Math 110 FINAL Exam

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Use big Blue Books!

	Problem	**Parts**	**?**	**= Points**
1:	Definitions	21	2	= 42
2:	LDU	5	5	= 25
3:	Triangle ABC	2	5	= 10
4:	$\sum e^2$	4	5	= 20
5:	e^{Bt}	1	10	= 10
6:	Circulant	3	5	= 15
7:	True/False	13	1	
	Prove/Disprove	13	3	= 52
8:	$A(x, y)$			
	determinant		6	
	Plot		4	
	Shade		3	= 15
	integers		2	
				Total = 189

1. Assume that fields, vector spaces, and matrices have already been defined. Then **define** each of the following. Be accurate and succinct.

A) linearly dependent
B) span
C) basis
D) dimension of vector space
E) rank
F) null space
G) inverse
H) transpose
I) hermite
J) trace
K) determinant of real matrix
L) eigenvalue
M) eigenvector
N) generalized eigenvector
O) similar matrices
P) diagonal matrix
Q) diagonalizable matrix
R) hermitian matrix
S) unitary matrix
T) Markov matrix
U) permutation matrix

2. (A) Factor the following real 3×3 matrix into $M = LDU$, where L is lower triangular, D is diagonal, U is upper triangular, and both L and U have all-ones on the main diagonal, and $M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

2(B). Find vectors $\vec{w}$, $\vec{v}$, and $\vec{x}$ s.t. $L\vec{w} = \vec{b}$, $D\vec{v} = \vec{w}$, $U\vec{x} = \vec{v}$.
Verify that $M\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

2(C). State whether or not the following equation can be solved by a real vector, $\vec{y}$, and **explain**.

$$\vec{y}^T M \vec{y} = -1$$

3. Consider the triangle ABC formed by these points in $\mathbb{R}^3$:

$$\begin{aligned} A &= (2,4,2) \\ B &= (4,6,4) \\ C &= (6,6,-4) \end{aligned}$$

3(A). Find the angles of this triangle.

3(B). Find the area of this triangle.

4. Let A be an $m \times n$ matrix, $\vec{b}$ an $m \times 1$ vector, and $\vec{x}$ an $n \times 1$ vector. Define $\vec{e} = A\vec{x} - \vec{b}$. A, b, x, and e are all real.

4(A). Express $\displaystyle\sum_{i=1}^{m} e_i^2$ in terms of A, $\vec{x}$, and $\vec{b}$.

4(B). Assume that $\vec{x}$ is a differentiable function of time, t, and that $\vec{v} = \dfrac{d\vec{x}}{dt}$. Express $\dfrac{d}{dt}\Big(\displaystyle\sum_{i=1}^{m} e_i^2\Big)$ in terms of A, $\vec{x}$, $\vec{b}$ and $\vec{v}$.

4(C). Find a sufficient condition which ensures that for all $\vec{v}$, $\frac{d}{dt}\Big(\displaystyle\sum_{i=1}^{m} e_i^2\Big) = 0$.

4(D). Prove this condition is necessary.

5. Let $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, and let t be a scalar. Compute the 2×2 matrix $e^{Bt} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(Bt)^n$.

6. Let C be this real 5×5 "circulant" matrix:

$$C = \begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}$$

6(A). Exhibit a permutation matrix, P, and a real polynomial, $f(x)$, s.t. $C = f(P)$.

6(B). Write down the 5×5 Fourier matrix, F, explicitly in terms of $w = e^{2\pi i/5}$, and verify that the columns of F are the eigenvectors of P.

6(C). Express the eigenvalues of C in terms of $f(x)$ and w.

7. (13 parts) For each of the following assertions, state whether it is **True or False**. Then **Prove or Disprove** the assertion. You may use the fact that the determinant is the product of the eigenvalues. You may use Perron's Theorem (which states that the largest eigenvalue of a positive matrix is positive). Otherwise avoid unnecessarily advanced assumptions.

Definitions for [A–E]. A "zero/one" matrix is a matrix each of whose entries is either a zero or a one. A permutation matrix is a zero/one matrix. There are 16 2×2 zero/one matrices.

A) A real 2×2 zero/one matrix must have real eigenvalues.

B) If the eigenvalues of a real 2×2 zero/one matrix are real, they must be integers.

C) If G and H are nonsingular real 2×2 zero/one matrices which have the same eigenvalues, then they must be similar.

Definitions for [D–E]. Let A be a **real** 3×3 zero/one matrix,
and let B be the same **binary** matrix. Real $1 + 1 = 2 > 0$; binary $1 + 1 = 0$.

D) If A is invertible, then B is invertible.

E) If B is invertible, then A is invertible.

F) If $CD = -DC$ then C or D must be singular.

G) Let G be a real symmetric 3×3 matrix whose entries are all positive, and for which

$$\det(G) = -1 \qquad \text{and} \qquad \text{Trace}(G) = +10$$

If the eigenvalues of G are λ_1, λ_2, and λ_3, and if $\lambda_1 \geq \lambda_2 \geq \lambda_3$, then $\lambda_3 < 0 < \lambda_2 < \lambda_1$.

H) The product of Hermitian matrices is Hermitian.

J) If A and B are diagonalizable, and if they share the same eigenvector matrix, S, such that $A = S\Lambda_A S^{-1}$ and $B = S\Lambda_B S^{-1}$, then $AB = BA$.

K) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$ are similar matrices.

M) There exists a 3×3 Markov matrix, M, such that $M \begin{bmatrix} .2 \\ .4 \\ .2 \end{bmatrix} = \begin{bmatrix} .1 \\ .2 \\ .1 \end{bmatrix}$

T) If T is triangular and $T^H T = T T^H$, then T is diagonal.

U) The product of unitary matrices is unitary.

8. Consider this real 3×3 symmetric matrix: $A = \begin{bmatrix} 1 & x & y \\ x & 1 & x \\ y & x & 1 \end{bmatrix}$.

A) Express determinant(A) as a polynomial in x and y.

B) Plot all points in the x, y plane where determinant$(A) = 0$.

C) On **another** plot of the x, y plane, **shade** the region(s) where A is positive definite.

D) List as many pairs of integers, (x, y), for which A is positive definite, as you can. (But do not list more than ten such pairs.)