Math 32 Midterm 2 Solutions

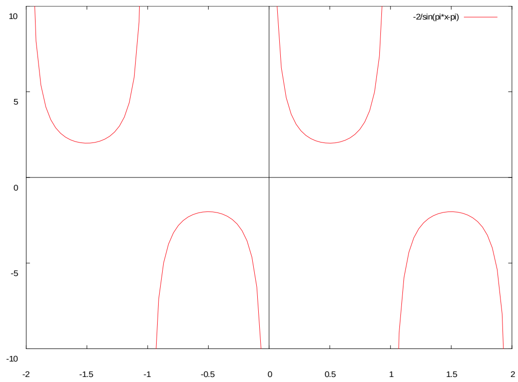
1. Prove the identity $\cot 2\theta = \frac{1}{2}(\cot\theta - \tan\theta)$.

Solution:

$$\begin{aligned}\frac{1}{2}(\cot\theta - \tan\theta) &= \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{1}{2}\left(\frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\right)\\ &= \frac{\cos 2\theta}{2\sin\theta\cos\theta}\\ &= \frac{\cos 2\theta}{\sin 2\theta}\\ &= \cot 2\theta\end{aligned}$$

2. Graph 2 periods of the function $y = -2\csc(\pi x - \pi)$. Label the asymptotes.

Solution: First factor the inside of the expression, $y = -2\csc(\pi(x-1))$. csc normally has a period of 2π therefore this function has a period of $2\pi/\pi = 2$. The -1 signifies that the graph is shifted to the right 1 unit, however since there is a negative sign in front of the expression it is also flipped about the x axis. The asymptotes in the section I chose are at $x = -2, -1, 0, 1, 2$. And the 2 in front changes how low or high the humps are in the graph.



3. Prove the identity

$$\frac{\cos\theta}{\sec\theta - 1} + \frac{\cos\theta}{\sec\theta + 1} = 2\cot^2\theta$$

Solution:

$$\begin{aligned}\frac{\cos\theta}{\sec\theta - 1} + \frac{\cos\theta}{\sec\theta + 1} &= \frac{\cos\theta(\sec\theta + 1) + \cos\theta(\sec\theta - 1)}{\sec^2\theta - 1}\\ &= \frac{\cos\theta\sec\theta - \cos\theta + \cos\theta\sec\theta + \cos\theta}{\tan^2\theta}\\ &= \frac{2}{\tan^2\theta}\\ &= 2\cot^2\theta\end{aligned}$$

4. Make and fill in a table for $\sin\theta, \cos\theta$ and $\tan\theta$ for $\theta = \pi, 7\pi/6, 5\pi/4, 4\pi/3, 3\pi/2$.

Solution: The table is:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
π	0	-1	0
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
$3\pi/2$	-1	0	undefined

5. Solve $9^x + 3(3^x) - 4 = 0$ for x.

Solution: This is a quadratic in terms of 3^x. We can observe this by noting that $9^x = (3^x)^2$. Then this problem is just a matter of factoring and solving the individual expressions from there.

$$\begin{aligned} 9^x + 3(3^x) - 4 &= 0 \\ (3^x)^2 + 3(3^x) - 4 &= 0 \\ (3^x + 4)(3^x - 1) &= 0 \end{aligned}$$

Now x is a solution if and only if it is a solution to one of the two factors. Note that the first factor $3^x + 4$ is always positive and therefore can not be zero. However the second factor has a solution, namely $x = 0$ since $3^0 - 1 = 1 - 1 = 0$.

6. Make and fill in a table for $\csc\theta, \sec\theta$ and $\cot\theta$ for $\theta = 0°, 30°, 45°, 60°, 90°$.

Solution: The table is:

θ	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0°$	undefined	1	undefined
$30°$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$45°$	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$90°$	1	undefined	0

7. a. Compute $\log_{81} 27$.
   b. Simplify the expression $\log_3 A + 4\log_9 B - 9\log_{27} C$, by writing it as a single logarithm with base 3 and leading coefficient 1.

