

MATH 74, FALL 2004, HOMEWORK 12 SOLUTIONS

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Due November 24

Assignment: 1,2,3, assigned in class

(1) Prove that Exercise 3.2.4(5) in your textbook is false as stated. Prove a correct version.

The exercise states that if $x < y$ with $x \neq 0$ and $y \neq 0$, then $y^{-1} < x^{-1}$. This statement is actually false for any ordered field whenever $x < 0$ and $y > 0$. The preceding claim is not difficult to prove directly; however, the easiest way to refute the exercise is just to cite a single counterexample. In the field \mathbb{R} , (which is an ordered field by an axiom), let $x = -1$ and let $y = 1$. Then we have $x < y$ with $x \neq 0$ and $y \neq 0$, but we do not have $y^{-1} < x^{-1}$, since $y^{-1} = 1^{-1} = 1$ and $x^{-1} = -1^{-1} = -1$, and it is not the case that $1 < -1$ (since $(-1) - 1 = -2 \notin P$). This shows that the exercise as stated is false.

One correct version says that if $x, y \neq 0$, and it is not the case that $x < 0$ and $y > 0$, then $x < y \Rightarrow y^{-1} < x^{-1}$.

For a more concise proof of the above, we'll first prove the following lemma:

Lemma: For any ordered field $\langle F, P \rangle$, and for any $x \in F$, $x \in P \Leftrightarrow x^{-1} \in P$

Proof. (\Rightarrow) Assume $x \in P$. Then $x \neq 0$, and so we cannot have $x^{-1} = 0$ (since we need $x^{-1} \cdot x = 1$). If $-(x^{-1}) \in P$, then $-(x^{-1}) \cdot x$ must be in P by (O3). But $-(x^{-1}) \cdot x = (-1 \cdot (x^{-1})) \cdot x = -1 \cdot ((x^{-1}) \cdot x) = -1 \cdot 1 = -1$ and we have already shown that $-1 \notin P$. This gives a contradiction, so $-(x^{-1}) \notin P$, and hence by trichotomy (O1), we must have $x^{-1} \in P$.

(\Leftarrow) The reverse direction is analogous. Simply replace every occurrence of x with x^{-1} and vice versa in the argument above. \square

Now to prove the corrected version of the exercise. Assume $x < y$, and consider three cases

(a) $x > 0$ and $y > 0$.

In this case, we have $y-x, x, y, xy \in P$. We need to show that $y^{-1} < x^{-1}$, or equivalently, that $x^{-1} - y^{-1} \in P$. Using more familiar notation, we have $x^{-1} - y^{-1} = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} = (y-x) \cdot xy^{-1} \in P$ (by the lemma $xy^{-1} \in P$, and we assumed $y-x \in P$, so the last statement follows by axiom (O3) for ordered fields).

(b) $x < 0$ and $y < 0$.

In this case, we have $y-x, -x, -y, xy \in P$. All we used in the argument (for the preceding case) to show $x^{-1} - y^{-1} \in P$ was that $y-x$ and xy^{-1} were both in P . Those two conditions also hold in this case, so the remainder of the proof in this case is the same as the proof in the previous case.

(c) $x > 0$ and $y < 0$. In this case, trichotomy gives us $y < x$, and thus our hypothesis $x < y$ can never hold. So in this case, the conditional $x < y \Rightarrow y^{-1} < x^{-1}$ is vacuously true.

The above cases are mutually exclusive and exhaustive under our assumptions, (i.e. $x, y \neq 0$, and it is not the case that $x < 0$ and $y > 0$), so this completes the proof of the corrected version. A simpler "correct version", that gives the result only when x and y are both positive or both have the same sign would also be sufficient for this homework problem.

(2) Exercise 3.3.4

(a) Does the conclusion of the NIP hold if the intervals J_n are open rather than closed?

NO! Even in \mathbb{R} , the nested sequence of open intervals $\langle (0, \frac{1}{n+1}) \rangle_{n \in \mathbb{N}}$ has empty intersection. These intervals are nested since for every $n \in \mathbb{N}$, $(0, \frac{1}{(n+1)+1}) \subseteq (0, \frac{1}{n+1})$. For every $x \leq 0$, and for every $n \in \mathbb{N}$, $x \notin (0, \frac{1}{n+1})$; and for every positive real number x , (by the Archimedean ordering property), there is an $n \in \mathbb{N}^*$ with $n \cdot 1 > \frac{1}{x}$, so that $x \notin (0, \frac{1}{n})$. So for every real number x , x fails to be in the intersection of all the intervals. So the intersection must be empty.

(b) Investigate the NIP result when there are only a finite number of intervals.

If $\langle J_n \rangle_{n=0, \dots, k}$ is a finite sequence of intervals, then the intersection $\bigcap_{n=0}^k J_n$ is exactly the last nested interval J_k . We have $J_k \subseteq \bigcap_{n=0}^k J_n$, since the intervals are nested; and we have $\bigcap_{n=0}^k J_n \subseteq J_k$ by the definition of intersection. Now in any ordered field, an interval J_k is always non-empty; hence by the above reduction, so is the intersection of any finite sequence of intervals. Note that we did not use any form of the NIP axiom to get this result. The nonempty intersection property of nested finite sequences of intervals holds in any ordered field, and it also holds when the intervals are open; since all we needed for the above proof was to show that the finite intersection of nested intervals is itself an interval, and note that all intervals in any ordered field are always non-empty.

(3) Exercise 3.3.14(1) Prove or Disprove:

(a) Each non-empty subset of \mathbb{Q} that is bounded above in \mathbb{Q} has a supremum in \mathbb{Q} .

The above statement is false. For a counterexample, the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is a non-empty subset of \mathbb{Q} that is bounded above in \mathbb{Q} , but has no supremum in \mathbb{Q} . A is non-empty since $0 \in A$. A is bounded above in \mathbb{Q} by (say) $\frac{3}{2}$, since for every $x \in \mathbb{Q}$ with $x > \frac{3}{2}$, we have $x^2 > \frac{9}{4} > 2$, so that $x \notin A$. A has no supremum in \mathbb{Q} because every upper bound b of A in \mathbb{Q} must be greater (in \mathbb{R}) than $\sqrt{2}$, so that (by the order density theorem) there is an element c of \mathbb{Q} with $\sqrt{2} <_{\mathbb{R}} c <_{\mathbb{R}} b$, showing that b is not the least upper bound of A in \mathbb{Q} .

(b) Each non-empty subset of \mathbb{Q} that is bounded above in \mathbb{Q} has a supremum in \mathbb{R} .

The above statement is true. It is an immediate consequence of the Order Completeness Theorem for \mathbb{R} , since any non-empty subset of \mathbb{Q} that is bounded above in \mathbb{Q} is also a non-empty subset of \mathbb{R} that is bounded above in \mathbb{R} , (since $\mathbb{Q} \subseteq \mathbb{R}$). Such a subset will have a supremum in \mathbb{R} by the Order Completeness Theorem for \mathbb{R} .

(c) Each non-empty subset of \mathbb{Q} that is bounded above in \mathbb{R} has a supremum in \mathbb{Q} .

The above statement is false. The counterexample from part (a) works here as well, and the argument is the same. In fact, since $\mathbb{Q} \subseteq \mathbb{R}$, it is easy to see that (c) logically implies (a). Since we've shown that (a) is false, it necessarily follows that (c) is false as well.