tropical variety as a set.

To check this, we compute the ideal $I(X*X)$ and the corresponding tropicalization. The elimination ideal was computed with `Singular`:

```
LIB = "elim.lib";
ring R = 0, (s,t,u,v,x,y,z,w,a,b,c,d), lp;
ideal I = (s-x*a, t-y*b, u-z*c, v-w*d, y^10-x*z*w^8, x^4+y^4+z^4+w^4,
b^10-a*c*d^8, a^4+b^4+c^4+d^4);
eliminate(I, a*b*c*d*x*y*z*w);

_[1]=suv8-t10
```

Since the generating polynomial is a binomial, we know that its tropicalization is a vector space, namely, its lineality space.

To confirm this and check the multiplicity structure, we plug in this equation into `gfan` to obtain the weighted fan structure:

```
 gfan_buchberger | gfan_tropicalstartingcone |gfan_tropicaltraverse
&>output.gfan.TropXXn4
Q[s,t,u,v]
{suv^8-t^10};
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
3

LINEALITY_DIM
3

RAYS

N_RAYS
0

LINEALITY_SPACE
0 0 8 -1
0 4 0 5
```

```
8 0 0 -1

ORTH_LINEALITY_SPACE
1 -10 1 8

F_VECTOR
1

CONES
{} # Dimension 3

MAXIMAL_CONES
{} # Dimension 3

PURE
1

MULTIPLICITIES
1 # Dimension 3
```

■

Corollary 1 (Expected dimension)**.** *Let X, Y be as in Proposition 4. Let L_X, L_Y be the lineality spaces in X and Y, and let $\dim X = d$, $\dim Y = d'$ denote their dimensions as* affine *varieties. Then the expected* affine *dimension of $\mathscr{T}(X*Y)$ is $\min\{(d-\dim L_X)+(d'-\dim L_Y)+\dim(L_X+L_Y), n\} = \min\{d+d'-\dim(L_X\cap L_Y), n\}$.*

Proof. By Proposition 4, we have $\dim \mathscr{T}(X*Y) \leq \min\{d+d'-\dim(L_X\cap L_Y)\}$, and since $\mathscr{T}(X*Y) \subset \mathbb{R}^n$, $\dim \mathscr{T}(X*Y) \leq n$.

Assume $d+d'-\dim(L_X\cap L_Y) \leq n$. To prove the other inequality, we need to find two maximal cones $\sigma \in \mathscr{T}X$, $\tau \in \mathscr{T}Y$ s.t. $\dim \sigma + \tau = d + d' - \dim(L_X + L_Y)$, and we need to do this if X and Y are generic irreducible varieties of dimensions d and d'.

If X and Y are generic, we have that $L_X = L_Y$ is 1-dimensional, generated by the all ones vector (recall that X, Y are projective). Therefore, $L_X = L_Y = L_X \cap L_Y$. In addition, if X and Y are generic, then the maximal cones in X and Y will also be generic d and d'-dimensional cones in $\mathbb{R}^n$. And therefore, the Minkowski sum of pair of cones in X and Y will have dimension $d+d'-1$, as we wanted.

On the other hand, if $d+d'-\dim(L_X\cap L_Y) > n$, and X and Y are sufficiently generic, we have that the Minkowski sum of maximal cones $\sigma \in \mathscr{T}X$ and $\tau \in \mathscr{T}Y$ will span the whole space ($d+d'-1 > n$ vectors in $\mathbb{R}^n$.) Therefore, the result also follows in this case. □

Before moving on to the next section, let us say a word about the dimension of our statistical model, as an application of the previous corollary. It is a well-known fact (see [10], Theorem 2.7) that the dimension of an irreducible variety in $\mathbb{C}^n$ coincides with the dimension of its tropicalization. Since our variety is the algebraic closure of the image of a polynomial map, we know it is irreducible.

If we specialize Corollary 1 to the case $X = Y$, we have that the expected dimension of $\mathscr{T}X * X$ is $\min\{2\dim X - \dim L_X, n\}$. In Section 5.1, we compute $\mathscr{T}X$ and we see that $\dim \mathscr{T}X = 10$, with $\dim L_X = 5$, so $X = \mathrm{Sec}(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ has the expected projective dimension (9). Hence, the expected dimension for $\mathscr{T}(X*X)$ is $\min\{15,16\} = 15$. By computing $\mathscr{T}X*X$ via Minkowski sums, we show that $\mathscr{T}(X*X) \subset \mathbb{R}^{16}$ is a (pure) polyhedral fan of dimension 15. This ensures that our model corresponds to a hypersurface in $\mathbb{P}^{15}$ (its associated affine cone is 15-dimensional). Therefore, the dimension predicted in [4] was correct.

4.1 Multiplicities

We now return to the question of equality as weighted polyhedral fans. Our aim is to study the relation between the multiplicities of maximal cones $\sigma \in \mathscr{T}X$, $\tau \in \mathscr{T}Y$ and the multiplicity of $\sigma+\tau \in \mathscr{T}(X*Y)$, if this sum is a maximal cone in $\mathscr{T}(X*Y)$. For simplicity, assume $X = Y$. Since the Minkowski sum map Φ satisfies $\Phi(\sigma,\tau) = \Phi(\tau,\sigma)$, we see that for maximal cones in $\operatorname{im}\Phi$ the fiber has even cardinality. We should expect our formula to involve half the cardinality of the fibers of maximal cones in $\mathscr{T}(X*X)$.

