

## MATH 74, FALL 2004, HOMEWORK 2 SOLUTIONS

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**Assignment:** 1.1.16, 1.1.17(b), 1.1.27(a)(b)(d), 1.2.6

$$1.1.16 \sum_{j=1}^{1000} j = \frac{1000(1000+1)}{2} = 500500$$

1.1.17(b) We need to show  $(\forall n \in \mathbb{N}^*)(\sum_{j=1}^n 2j = n(n+1))$ .

*Proof.* Let  $P(n)$  say  $\sum_{j=1}^n 2j = n(n+1)$ .

(1)  $P(n)$  is a predicate in the variable  $n$ .

(2)  $P(1)$  is true since  $\sum_{j=1}^1 2j = 2 \cdot 1 = 2 = 1(1+1)$ .

(3) Let  $n \in \mathbb{N}^*$ , and assume  $P(n)$  is true. Then  $\sum_{j=1}^n 2j = n(n+1)$ . We need to show  $P(n+1)$  is true.  $P(n+1)$  says  $\sum_{j=1}^{n+1} 2j = (n+1)(n+1+1)$ . We have

$$\begin{aligned} & \sum_{j=1}^{n+1} 2j \\ &= \sum_{j=1}^n 2j + 2(n+1) \text{ (basic algebra)} \\ &= n(n+1) + 2(n+1) \text{ (by our induction hypothesis)} \\ &= (n+2)(n+1) \text{ (basic algebra)} \\ &= (n+1)(n+1+1). \end{aligned}$$

So  $P(n+1)$  is true. So  $(\forall n \in \mathbb{N}^*)(P(n) \Rightarrow P(n+1))$  is true. So by induction,  $(\forall n \in \mathbb{N}^*)(P(n))$  is true, i.e.  $(\forall n \in \mathbb{N}^*)(\sum_{j=1}^n 2j = n(n+1))$ . □

1.1.27(a) We first need to discover a formula for the sum of the interior angles of a convex  $n$ -gon. For notation, let  $C(n)$  = The sum of the interior angles of a convex  $n$ -gon, measured in radians. For  $n = 3$ , a 3-gon is a triangle, and the sum of the interior angles in any triangle is  $180^\circ$  or  $\pi$  radians. So  $C(3) = \pi$ . For a convex 4-gon, or convex quadrilateral, we can divide the figure into two triangles by connecting any two non-adjacent vertices. In doing so, we see that the sum of the interior angles of a convex quadrilateral is the sum of all the interior angles of two triangles, which gives  $360^\circ$  or  $2\pi$  radians. So  $C(4) = 2\pi$ . By considering a few more small examples, one sees easily that  $C(5) = 3\pi$ ,  $C(6) = 4\pi$ , and we conjecture that  $C(n) = (n-2)\pi$  for every  $n \geq 3$ .

Let's try to determine why this might be true in general. Consider a convex  $n+1$ -gon (where  $n \geq 3$ ). By taking any two adjacent edges, and connecting the outside vertices, we can divide the  $n+1$ -gon into a triangle and a convex  $n$ -gon. This construction works for any value of  $n \geq 3$ , and the sum of the interior angles in the  $n+1$ -gon is just the sum of the angles in the convex  $n$ -gon plus the sum of the interior angles in the triangle. In equational form, the preceding statement just says  $C(n+1) = C(n) + C(3)$ . The above argument is a proof of the general statement defining a recurrence relation for  $C(n)$ ,  $(\forall n \in \mathbb{N}_3)(C(n+1) = C(n) + C(3))$ . We now have the tools necessary to prove  $(\forall n \in \mathbb{N}_3)(C(n) = (n-2)\pi)$  by induction.

*Proof.* Let  $P(n)$  say  $C(n) = (n-2)\pi$ .

- (1)  $P(n)$  is a predicate in the variable  $n$ .
- (2)  $P(3)$  is true since we already observed that  $C(3) = \pi$ , and  $\pi = (3 - 2)\pi$ .
- (3) Let  $n \in \mathbb{N}_3$ , and assume  $P(n)$  is true. Then  $n \geq 3$  and  $C(n) = (n - 2)\pi$ . We need to show  $P(n+1)$  is true.  $P(n+1)$  says  $C(n + 1) = ((n + 1) - 2)\pi$ . We have

$$\begin{aligned}
 & C(n + 1) \\
 &= C(n) + C(3) \text{ (by the recurrence relation we proved in the paragraph above)} \\
 &= (n - 2)\pi + C(3) \text{ (by our induction hypothesis)} \\
 &= (n - 2)\pi + \pi \text{ (since, as we have already shown, } C(3) = \pi) \\
 &= ((n + 1) - 2)\pi \text{ (basic algebra).}
 \end{aligned}$$

So  $P(n + 1)$  is true. So  $(\forall n \in \mathbb{N}_3)(P(n) \Rightarrow P(n + 1))$  is true. So by induction,  $(\forall n \in \mathbb{N}_3)(P(n))$  is true, i.e.  $(\forall n \in \mathbb{N}_3)(C(n) = (n - 2)\pi)$ . □

1.1.27(b) Let  $H(n)$  = the number of possible handshakes among  $n$  people. By considering a few small values for  $n$ , we find that  $H(0) = 0$ ,  $H(1) = 0$ ,  $H(2) = 1$ ,  $H(3) = 3$ ,  $H(4) = 6$ ,  $H(5) = 10$  and  $H(6) = 15$ . We might already notice a pattern at this point,  $H(n) = \sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}$ . But we don't have enough for a proof by induction yet. We need to consider how the number of handshakes among  $n + 1$  people relates to the number of handshakes among  $n$  people.