Solution:
   a. $\log_{81} 27 = \log_{81} 3^3 = 3\log_{81} 3 = 3/4$
   b.

$$\begin{aligned} \log_3 A + 4\log_9 B - 9\log_{27} C &= \log_3 A + 4\frac{\log_3 B}{\log_3 9} - 9\frac{\log_3 C}{\log_3 27} \\ &= \log_3 A + 2\log_3 B + 3\log_3 C \\ &= \log_3\left(\frac{AB^2}{C^3}\right) \end{aligned}$$

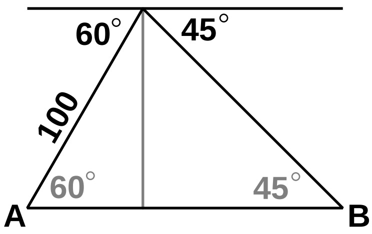
8. Simplify the expression as much as possible, your final answer should not contain any fractions.

$$\frac{\cos(90^\circ - B)}{1+\cos(B)} + \frac{1+\sin(90^\circ - B)}{\sin(B)}$$

Solution:

$$\begin{aligned}
\frac{\cos(90^\circ - B)}{1+\cos(B)} + \frac{1+\sin(90^\circ - B)}{\sin(B)} &= \frac{\sin B}{1+\cos B} + \frac{1+\cos B}{\sin B} \\
&= \frac{\sin^2 B + (1+\cos B)^2}{\sin B(1+\cos B)} \\
&= \frac{\sin^2 B + 1 + 2\cos B + \cos^2 B}{\sin B(1+\cos B)} \\
&= \frac{2+2\cos B}{\sin B(1+\cos B)} \\
&= \frac{2}{\sin B} \\
&= 2\csc B
\end{aligned}$$

9. Find the distance from point A to point B. The two horizontal lines are parallel.

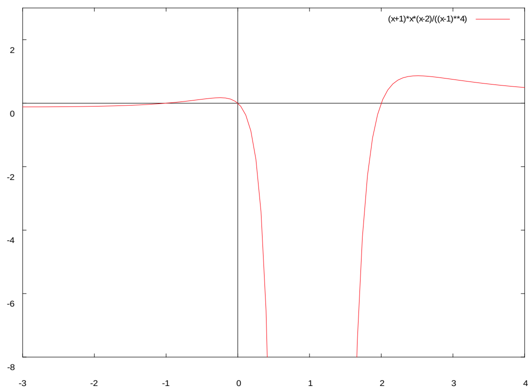


Solution: Consider the left triangle first. Denote its height by h and and its base by b. Then $h/100 = \sin 60^\circ = \sqrt{3}/2$, solving for h gives $h = 50\sqrt{3}$. Likewise $b/100 = \cos 60^\circ = 1/2$ and solving for b gives $b = 50$. Now consider the triangle on the right, it is a $45-45-90$ and therefore as it is drawn its base is equal to its height. We know its height already to be $50\sqrt{3}$. Therefore the distance from A to B is $50 + 50\sqrt{3}$.

10. a. Graph the function $f(x) = \frac{(x+1)x(x-2)}{(x-1)^4}$. State any and all horizontal and vertical asymptotes.

b. Find the domain of the function $\sqrt{f(x)}$ where $f(x)$ is the function defined in part $10.a$.

Solution: a. The critical values are $x = -1, 0, 1, 2$. We know there are zeroes at $x = -1, 0, 2$ and an asymptote at $x = 1$ by examining the roots of numerator and denominator of $f(x)$. We can see what f looks like locally at each of these points. At $x = -1$ we have that $f(x) \sim (3/16)(x+1)$, likewise around $x = 0$ gives $f(x) \sim -2x$, around $x = 1$ gives $f(x) \sim -2/(x-1)^4$ and finally around $x = 2$ gives $f(x) \sim 6(x-2)$. Additionally we know that the numerator has degree 3 and the denominator has degree 4, so for very large positive or negative values of x this function tends towards 0. Which gives us the only other asymptote $y = 0$.



b. The function $\sqrt{f(x)}$'s domain is precisely those points where $f(x) \geq 0$. By examination of the graph the domain is $[-1, 0] \cup [2, \infty)$.