We first consider Example 1. As we can see, the Minkowski sum map has only fibers of cardinality 2. However, in this case, our maximal cones in $\mathscr{T}(X*X)$ (namely, the unique linear space) has empty fiber under the map Φ. This will be the case if $\mathscr{T}(X*X)$ is a linear subspace of $\mathbb{R}^n$ (with the coarsest fan structure) and X is an irreducible generic projective variety of affine dimension $d \leq \lceil (n-r)/2 \rceil$ because we need $n - \dim L_X + 1$ points to express $L_X^{\perp} \subset \mathbb{R}^n$ as a cone (namely $v_1, \ldots, v_{n-r}, -(v_1 + \ldots + v_{n-r})$, were $r = \dim L_X$.)

In this case, the fan structure will consist of only 1 cone (the entire linear space) and the multiplicity will be 1 (because the tropical variety coincides with the homogeneous space, so for any w in the relative interior of the cone (i.e. any w in the linear space) $\in_w (I) = I$. Since $I(X*X) = I$ is a prime ideal in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we have $m_{\sigma,\mathscr{T}(X*X)} = 1$.

The case of a tropical variety being a linear space (with more general fan structure) was studied in detail by Sturmfels in [14], Lemma 9.9:

Lemma 2. *Let $X \subset \mathbb{R}^n$ be an irreducible proyective variety of dimension d not contained in any coordinate hyperplane, and suppose $\mathscr{T}(X*X)$ is a linear subspace of $\mathbb{R}^n$ of dimension r. Then there is an r-dimensional subtorus $T \subset (\mathbb{C}^*)^n$, such that $X*X$ consists of finitely many T-orbits.*

Note that since $X*X$ is not contained in any coordinate hyperplane and is irreducible, then $X*X \cap (\mathbb{C}^*)^n$ is irreducible and dense in $X*X$. In addition, if X is generic we expect r to equal $\min\{2d - \dim L_X, n\}$, were L_X denotes the lineality space of $\mathscr{T}X$.

We now consider an example where Φ maps pairs of maximal cones in $\mathscr{T}X$ onto maximal cones in $\mathscr{T}(X*X)$.

Example 2. Let $X = Y = V(y^4 - xzw^2 + xyzw, x^2 + y^2 + z^2 + w^2) \subset \mathbb{C}^4$. We know that the corresponding ideal is given by these two polynomials and it is a prime ideal (we compute a primary decomposition of it and we obtain one component). Therefore, we ensure that `gfan` will give us the right answer.

We compute the tropicalization of X: it consists of 4 rays and a 1-dimensional lineality space:

```
gfan_buchberger |gfan_tropicalstartingcone |gfan_tropicaltraverse
&>output.gfan.n4.bis.bis

Q[x,y,z,w]
{y^4-x*z*w^2+z*x*w*y, x^2+y^2+z^2+w^2}
Hypersurfaces to go:5
Max dimension: 4
Hypersurfaces to go:4
Max dimension: 3
Hypersurfaces to go:3
Max dimension: 3

_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
2
```

```
LINEALITY_DIM
1

RAYS
-4 -1 0 0 # 0
1 0 1 0 # 1
3 2 3 0 # 2
0 -1 -4 0 # 3

N_RAYS
4

LINEALITY_SPACE
1 1 1 1

ORTH_LINEALITY_SPACE
0 0 1 -1
0 1 0 -1
1 0 0 -1

F_VECTOR
1 4

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}

MAXIMAL_CONES
{0} # Dimension 2
{1}
{2}
{3}

PURE
1

MULTIPLICITIES
2 # Dimension 2
2
2
2
```

After moding out by the lineality space, we can identify our fan with a fan in $\mathbb{R}^3$ by proyection onto the first three coordinates. As a set, we'll have six 2-dimensional maximal cones, which pairwise span the whole space. Hence, the only possibility to get the fan structure would be to subdivide our 2-dimensional cones. In particular, the tropical variety $\mathscr{T}(X * X)$ has dimension 3, so we know $X * X$ is a hypersurface. We compute its implicit equation with `Singular`:

```
LIB = "elim.lib";
ring R = 0, (s,t,u,v,x,y,z,w,a,b,c,d), lp;
ideal I =(s-x*a, t-y*b, u-z*c, v-w*d, y^4-x*z*w^2+z*x*w*y, x^2+y^2+z^2+w^2,
```

```
b^4-a*c*d^2+ c*a*d*b , a^2+b^2+c^2+d^2);
eliminate(I, a*b*c*d*x*y*z*w);

_[1]=s10t8u6v8-2s8t10u6v8-2s8t8u8v8-2s8t8u6v10+s8t6u8v10-2s8t5u8v11+3s8t4u8v12-
4s8t3u8v13+3s8t2u8v14-2s8tu8v15+s8u8v16+2s7t13u5v7+2s7t12u5v8+2s7t9u7v9-4s7t8u7v10+
8s7t7u7v11-12s7t6u7v12+6s7t5u7v13-8s7t4u7v14-s6t16u4v6+4s6t14u6v6+8s6t13u6v7+
15s6t12u6v8+12s6t11u6v9-2s6t10u8v8+35s6t10u6v10+6s6t9u6v11+s6t8u10v8-2s6t8u8v10+
28s6t8u6v12-16s5t17u5v5-40s5t16u5v6-60s5t15u5v7-80s5t14u5v8+2s5t13u7v7-50s5t13u5v9+
2s5t12u7v8-56s5t12u5v10+28s4t20u4v4+72s4t19u4v5+105s4t18u4v6+90s4t17u4v7-s4t16u6v6+
70s4t16u4v8-28s3t23u3v3-68s3t22u3v4-78s3t21u3v5-56s3t20u3v6+17s2t26u2v2+34s2t25u2v3+
28s2t24u2v4-6st29uv-8st28uv2+t32
```