If  $n + 1$  people need to mutually shake hands, and the first  $n$  people have already mutually shaken hands, then the  $(n + 1)^{st}$  person needs to shake hands with each of the other  $n$  people. After this happens, all  $n + 1$  people will have mutually shaken hands. As an equation, the preceding two sentences simply say  $H(n + 1) = H(n) + n$ . The above argument provides proof for the statement defining a general recurrence relation for  $H(n)$ ,  $(\forall n \in \mathbb{N})(H(n + 1) = H(n) + n)$ . We now have the tools necessary to prove the statement  $(\forall n \in \mathbb{N})(H(n) = \frac{n(n-1)}{2})$  using mathematical induction.

*Proof.* Let  $P(n)$  say  $H(n) = \frac{n(n-1)}{2}$ .

- (1)  $P(n)$  is a predicate in the variable  $n$ .
- (2)  $P(0)$  is true since, as we've already seen,  $H(0) = 0$ , and  $0 = \frac{0(0-1)}{2}$ .
- (3) Let  $n \in \mathbb{N}$ , and assume  $P(n)$  is true. Then  $H(n) = \frac{n(n-1)}{2}$ . We need to show  $P(n+1)$  is true.  $P(n+1)$  says  $H(n + 1) = \frac{(n+1)(n+1-1)}{2}$ . We have

$$\begin{aligned}
 & H(n + 1) \\
 &= H(n) + n \text{ (by the recurrence relation we proved in above)} \\
 &= \frac{n(n-1)}{2} + n \text{ (by our induction hypothesis)} \\
 &= n\left(\frac{n-1}{2} + 1\right) \text{ (basic algebra)} \\
 &= n\left(\frac{n+1}{2}\right) \text{ (basic algebra).} \\
 &= \frac{(n+1)(n+1-1)}{2} \text{ (basic algebra)}
 \end{aligned}$$

So  $P(n + 1)$  is true. So  $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n + 1))$  is true. So by induction,  $(\forall n \in \mathbb{N})(P(n))$  is true, i.e.  $(\forall n \in \mathbb{N})(H(n) = \frac{n(n-1)}{2})$ . □

1.1.27(d) Let  $S(n)$  = the number of subsets of a set with  $n$  elements. By considering a few small values for  $n$ , we find that  $S(0) = 1$ ,  $S(1) = 2$ ,  $S(2) = 4$ ,  $S(3) = 8$ , and  $S(4) = 16$ . (For example, if  $A = \{a, b, c\}$  is a set with three elements, then the subsets of  $A$  are  $\phi$ ,  $\{a\}$ ,  $\{b\}$ ,  $\{c\}$ ,  $\{a,b\}$ ,  $\{a,c\}$ ,  $\{b,c\}$ , and  $\{a,b,c\}$ . There are eight of these.) We might already notice a pattern at this point,  $S(n) = 2^n$ . But for a proof, we need to consider how the number subsets of a set with  $n + 1$  elements relates to the number of subsets of a set with  $n$  elements.

If  $B$  is any set with  $n + 1$  elements, then let's write  $B = A \cup \{c\}$ , where  $A$  is a set with  $n$  elements. Notice that for each subset  $P$  of  $A$ , there are exactly two ways to extend  $P$  to a

subset of  $B = A \cup \{c\}$ , namely  $P$  and  $P \cup \{c\}$ . Conversely, for any subset  $Q$  of  $B$ , either  $Q$  contains  $c$  or it does not. In either case, there is a determined subset  $P$  of  $A$ , for which  $Q$  is either equal to  $P$  or  $P \cup \{c\}$ . The above argument shows that there must be precisely twice as many subsets of  $B$  as there are subsets of  $A$ . In equational form, the preceding sentence yields  $S(n+1) = 2S(n)$ . The above argument provides proof for the statement defining a general recurrence relation for  $S(n)$ ,  $(\forall n \in \mathbb{N})(S(n+1) = 2S(n))$ . We now have the tools necessary to prove the statement  $(\forall n \in \mathbb{N})(S(n) = 2^n)$  using mathematical induction.

*Proof.* Let  $P(n)$  say  $S(n) = 2^n$ .

