WHAT IS … a spectrahedron?

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A spectrahedron is a convex set that appears in a range of applications. Introduced in [3], the name "spectra" is used because its definition involves the eigenvalues of a matrix and "-hedron" because these sets generalize polyhedra.

First we need to recall some linear algebra. All the eigenvalues of a real symmetric matrix are real and if these eigenvalues are all non-negative then the matrix is *positive semidefinite*. The set of positive semidefinite matrices is a convex cone in the vector space of real symmetric matrices.

A *spectrahedron* is the intersection of an affine-linear space with this convex cone of matrices. An n-dimensional affine-linear space of real symmetric matrices can be parametrized by

$$A(x) = A_0 + x_1A_1 + \ldots + x_nA_n$$

as $x = (x_1, \ldots, x_n)$ ranges over $\mathbb{R}^n$, where $A_0, \ldots, A_n$ are real symmetric matrices. This writes our spectrahedron as the set of x in $\mathbb{R}^n$ for which the matrix $A(x)$ is positive semidefinite. This condition, denoted $A(x) \succeq 0$, is commonly known as a *linear matrix inequality*.

For example, we can write the cylinder,

$$\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 \leq 1,\ -1 \leq z \leq 1\},$$

as a spectrahedron. To do this, parametrize a 3-dimensional affine space of 4×4 matrices by

$$\begin{pmatrix} 1+x & y & 0 & 0 \\ y & 1-x & 0 & 0 \\ 0 & 0 & 1+z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix}.$$

This matrix is clearly positive definite at the point $(x, y, z) = (0, 0, 0)$. In fact, it is positive semidefinite exactly for points in the cylinder.

This matrix has rank four at points in the interior of the cylinder, rank three at most points on the boundary, and rank two for points on the two circles on the top and bottom. Here we start to see the connection between the geometry of spectrahedra and rank. The boundary is "more pointy" at matrices of lower rank.

Another example is a *polyhedron*, which is the intersection of the non-negative orthant with an affine-linear space. This is a spectrahedron parametrized by diagonal matrices, since a diagonal matrix is positive semidefinite exactly when the diagonal entries are non-negative.

Like polyhedra, spectrahedra have faces cut out by tangent hyperplanes, but they may have infinitely many. For example, one can imagine rolling a cylinder on the floor along the one-dimensional family of its edges.

This brings us to one of the main motivations for studying spectrahedra: optimization. The well-studied problem of maximizing a linear function over a polyhedron is known as a *linear program*. Generalizing polyhedra to spectrahedra leads to *semidefinite programming*, the problem of maximizing a linear function over a spectrahedron. Semidefinite programs can be solved in polynomial time using *interior-point methods* and form a broad and powerful tool in optimization.

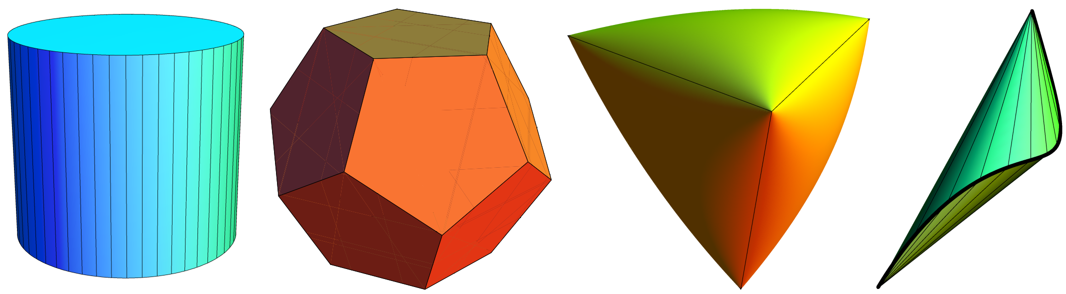
Angles, statistics, and graphs.

Semidefinite programs have been used to relax many "hard" problems in optimization, meaning that they give a bound on the true solution. This has been most successful when the geometry of the underlying spectrahedron reveals that these bounds are close to the true answer.