At last, we obtain the fan structure of $\mathscr{T}(X * X)$ with `gfan`. This confirms that our fan has 6 maximal cones, so Φ is also onto in this case. Moreover, each fiber has cardinality two, and consists of pairs $\{(\sigma,\tau),(\tau,\sigma)\}$.

```
gfan_buchberger | gfan_tropicalstartingcone | gfan_tropicaltraverse
&>output.gfan.TropXXn4.bis.bis
Q[s,t,u,v]
{s10t8u6v8-2s8t10u6v8-2s8t8u8v8-2s8t8u6v10+s8t6u8v10-2s8t5u8v11+3s8t4u8v12-
4s8t3u8v13+3s8t2u8v14-2s8tu8v15+s8u8v16+2s7t13u5v7+2s7t12u5v8+2s7t9u7v9-4s7t8u7v10+
8s7t7u7v11-12s7t6u7v12+6s7t5u7v13-8s7t4u7v14-s6t16u4v6+4s6t14u6v6+8s6t13u6v7+
15s6t12u6v8+12s6t11u6v9-2s6t10u8v8+35s6t10u6v10+6s6t9u6v11+s6t8u10v8-2s6t8u8v10+
28s6t8u6v12-16s5t17u5v5-40s5t16u5v6-60s5t15u5v7-80s5t14u5v8+2s5t13u7v7-50s5t13u5v9+
2s5t12u7v8-56s5t12u5v10+28s4t20u4v4+72s4t19u4v5+105s4t18u4v6+90s4t17u4v7-s4t16u6v6+
70s4t16u4v8-28s3t23u3v3-68s3t22u3v4-78s3t21u3v5-56s3t20u3v6+17s2t26u2v2+34s2t25u2v3+
28s2t24u2v4-6st29uv-8st28uv2+t32}

_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
3

LINEALITY_DIM
1

RAYS
-4 -1 0 0 # 0
1 0 1 0 # 1
3 2 3 0 # 2
0 -1 -4 0 # 3

N_RAYS
4

LINEALITY_SPACE
1 1 1 1

ORTH_LINEALITY_SPACE
0 0 1 -1
```

```
0 1 0 -1
1 0 0 -1

F_VECTOR
1 4 6

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}
{0 1} # Dimension 3
{0 3}
{1 2}
{1 3}
{0 2}
{2 3}

MAXIMAL_CONES
{0 1} # Dimension 3
{0 3}
{1 2}
{1 3}
{0 2}
{2 3}

PURE
1

MULTIPLICITIES
2 # Dimension 3
8
4
2
2
2
```

■

Therefore, we see that the relationship between multiplicities in $\mathscr{T}(X*Y)$ and $\mathscr{T}X$, $\mathscr{T}Y$ is far from being clear. Moreover, we shouldn't expect any general formula involving these quantities. However, because the model we're studying in this paper is highly symmetric we should expect all multiplicities to be the same. This will be the case for $\mathscr{T}X$, as we shall see in next section.

One last example will show us that the fan structure can come from subdividing maximal cones obtained via Minkowski sums:

Example 3. Let $X = Y = V(xz - yw + zy + xw, x^3 + y^3 + z^3 + w^3) \subset \mathbb{C}[x, y, z, w]$. It's associated ideal is generated by these two polynomials (via a primary decomposition computation). The tropicalization of X will consist of 6 rays plus a lineality space generated by the all 1's vector.

```
gfan_buchberger | gfan_tropicalstartingcone | gfan_tropicaltraverse

Q[x,y,z,w]
{xz-yw+zy+xw, x^3+y^3+z^3+w^3}
```

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
2

LINEALITY_DIM
1

RAYS
-1 0 0 0 # 0
-1 -1 0 0 # 1
1 1 0 0 # 2
-1 -4 0 0 # 3
0 0 -1 0 # 4
4 4 3 0 # 5

N_RAYS
6

LINEALITY_SPACE
1 1 1 1

ORTH_LINEALITY_SPACE
0 0 1 -1
0 1 0 -1
1 0 0 -1

F_VECTOR
1 6

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}
{4}
{5}

MAXIMAL_CONES
{0} # Dimension 2
{1}
{2}
{3}
{4}
{5}
```

```
PURE
1

MULTIPLICITIES
3 # Dimension 2
2
2
1
3
1
```

We mod out by the lineality space and we get a fan structure consisting of six rays $\{e+0,\ldots,e_n\}$ in $\mathbb{R}^3$. They are grouped into two sets: $\{e_0,e_1,e_2,e_3\}$ and $\{e_1,e_2,e_4,e_5\}$. The union of the pairwise Minkowski sums on each set gives us two vector spaces, namely ($z=0$ and ($x-y=0$). To finish computing $\mathscr{T}(X*X)$ as a set, we need to add the cones $(e_0,e_4),(e_0,e_5),(e_3,e_4)$ and (e_3,e_5). However, this will *not* be the fan decomposition of $\mathscr{T}(X*X)$, partly because the intersection of the two vector spaces above won't be a face of each one, contradicting the definition of a polyhedral fan. We compute the implicit equation of $X*X$ with `Singular`:

```
LIB = "elim.lib";
ring R = 0, (s,t,u,v,x,y,z,w,a,b,c,d), lp;
ideal I =(s-x*a, t-y*b, u-z*c, v-w*d,
x*z-y*w+z*y+x*w, x^3+y^3+z^3+w^3,a*c-b*d+c*b+a*d, a^3+b^3+c^3+d^3);
eliminate(I, a*b*c*d*x*y*z*w);
_[1]=3s6u2v+s6v3+15s5tu2v-6s5tuv2+3s5tv3+33s4t2u2v-24s4t2uv2+3s4t2v3+42s3t3u2v-
36s3t3uv2+2s3t3v3+33s2t4u2v-24s2t4uv2+3s2t4v3-3s2tu6-15s2tu5v-33s2tu4v2-42s2tu3v3-
33s2tu2v4-15s2tuv5-3s2tv6+15st5u2v-6st5uv2+3st5v3+6st2u5v+24st2u4v2+36st2u3v3+
24st2u2v4+6st2uv5+3t6u2v+t6v3-t3u6-3t3u5v-3t3u4v2-2t3u3v3-3t3u2v4-3t3uv5-t3v6
```

To finish, we compute the tropicalization of this variety with `gfan`, which provides the weighted polyhedral complex structure:

```
gfan_buchberger | gfan_tropicalstartingcone | gfan_tropicaltraverse
&>outputXX.gfan.cube

Q[s,t,u,v]
{3s6u2v+s6v3+15s5tu2v-6s5tuv2+3s5tv3+33s4t2u2v-24s4t2uv2+3s4t2v3+42s3t3u2v-
36s3t3uv2+2s3t3v3+33s2t4u2v-24s2t4uv2+3s2t4v3-3s2tu6-15s2tu5v-33s2tu4v2-42s2tu3v3-
33s2tu2v4-15s2tuv5-3s2tv6+15st5u2v-6st5uv2+3st5v3+6st2u5v+24st2u4v2+36st2u3v3+
24st2u2v4+6st2uv5+3t6u2v+t6v3-t3u6-3t3u5v-3t3u4v2-2t3u3v3-3t3u2v4-3t3uv5-t3v6}

_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
4

DIM
3

LINEALITY_DIM
1

RAYS
```

```
-1 0 0 0 # 0
-1 -1 0 0 # 1
1 1 0 0 # 2
-1 -4 0 0 # 3
0 0 -1 0 # 4
4 4 3 0 # 5

N_RAYS
6

LINEALITY_SPACE
1 1 1 1

ORTH_LINEALITY_SPACE
0 0 1 -1
0 1 0 -1
1 0 0 -1

F_VECTOR
1 6 12

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}
{4}
{5}
{1 3} # Dimension 3
{0 1}
{0 2}
{2 3}
{0 4}
{0 5}
{1 5}
{1 4}
{2 4}
{2 5}
{3 5}
{3 4}

MAXIMAL_CONES
{1 3} # Dimension 3
{0 1}
{0 2}
{2 3}
{0 4}
{0 5}
{1 5}
{1 4}
{2 4}
{2 5}
{3 5}
```

```
{3 4}

PURE
1

MULTIPLICITIES
6 # Dimension 3
6
2
2
3
1
2
2
6
6
1
1
```

In this case we see that the vector space $(z = 0) \subset \mathscr{T}(X * X)$ is decomposed into the union of the cones $(e_0, e_1), (e_1, e_3), (e_3, e_2)$ and (e_2, e_0). In particular, the cone (e_0, e_3) which appeared in our Minkowski sum construction, decomposes into the union of (e_0, e_1) and (e_1, e_3). Therefore, in this case, the fan structure is given by the minimal cones constructed in our Minkowski sum. ■

5 Computations

5.1 Computation of $\mathscr{T}(X * X)$

We now turn into the question of how do we get a handle on our tropical computations. According to our key result, we only need to work with the right-hand side of our equality. Recall that in our case, our varieties are $X = Y = \mathrm{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, i.e. the first secant of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

By our discussions in previous sections, we know how to compute the tropical variety associated to X: we need to consider the tropical variety of all three 4×4-flattenings. To compute this variety we use `gfan`, and to compute the ideal associated to each flattening we use `macaulay2`. More precisely:

```
i1 : R =
QQ[p0000,p0001,p0010,p0011,
p0100,p0101,p0110,p0111,
p1000,p1001,p1010,p1011,
p1100,p1101,p1110,p1111]

I1 = minors(3, matrix({{
p0000,p0001,p0010,p0011},{
p0100,p0101,p0110,p0111},{
p1000,p1001,p1010,p1011},{
p1100,p1101,p1110,p1111
}}))


R =
QQ[p0000,p0001,p0010,p0011,
p0100,p0101,p0110,p0111,
```

```
p1000,p1001,p1010,p1011,
p1100,p1101,p1110,p1111]

I1 = minors(3, matrix({{
p0000,p0001,p0010,p0011},{
p0100,p0101,p0110,p0111},{
p1000,p1001,p1010,p1011},{
p1100,p1101,p1110,p1111
}}))

I2 = minors(3, matrix({{
p0000,p0010,p1000,p1010},{
p0001,p0011,p1001,p1011},{
p0100,p0110,p1100,p1110},{
p0101,p0111,p1101,p1111
}}))

I3 = minors(3, matrix({{
p0000,p0001,p1000,p1001},{
p0010,p0011,p1010,p1011},{
p0100,p0101,p1100,p1101},{
p0110,p0111,p1110,p1111
}}))