- (1)  $P(n)$  is a predicate in the variable  $n$ .
- (2)  $P(0)$  is true since, as we've already seen,  $S(0) = 1$ , and  $1 = 2^0$ .
- (3) Let  $n \in \mathbb{N}$ , and assume  $P(n)$  is true. Then  $S(n) = 2^n$ . We need to show  $P(n+1)$  is true.  $P(n+1)$  says  $S(n+1) = 2^{n+1}$ . We have

$$\begin{aligned} S(n+1) &= 2S(n) \text{ (by the recurrence relation we proved in above)} \\ &= 2 \cdot 2^n \text{ (by our induction hypothesis)} \\ &= 2^{n+1} \text{ (basic algebra)} \end{aligned}$$

So  $P(n+1)$  is true. So  $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$  is true. So by induction,  $(\forall n \in \mathbb{N})(P(n))$  is true, i.e.  $(\forall n \in \mathbb{N})(S(n) = 2^n)$ . □

1.2.6 Let  $T(n)$  = the least number of moves necessary to solve this restricted Tower of Hanoi problem. We have  $T(0) = 0$  since the 0-disk puzzle is already solved, and  $T(1) = 2$  since it now takes two moves to get one disk from peg A to peg C. Similar experimentation yields  $T(2) = 8$  and  $T(3) = 26$ . To prove the general case, we need to find a recurrence relation for  $T$ .

Consider the problem of solving this restricted Tower of Hanoi puzzle with  $n+1$  disks. We first have to move the top  $n$  disks from peg A to peg C. Then we have to move the bottom disk to peg B. Then we have to move the stack of  $n$  disks from peg C back to peg A. Then we move the bottom disk to peg C; and finally, we move the stack of  $n$  disks back again to peg C. This solution method is optimal because all the above moves are required to get the bottom disk moved to peg C. If we add up all the above moves, and write the result as an equation, we get a recurrence relation  $T(n+1) = 3T(n) + 2$ .

Plugging in for more small values of  $n$  gives  $T(4) = 3T(3) + 2 = 3 \cdot 26 + 2 = 80$ , and  $T(5) = 3T(4) + 2 = 3 \cdot 80 + 2 = 242$ . We might notice a pattern at this point, from the numbers 0, 2, 8, 26, 80, 242, and guess that  $T(n) = 3^n - 1$ . In case we didn't notice this pattern so soon, we could try to use a more general method of solving the recurrence relation.

$$\begin{aligned} T(n) &= 3T(n-1) + 2 \\ &= 3[3T(n-2) + 2] + 2 = 3^2T(n-2) + 3 \cdot 2 + 2 \\ &= 3^2[3T(n-3) + 2] + 3 \cdot 2 + 2 = 3^3T(n-3) + 3^2 \cdot 2 + 3 \cdot 2 + 2 \\ &= 3^3[3T(n-4) + 2] + 3^2 \cdot 2 + 3 \cdot 2 + 2 = 3^4T(n-4) + 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2 \\ &\vdots \\ &= 3^nT(n-n) + 3^{n-1} \cdot 2 + 3^{n-2} \cdot 2 + \dots + 3^2 \cdot 2 + 3 \cdot 2 + 2 \\ &= 0 + 2[3^{n-1} + 3^{n-2} + \dots + 3^2 + 3^1 + 3^0] \\ &= 2 \sum_{j=0}^{n-1} 3^j \\ &= 2 \cdot \frac{1-3^n}{1-3} \\ &= 3^n - 1 \end{aligned}$$

The above method will work for most easy recurrences, but not for all recurrences, as it requires recognizing a distinct pattern in the expressions. For most easy recurrences, the

faster way to figure out a closed form formula is to just compute the first several values, and compare them to the values of functions with which you are familiar. In, any case, once we know what the formula is supposed to be, and we have a proof for the recurrence relation, we can prove that our formula is correct by using mathematical induction. We'll prove  $(\forall n \in \mathbb{N})(T(n) = 3^n - 1)$  by induction.

*Proof.* Let  $P(n)$  say  $T(n) = 3^n - 1$ .

(1)  $P(n)$  is a predicate in the variable  $n$ .

(2)  $P(0)$  is true since  $T(0) = 0 = 3^0 - 1$ .

(3) Let  $n \in \mathbb{N}$ , and assume  $P(n)$  is true. Then  $T(n) = 3^n - 1$ . We need to show  $P(n+1)$  is true.  $P(n+1)$  says  $T(n+1) = 3^{n+1} - 1$ . We have

$$\begin{aligned} T(n+1) &= 3T(n) + 2 \text{ (by the recurrence relation we proved previously)} \\ &= 3(3^n - 1) + 2 \text{ (by our induction hypothesis)} \\ &= 3^{n+1} - 3 + 2 \text{ (basic algebra)} \\ &= 3^{n+1} - 1 \end{aligned}$$

So  $P(n+1)$  is true. So  $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$  is true. So by induction,  $(\forall n \in \mathbb{N})(P(n))$  is true, i.e.  $(\forall n \in \mathbb{N})(T(n) = 3^n - 1)$ .

□