MATH 74, FALL 2004, HOMEWORK 13 SOLUTIONS

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Due December 1

Assignment: 3.4.4; 3.4.8

3.4.4 Let $f : F \to K$ be a field isomorphism (using the book's definition).

(1) Show $f(0_F) = 0_K$ and $f(1_F) = 1_K$.

Proof. Let $y \in K$. Since f is surjective, there is $x \in F$ with $f(x) = y$. We have

$$\begin{aligned} f(0_F) +_K y &= f(0_F) +_K f(x) \text{ (since } y = f(x)) \\ &= f(0_F +_F x) \text{ (since } f \text{ preserves } +) \\ &= f(x) \text{ (since } 0_F \text{ is an additive identity element for } F) \\ &= y \text{ (since } y = f(x)) \end{aligned}$$

This shows that $f(0_F)$ is a left additive identity element for K. Since the additive identity element for a field is unique, $f(0_F) = 0_K$.

Similarly, we have

$$\begin{aligned} f(1_F) \cdot_K y &= f(1_F) \cdot_K f(x) \text{ (since } y = f(x)) \\ &= f(1_F \cdot_F x) \text{ (since } f \text{ respects } \cdot) \\ &= f(x) \text{ (since } 1_F \text{ is a multiplicative identity element for } F) \\ &= y \text{ (since } y = f(x)) \end{aligned}$$

This shows that $f(1_F)$ is a left multiplicative identity element for K. Since the multiplicative identity element for a field is unique, $f(1_F) = 1_K$.

□

(2) $f(-_F x) = -_K f(x)$ for each $x \in F$

Proof. Let $x \in F$. Then

$$\begin{aligned} f(-_F x) +_K f(x) &= f(-_F x +_F x) \text{ (since } f \text{ respects } +) \\ &= f(0_F) \text{ (since } -_F x \text{ is an additive inverse for } x \text{ in } F) \\ &= 0_K \text{ (by part (1) above)} \end{aligned}$$

This shows that $f(-_F x)$ is a left additive inverse for $f(x)$ in K. Since the additive inverse for $f(x)$ in K is unique, $f(-_F x) = -_K f(x)$. Since x was an arbitrary element of F, the result holds for every $x \in F$. □

(3) $f(x^{-1_F}) = (f(x))^{-1_K}$ for each $x \in F$ with $x \neq 0$

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Proof. Let $x \in F$ with $x \neq 0$. Then

$$\begin{aligned} f(x^{-1_F}) \cdot_K f(x) &= f(x^{-1_F} \cdot_F x) \text{ (since } f \text{ respects } \cdot) \\ &= f(1_F) \text{ (since } x \neq 0_F \text{ and thus } x^{-1_F} \text{ is a multiplicative inverse for } x \text{ in } F) \\ &= 1_K \text{ (by part (1) above)} \end{aligned}$$

This shows that $f(x^{-1_F})$ is a left multiplicative inverse for $f(x)$ in K. Since the multiplicative inverse for $f(x)$ in K is unique, $f(x^{-1_F}) = (f(x))^{-1_K}$. Since x was an arbitrary nonzero element of F, the result holds for every $x \in F$ with $x \neq 0_F$. □

3.4.8 Let $f : F \to K$ be a surjection from a field F onto a field K that preserves addition and multiplication. Suppose $(\forall x \in F)(f(x) = 0 \Rightarrow x = 0)$. Show f is a field isomorphism.

Proof. From the book's definition of field isomorphism, it remains only to show that f is injective. Let $x, y \in F$, and suppose that $f(x) = f(y)$. Then

$$\begin{aligned} f(x +_F (-_F y)) &= f(x) +_K f(-_F y) \text{ (since } f \text{ preserves addition)} \\ &= f(x) +_K (-_K f(y)) \text{ (by exercise 3.4.4(2), } f(-_F y) = -_K f(y)) \\ &= f(x) +_K (-_K f(x)) \text{ (since } f(x) = f(y)) \\ &= 0_K \text{ (since } -_K f(x) \text{ is an additive inverse for } f(x) \text{ in } K) \end{aligned}$$

Now since $f(x +_F (-_F y)) = 0_K$, our assumption tells us that $x +_F (-_F y) = 0_F$. Adding y to both sides of this equation yields $x = y$. We've shown that $(\forall x, y \in F)(f(x) = f(y) \Rightarrow x = y)$. So f is injective and hence is a field isomorphism.

□