o1 = R

o1 : PolynomialRing

i2 :
o2 = ideal (- p0010*p0101*p1000 + p0001*p0110*p1000 + p0010*p0100*p1001 - p0000*p0110*p1001 -
p0001*p0100*p1010 + p0000*p0101*p1010, - p0010*p0101*p1100 + p0001*p0110*p1100 +p0010*p0100*
p1101 - p0000*p0110*p1101 - p0001*p0100*p1110 + p0000*p0101*p1110, - p0010*p1001*p1100 +
p0001*p1010*p1100 + p0010*p1000*p1101 - p0000*p1010*p1101 - p0001*p1000*p1110 + p0000*p1001*
p1110, - p0110*p1001*p1100 + p0101*p1010*p1100 + p0110*p1000*p1101 - p0100*p1010*p1101 -
p0101*p1000*p1110 + p0100*p1001*p1110, - p0011*p0101*p1000 + p0001*p0111*p1000 + p0011*p0100*
p1001 - p0000*p0111*p1001 - p0001*p0100*p1011 + p0000*p0101*p1011, - p0011*p0101*p1100 +
p0001*p0111*p1100 + p0011*p0100*p1101 - p0000*p0111*p1101 - p0001*p0100*p1111 + p0000*p0101*
p1111, - p0011*p1001*p1100 + p0001*p1011*p1100 + p0011*p1000*p1101 - p0000*p1011*p1101 -
p0001*p1000*p1111 + p0000*p1001*p1111, - p0111*p1001*p1100 + p0101*p1011*p1100 + p0111*p1000*
p1101 - p0100*p1011*p1101 - p0101*p1000*p1111 + p0100*p1001*p1111, - p0011*p0110*p1000 +
p0010*p0111*p1000 + p0011*p0100*p1010 - p0000*p0111*p1010 - p0010*p0100*p1011 + p0000*p0110*
p1011, - p0011*p0110*p1100 + p0010*p0111*p1100 + p0011*p0100*p1110 - p0000*p0111*p1110 -
p0010*p0100*p1111 + p0000*p0110*p1111, - p0011*p1010*p1100 + p0010*p1011*p1100 + p0011*p1000*
p1110 - p0000*p1011*p1110 - p0010*p1000*p1111 + p0000*p1010*p1111, - p0111*p1010*p1100 +
p0110*p1011*p1100 + p0111*p1000*p1110 - p0100*p1011*p1110 - p0110*p1000*p1111 + p0100*p1010*
p1111, - p0011*p0110*p1001 + p0010*p0111*p1001 + p0011*p0101*p1010 - p0001*p0111*p1010 -
p0010*p0101*p1011 + p0001*p0110*p1011, - p0011*p0110*p1101 + p0010*p0111*p1101 + p0011*p0101*
p1110 - p0001*p0111*p1110 - p0010*p0101*p1111 + p0001*p0110*p1111, - p0011*p1010*p1101 +
p0010*p1011*p1101 + p0011*p1001*p1110 - p0001*p1011*p1110 - p0010*p1001*p1111 + p0001*p1010*
p1111, - p0111*p1010*p1101 + p0110*p1011*p1101 + p0111*p1001*p1110 - p0101*p1011*p1110 -
p0110*p1001*p1111 + p0101*p1010*p1111)

o2 : Ideal of R
```

```
o2 = ideal (- p0011*p0100*p1000 + p0001*p0110*p1000 + p0010*p0100*p1001 - p0000*p0110*p1001 -
p0001*p0010*p1100 + p0000*p0011*p1100, - p0011*p0101*p1000 + p0001*p0111*p1000 + p0010*p0101*
p1001 - p0000*p0111*p1001 - p0001*p0010*p1101 + p0000*p0011*p1101, - p0101*p0110*p1000 +
p0100*p0111*p1000 + p0010*p0101*p1100 - p0000*p0111*p1100 - p0010*p0100*p1101 + p0000*p0110*
p1101, - p0101*p0110*p1001 + p0100*p0111*p1001 + p0011*p0101*p1100 - p0001*p0111*p1100 -
p0011*p0100*p1101 + p0001*p0110*p1101, - p0011*p0100*p1010 + p0001*p0110*p1010 + p0010*p0100*
p1011 - p0000*p0110*p1011 - p0001*p0010*p1110 + p0000*p0011*p1110, - p0011*p0101*p1010 +
p0001*p0111*p1010 + p0010*p0101*p1011 - p0000*p0111*p1011 - p0001*p0010*p1111 + p0000*p0011*
p1111, - p0101*p0110*p1010 + p0100*p0111*p1010 + p0010*p0101*p1110 - p0000*p0111*p1110 -
p0010*p0100*p1111 + p0000*p0110*p1111, - p0101*p0110*p1011 + p0100*p0111*p1011 + p0011*p0101*
p1110 - p0001*p0111*p1110 - p0011*p0100*p1111 + p0001*p0110*p1111, - p0100*p1001*p1010 +
p0100*p1000*p1011 + p0001*p1010*p1100 - p0000*p1011*p1100 - p0001*p1000*p1110 + p0000*p1001*
p1110, - p0101*p1001*p1010 + p0101*p1000*p1011 + p0001*p1010*p1101 - p0000*p1011*p1101 -
p0001*p1000*p1111 + p0000*p1001*p1111, - p0101*p1010*p1100 + p0100*p1010*p1101 + p0101*p1000*
p1110 - p0000*p1101*p1110 - p0100*p1000*p1111 + p0000*p1100*p1111, - p0101*p1011*p1100 +
p0100*p1011*p1101 + p0101*p1001*p1110 - p0001*p1101*p1110 - p0100*p1001*p1111 + p0001*p1100*
p1111, - p0110*p1001*p1010 + p0110*p1000*p1011 + p0011*p1010*p1100 - p0010*p1011*p1100 -
p0011*p1000*p1110 + p0010*p1001*p1110, - p0111*p1001*p1010 + p0111*p1000*p1011 + p0011*p1010*
p1101 - p0010*p1011*p1101 - p0011*p1000*p1111 + p0010*p1001*p1111, - p0111*p1010*p1100 +
p0110*p1010*p1101 + p0111*p1000*p1110 - p0010*p1101*p1110 - p0110*p1000*p1111 + p0010*p1100*
p1111, - p0111*p1011*p1100 + p0110*p1011*p1101 + p0111*p1001*p1110 - p0011*p1101*p1110 -
p0110*p1001*p1111 + p0011*p1100*p1111)