For a flavor of these applications, consider the spectrahedron of 3×3 matrices with 1's along the diagonal (shown in yellow below):

$$\left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \begin{matrix} \text{is positive} \\ \text{semidefinite} \end{matrix} \right\}.$$

This spectrahedron consists of points $(x, y, z) = (\cos(\alpha), \cos(\beta), \cos(\gamma))$ where α, β, γ are the pairwise angles between three length-one vectors in $\mathbb{R}^3$. To see this, note that we can factor any positive semidefinite matrix A as a real matrix times its transpose, $A = VV^T$. The entries of A are then the inner products of the row vectors of V.



The four rank-one matrices on this spectrahedron occur exactly when these row vectors lie on a common line. They correspond to the four ways of partitioning the three vectors into two sets.

This *elliptope* appears in statistics as a set of correlation matrices and in the remarkable Goemans-Williamson semidefinite relaxation for finding the maximal cut of a graph (see [2]).

This spectrahedron sticks out at its rank-one matrices, meaning that a random linear function often (but not always) achieves its maximum at one of these points. This is good news for the many applications that favor low-rank matrices.

Sums of squares and moments.

Another important application of semidefinite programming is to polynomial optimization [1, Chapter 3]. For example, given a multivariate polynomial $p(x)$, one can bound (from below) its global minimum by the maximum value of λ in $\mathbb{R}$ so that the polynomial $p(x) - \lambda$ can be written as a *sum of squares* of real polynomials. (Sums of squares are guaranteed to be globally non-negative!) Finding this λ is a semidefinite program and the expressions of a polynomial as a sum of squares form a spectrahedron.

For example, take the univariate polynomial $p(t) = t^4 + t^2 + 1$. For any choice of the parameter a in $\mathbb{R}$ we can write our polynomial as

$$p(t) \;=\; \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1-2a & 0 \\ a & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.$$

When this 3×3 matrix is positive semidefinite, it gives a representation of $p(t)$ as a sums of squares. Indeed, if it has rank r we can write it as a sum of r rank-one matrices $\sum_{i=1}^{r} v_i v_i^T$. Multiplying both sides by the vector of monomials $(1, t, t^2)$ writes $p(t)$ as the sum of squares $\sum_{i=1}^{r}((1, t, t^2) \cdot v_i)^2$.

Here the spectrahedron is a line segment parametrized by $a \in [-1, 1/2]$. Its two rank-two end points correspond to the two representations of $p(t)$ as a sum of two squares:

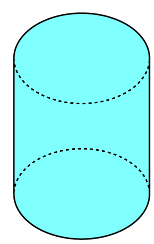
$$(t^2 - 1)^2 + (\sqrt{3}t)^2 \quad \text{and} \quad (t^2 + 1/2)^2 + (\sqrt{3}/2)^2.$$

This idea extends to relaxations for optimization of a multivariate polynomial over any set defined by polynomial equalities and inequalities.

Dual to this theory is the study of *moments*, which come with their own spectrahedra. The convex hull of the curve $\{(t, t^2, t^3) : t \in [-1, 1]\}$ (a spectrahedron) is an example shown above.

A non-example.

To finish, let us return to the question of what a spectrahedron is with a non-example.



Projecting our original cylinder onto the plane $x + 2z = 0$ results in the convex hull of two ellipses. This convex set is not a spectrahedron! Any spectrahedron is cut out by finitely many polynomial inequalities, namely all of the diagonal minors of the matrix being non-negative. However the projection cannot be written this way. This shows that, unlike polyhedra, the class of spectrahedra is not closed under taking projections.

Spectrahedral conclusions.

The study of spectrahedra brings together optimization, convexity, real algebraic geometry, statistics, and combinatorics, among others. There are effective computer programs like `cvx` and `YALMIP` (both for `MATLAB`) that work with spectrahedra and solve semidefinite programs.

Spectrahedra are beautiful convex bodies and fundamental objects in optimization and matrix theory. By understanding the geometry of spectrahedra (and their projections) we can fully explore the potential of semidefinite programming and its many applications.

References

[1] Semidefinite Optimization and Convex Algebraic Geometry, Editors: Grigoriy Blekherman, Pablo A. Parrilo and Rekha R. Thomas, MOS-SIAM Series on Optimization 13 , 2012.

[2] Monique Laurent. Cuts, matrix completions and graph rigidity. Math. Programming, 79, 255–283, 1997.

[3] Motakuri Ramana and A. J. Goldman. Some geometric results in semidefinite programming. J. Global Optim. 7(1), 33–50, 1995.