o2 : Ideal of R

// Let us check the basic geometric invariants for this ideal

i3 : degree I1

o3 = 20

i4 : dim I1

o4 = 12

i5 : I = I1+ I2+ I3;

o5 : Ideal of R

// Basic geometric invariants for the ideal of flattenings

i6 : dim I

o6 = 10

i7 : degree I

o7 = 64

i8 : gens prune I

i9 : toString gens gb I
```

This last line will enable us to compute the corresponding tropical variety using a set of generators of I.

From these computations we see that the ideal generated by the flattenings has the expected dimension, if we consider the affine cone over the projective variety $\text{Sec}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Now that we gave a generating set for the ideal corresponding to our secant variety X, we can proceed to compute its associated tropical variety. For this we make use of `gfan`.

An important remark must be made at this stage. From our previous discussion, we know that the ideal associated to X inside $\mathbb{C}[p_{0000}, \ldots, p_{1111}]$ is invariant under the action of $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4$ given by

$$(\sigma, \tau_1, \tau_2, \tau_3, \tau_4) \cdot (p_{i_0,i_1,i_2,i_3}) = p_{\tau_0(i_{\sigma^{-1}(0)}),\tau_1(i_{\sigma^{-1}(1)}),\tau_2(i_{\sigma^{-1}(2)}),\tau_3(i_{\sigma^{-1}(3)})}.$$

If we identify the index $(ijkl)$ with the binary expansion of numbers from 0 to 15 (i.e. $(ijkl) \leftrightarrow l2^3 + k2^2 + j2^1 + i2^0$), then the action translates to this new setting as follows:

$$(\sigma, \tau) * (\sum_i a_i 2^i) = \sum_i \tau_i(a_{\sigma^{-1}(i)}) 2^i$$

i.e. via embedding σ and τ into $\mathbb{S}_{16}$. The semidirect product structure will be $(\sigma, \tau) * (\sigma', \tau') = (\sigma \circ \sigma', \delta)$, were $\delta_i = \tau_i \circ \tau'_{\sigma^{-1}(i)}$. Therefore our semidirect group structure is $G = \mathbb{S}_4 \ltimes_\varphi (\mathbb{S}_2)^4$, where $\varphi : \mathbb{S}_4 \to Aut((\mathbb{S}_2)^4)$, $\sigma \mapsto (\varphi(\sigma))(\tau) = \tau_{\sigma^{-1}}$. So $(\sigma, \tau) * (\sigma', \tau')$ becomes $\sigma\sigma', \tau\varphi(\sigma)(\tau')$.

In particular, if i, j, k, l denote four distinct elements in $\{0, \ldots, 3\}$, then for $\sigma_{ij} = (i\ j)$ and $\tau_l = (0\ 1)$ we get $\sigma_{ij}\tau_l = \tau_k$ for $k \neq i, j, l$. Therefore, the group $\langle \sigma_{ij}, \tau_k : i, j, k \in \{0, \ldots, 3\} \rangle$ equals the set $\{\tau_0^{i_0} \circ \ldots \tau_3^{i_3} \circ \sigma : i_j \in \{0, 1\}, \sigma \in \mathbb{S}_4\}$.

Luckily, `gfan` is equiped with an option that allows us to exploit any symmetry of a variety determined by an action of a subgroup of the symmetric group of $\mathbb{S}_n$, in this case $n = 16$. For this we need to proving a set of generators as part of the input of the program. The program checks that the ideal stays fixed when permuting the variables with respect to elements in this group. The program uses breadth first search to compute the set of reduced Gröbner bases up to symmetry with respect to the specified generators of the subgroup. The program will group cones together according to the orbits under group action. This will become very useful in subsequent steps of the construction.

In our case, the group will be generated by the transpositions $\sigma_{01} = (0\,1)$, $\sigma_{12} = (1\,2)$ and $\sigma_{23} = (2\,3)$ corresponding to $\mathbb{S}_4$ and by $\tau = ((0\ 1), id, id, id) \in (\mathbb{S}_2)^4$, where we should consider both subgroups embedded in $\mathbb{S}_{16}$. Note that `gfan` requires as input vectors of the form $(\delta(0), \ldots, \delta(15))$ encoding the element δ of our group.

```
gfan_tropicalstartingcone --symmetry | <flatten.gfan.start |
gfan_tropicaltraverse >flatten.gfan.output
```

The file flatten.gfan.start contains a Gröbner basis for the ideal of flattenings (which was computed with the command `gfan_buchberger`) and the list of generators of our group acting on our ideal. The input file can be found in

http://math.berkeley.edu/˜macueto/flatten.gfan.start

In this case, the group is described by the list of generators $\{\sigma_{01}, \sigma_{12}, \sigma_{23}, \tau\} \subset \mathbb{S}_{16}$:

```
{(0,1,2,3,8,9,10,11,4,5,6,7,12,13,14,15),
 (0,1,8,9,4,5,12,13,2,3,10,11,6,7,14,15),
 (0,8,2,10,4,12,6,14,1,9,3,11,5,13,7,15),
 (8,9,10,11,12,13,14,15,0,1,2,3,4,5,6,7)}
```

After 28 hours of computations on a Pentium 4, 3.4 GHz, 4GB RAM, linux 2.6.20 , `gfan` outputs the desired tropical variety $\mathscr{T}X$. The output file is available at:

http://math.berkeley.edu/˜macueto/flatten.gfan.output

We summarized some information that will be relevant for the Minkowski sum computation.

```
AMBIENT_DIM
16

DIM
10

LINEALITY_DIM
5

N_RAYS
382

LINEALITY_SPACE
0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 0 0 -1 0 -1 -1 -2 0 -1 -1 -2 -1 -2 -2 -3

F_VECTOR
1 382 3436 11236 15640 7680
```

An important fact is worth pointing out. As indicated in the output file, `gfan` captures the group action and also the multiplicities of each maximal cone of our fan. In this case, *all* multiplicities equal one. Regarding the orbits, there are 13 rays up to symmetry and 49 maximal cones up to symmetry, and with each collection, there are several choices for the cardinality of the orbits. This fact will be important for understanding the structure of $\mathscr{T}(X * X)$.

According to our key result, we need to compute $\mathscr{T}X$ and the Minkowski sum of this polyhedral fan with itself. Since we know that this will result in a pure polyhedral fan, we only need to compute all Minkowski sums between pairs of cones of *maximal* dimension.

For this step we'll use the action of our subgroup of $\mathbb{S}_{16}$. Since we are taking the Minkowski sum of $\mathscr{T}X$ with itself, there is a natural (coordinatewise) action of $G \times G$ on $\mathscr{T}X \times \mathscr{T}X$ that is carried onto $\mathscr{T}X + \mathscr{T}X$. Therefore, to compute the Minkowski sum of maximal cones we need only to consider 49×7680 pairs and then let our group act on these cones to obtain all maximal cones in $\mathscr{T}(X * X)$. We will take into account the flips after we have our output of $376, 320$. In particular, this says that we have an upper bound of $49 \times 7680 = 376, 320$ orbits corresponding to the maximal cones in $\mathscr{T}(X * X)$.

If we are lucky enough (and indeed we are!) a lot of these sums won't have maximal dimension, so we can forget about them. In our case, this maximal dimension will be 10 after moding out by the lineality space. The way to filter out the maximal cones is as follows. We concatenate our two lists of 5 rays and check its dimension. If it is 10, we save the new list of 10 vectors. To finish, we add the lineality space and check if the final dimension is 15. This computation is carried out by a Python code, which is available at

http://math.berkeley.edu/~macueto/processRays.py

After this we see that the total number of maximal cones computed is 92,469. It is important to say that all $92, 469$ are all distinct. This follows because all rays in the tropical variety are extremal rays generating the maximal cones.

By construction, among the $92, 469$ cones above are all representatives of the orbits of maximal cones in $\mathscr{T}(X * X)$. Fortunately, since not all orbits of the set of rays have equal cardinality, this list of $92, 469$ cones won't be a list of distinct representatives for the orbits of maximal cones in $\mathscr{T}(X * X)$.

More precisely, there are three types or orbits: they have cardinalities 96, 192 and 384. The number of each one of them is given in the following table:

k	96	192	384	TOTAL
#	9	2,171	16,792	18,972

A complete description of the orbits can be found in:

http://math.berkeley.edu/˜macueto/Orbits/

(There are three files, one per size of orbit.)

Concerning multiplicities, we see among our list on $92,469$ maximal cones (containing representatives of all orbits) there are only 37 cones that appear in it twice. And there are no maximal cones appearing more that twice. The file describing this is available at

http://math.berkeley.edu/˜macueto/CombinationsGivingMaximalCones.zip

The first entry on each row indicates the maximal cone, whereas the second one is a list of the original cones in $\mathscr{T}X$ summing to it. Finally, the last number indicates the cardinality of this list.

This gives us the characterization of the fiber of each maximal cone $\gamma \in \mathscr{T}(X * X)$ under the Minkowski sum map $\phi : \mathscr{T}X \times \mathscr{T}X \to \mathscr{T}(X*X)$: namely $\phi^{-1}(\gamma) = \{(\mu,\nu),(\nu,\mu)\}$ for unique maximal cones $\mu,\nu \in \mathscr{T}X$. Namely, two *distinct* pairs of maximal cones (σ,τ), (σ',τ') in $\mathscr{T}X$ will give the same maximal cone in $\mathscr{T}(X*X)$ when computing each Minkowski sum iff the lists $\{\tau,\sigma\},\{\tau',\sigma'\}$ of 10 rays agree. By the computation we've just mentioned, the only way this can occur is if $\tau' = \sigma$ and $\sigma' = \tau$.

Therefore, as we can see from the construction, the maximal cones in $\mathscr{T}(X * X)$ come from interchanging the representative of two maximal cones in $\mathscr{T}X$ that sum to the previous 15-dimensional cone. Thus, we should expect all maximal cones in our Minkowski sum will have "multiplicity" two, and we need to rule out this to get the right expected multiplicities.

Remark 3. *From the construction above, we have 18,972 orbits (of cardinalities 96, 192 and 384), and a total number of 6,865,824 maximal cones. We guess that all maximal cones in* $\mathscr{T}(X * X)$ *will have multiplicity 1.*

5.2 Computation of the Newton polytope of the implicit equation

Now that we know the tropicalization of our hypersurface $(f = 0)$ in $\mathbb{P}^{15}$, we wish to turn things upside down, and describe the equation f. In particular, we're interested in computing it's degree and, moreover, its Newton polytope. As we will explain, there is a strong connection between $\mathscr{T}(f)$ and $\mathrm{NP}(f)$.

Let's give some insight concerning this inverse problem. First of all, w.l.o.g. we can work over the ring of Laurent polynomials, since f is not a monomial and the tropicalization remains the same. Since $I = (f) \subset \mathbb{C}[x_0^{\pm 1},\ldots,x_{15}^{\pm 1}]$ is a principal ideal, then

$$\mathscr{T}I = \{w \,|\, \mathrm{in}_w I \text{ contains no monomial}\} = \{w \,|\, \mathrm{in}_w(f) \text{ is not a monomial}\}$$

Hence, by definition of the tropicalization, the maximal cones in the tropical variety $\mathscr{T}I$ form the normal fan of all normal cones at the edges of our Newton polytope. Why? By the H-representation of the Newton polytope of f, the only way to have more than two terms in $\mathrm{in}_w(f)$ is if the terms involve in the initial form lie in the span of an edge of the polytope. In particular, if the Newton polytope is fully dimensional, then from each maximal cone in $\mathscr{T}(f)$ we can obtain the vector generating the corresponding edge in $\mathrm{NP}(f)$, although we still need to determine its length. This magnitude will correspond to the *multiplicity* of the associated maximal cone in $\mathscr{T}(f)$.

Unfortunately, in our situation the Newton polytope won't be fully dimensional. More precisely, by inspection we see that the lineality space of our tropical variety is invariant under the action of our group $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4 \subset \mathbb{S}_{16}$. Therefore, its orthogonal complement (computed as the Orthogonal lineality space in `gfan`) will also be invariant under this action. Moreover, we know that our Newton polytope lives inside an affine translate of this orthogonal lineality space, and it is fully dimensional inside this subspace of $\mathbb{R}^{16}$. Thus, we can still recover the direction of the edge by means of each maximal cone in $\mathscr{T}I$.

The general algorithm to determine $\mathrm{NP}(f)$ by means of $\mathscr{T}(f)$ was developed in [9], Theorem 1.2 (see also [12] for several numerical examples). It allows us to recover the vertices of the Newton polytope of f, where $f \in \mathbb{C}[x_0^{\pm 1},\ldots,x_{15}^{\pm 1}]$ is a Laurent polynomial and $\mathrm{NP}(f)$ lies in the positive orthant and touches all coordinate axis. Thus, the polytope NP(f) can be reconstructed uniquely, up to translation, from the tropical hypersurface $\mathscr{T}(f)$ (i.e. from the set together with the information provided by the multiplicities of its maximal cones). In particular, by computing one single vertex of NP(f) by this method we'll obtain the degree of f. The algorithm is known as the "ray shooting" method and it allows us to find vertices of the polytope in a given direction. In our case, these directions will be given by the directions of the edges, which we can read of from the tropical variety.

We now describe the algorithm. Assume we are given the normal fan of an unknown polytope, and the multiplicity of each codim-1 cone (dual to an edge of the polytope), i.e. the lattice length of the edge. We want to find the unique polytope, lying in the positive orthant and touching all coordinate hyperplanes, whose normal fan is the given fan. Pick a generic vector w. Then, the face determined by w will be a vertex of NP(f), so w will belong to a fully-dimensional cone of the normal fan, namely, the dual cone to this vertex. Our goal is to compute this vertex.

To obtain its i-th coordinate, we proceed in a recursive way, walking from vertex to vertex towards the i-th coordinate hyperplane, at each step keeping track of how far we go in the i-th direction. This construction has a correspondence in the dual world: we walk from chamber to chamber (fully-dimensional cones) passing through walls (codimension 1 cones) at each step. Each time we pass through a wall we need to record the lattice length of the dual edge in i-th direction. To do this systematically, we shoot a ray from the head of the vector w in the direction $-e_i$, and every time the ray meets a wall, we record the "intersection multiplicity": the absolute value of the determinant of the $n \times n$ matrix where the first column is $-e_i$ and the other $n-1$ columns form a lattice basis for the wall. The sum of those intersection multiplicities (times the multiplicities of the wall) will be the i-th coordinate of the vertex.

The big advantage of this method is that we don't need to know the fan structure of the normal fan. The normal fan can be given as a pile of cones that may or may not intersect nicely. We can do ray-shooting with a list of cones. This matches our setting, since we only computed the maximal cones of $\mathscr{T}(f) = \mathscr{T}(X * X)$ via Minkowski sums, disregarded lower dimensional cones. Moreover, our maximal cones might get subdivided to give the fan structure of $\mathscr{T}(X * X)$.

Note that if we are able to compute one single vertex of $NP(f)$ by means of this method, then we would have the degree of the homogeneous polynomial f as the sum of the coordinates of this vertex.

Concerning the geometry of the Newton polytope, we should remark that this object has lots of symmetries coming also from the action of the group $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4$. More explicitly, we know that the set of vertices of the Newton polytope of f is invariant under the action of the group. This follows because the ideal $T = \langle (z_{ii} - x_i y_i, f_t(\underline{x}), g_l(\underline{y}) : i, j \in [n], t = 1, \ldots, s, l = 1, \ldots, r) \rangle \subset \mathbb{C}[\underline{x}, \underline{y}, \underline{z}]$ in Proposition 1 is invariant under the action of $\left(\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4\right)^3 \subset (\mathbb{S}^{16})^3$. Therefore, the ideal $I(X * X) = T \cap \mathbb{C}[\underline{z}]$ also has this same property. In particular, it's defining equation $I(X * X) = (f)$ is invariant under the action of the group $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4$ up to scaling by a non-zero constant. As a consequence of this, we should make use of this symmetry to construct the Newton polytope.

Therefore, we see that the Newton polytope has also a symmetric structure determined by $\mathbb{S}_4 \ltimes (\mathbb{S}_2)^4 \subset \mathbb{S}^{16}$ and NP(f)$\subset H + v$ (i.e. an affine translate of H) is fully dimensional, where H is the Orthogonal lineality space. The ray-shooting method will give us a particular translate: the one positioning the Newton polytope in the positive orthant and touching all coordinate hyperplanes.

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