MATH 74, FALL 2004, HOMEWORK 10 SOLUTIONS

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Due November 10

Assignment: (Given in Class - 1,2,3,4,5)

(1) Show $+_{\mathbb{Z}}$ is well-defined on $\mathbb{Z}$.

Recall that $\equiv_{\mathbb{Z}}$ was defined on $\mathbb{N}\times\mathbb{N}$ via, for $a,b,c,d \in \mathbb{N}$, $(a,b) \equiv_{\mathbb{Z}} (c,d)$ iff $a+_{\mathbb{N}} d = b +_{\mathbb{N}} c$; $\mathbb{Z}$ was defined as $\{[(a,b)]_{\mathbb{Z}} : (a,b) \in \mathbb{N}\times\mathbb{N}\}$; and $+_{\mathbb{Z}}$ was defined on $\mathbb{Z}$ via, for $a,b,c,d \in \mathbb{N}$, $[(a,b)]_{\mathbb{Z}} +_{\mathbb{Z}} [(c,d)]_{\mathbb{Z}} = [(a +_{\mathbb{N}} c, b +_{\mathbb{N}} d)]_{\mathbb{Z}}$.

Proof. For clarity, I'll omit the subscripts $_{\mathbb{N}}$ under the + signs. Suppose $(a',b') \in [(a,b)]_{\mathbb{Z}}$, and that $(c',d') \in [(c,d)]_{\mathbb{Z}}$. We need to show that $[(a'+c', b'+d')]_{\mathbb{Z}} = [(a+c, b+d)]_{\mathbb{Z}}$. We have

$$\begin{aligned}
(a',b') \in [(a,b)]_{\mathbb{Z}} \wedge (c',d') \in [(c,d)]_{\mathbb{Z}} &\Leftrightarrow (a,b) \equiv_{\mathbb{Z}} (a',b') \wedge (c,d) \equiv_{\mathbb{Z}} (c',d') \\
&\Leftrightarrow a + b' = b + a' \wedge c + d' = d + c' \\
&\Rightarrow a + b' + c + d' = b + a' + d + c' \\
&\Leftrightarrow (a+c) + (b'+d') = (b+d) + (a'+c') \\
&\Leftrightarrow (a+c, b+d) \equiv_{\mathbb{Z}} (a'+c', b'+d') \\
&\Leftrightarrow [(a'+c', b'+d')]_{\mathbb{Z}} = [(a+c, b+d)]_{\mathbb{Z}}
\end{aligned}$$

So $+_{\mathbb{Z}}$ is well-defined on $\mathbb{Z}$. □

(2) Show $\equiv_{\mathbb{Q}}$ is not transitive on $\mathbb{Z}\times\mathbb{Z}$.

Recall that $\equiv_Q$ was defined on $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$ via $(a,b) \equiv_{\mathbb{Q}} (c,d)$ iff $a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$

Proof. Consider the pairs of integers $(0,1)$, $(0,0)$, and $(1,0)$. We have $(0,1) \equiv_Q (0,0)$ since $0 \cdot_{\mathbb{Z}} 0 = 1 \cdot_{\mathbb{Z}} 0$; and we have $(0,0) \equiv_Q (1,0)$ since $0 \cdot_{\mathbb{Z}} 0 = 0 \cdot_{\mathbb{Z}} 1$; but we do not have $(0,1) \equiv_Q (1,0)$ since $0 \cdot_{\mathbb{Z}} 0 \neq 1 \cdot_{\mathbb{Z}} 1$. This example shows that $\equiv_{\mathbb{Q}}$ is not transitive on $\mathbb{Z}\times\mathbb{Z}$. □

(3) Show $\equiv_{\mathbb{Q}}$ is transitive on $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$.

Proof. (Corrected version, idea submitted by Jacob Chestnut) Let $(a,b),(c,d),(e,f) \in \mathbb{Z}\times\mathbb{Z}\setminus\{0\}$. Assume $(a,b) \equiv_Q (c,d)$ and $(c,d) \equiv_Q (e,f)$. Then $a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$ and $c \cdot_{\mathbb{Z}} f = d \cdot_{\mathbb{Z}} e$. (Omitting the subscript $_{\mathbb{Z}}$ for clarity, $ad = bc$ and $cf = de$). Multiplying the first equation by f, we obtain $adf = bcf$. Now since $cf = de$, we get $adf = bde$. Since $d \neq 0$, the cancellation law for $\mathbb{Z}$ applies and $af = be$. So $(a,b) \equiv_Q (e,f)$. Thus $\equiv_Q$ is transitive on $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$. □

(4) Show $\equiv_{\mathbb{Z}}$ is symmetric on $\mathbb{N}\times\mathbb{N}$.

Proof. Let $(a,b),(c,d) \in \mathbb{N}\times\mathbb{N}$, and suppose $(a,b) \equiv_{\mathbb{Z}} (c,d)$. Then $a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$. Since $+_{\mathbb{N}}$ is commutative on $\mathbb{N}$, (and $=$ is symmetric on every set), $c +_{\mathbb{N}} b = d +_{\mathbb{N}} a$. So $(c,d) \equiv_{\mathbb{Z}} (a,b)$. So $\equiv_{\mathbb{Z}}$ is symmetric on $\mathbb{N}\times\mathbb{N}$. □

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(5) Show ${}^{-1}{}_{\mathbb{Q}}$ is well-defined on $\mathbb{Q}$.

Recall that $\mathbb{Q}$ was defined as $\{[(a,b)]_{\mathbb{Q}} : (a,b) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}\}$; and ${}^{-1}{}_{\mathbb{Q}}$ was defined on $\mathbb{Q}$ via $[(a,b)]_{\mathbb{Q}}^{-1} = \begin{cases} [(b,a)]_{\mathbb{Q}} & \text{if } [(a,b)]_{\mathbb{Q}} \neq 0_{\mathbb{Q}} \\ 0_{\mathbb{Q}} & \text{if } [(a,b)]_{\mathbb{Q}} = 0_{\mathbb{Q}} \end{cases}$.

Proof. Let $[(a,b)]_{\mathbb{Q}} \in \mathbb{Q}$. Suppose $(a',b') \in [(a,b)]_{\mathbb{Q}}$. We need to show that $[(a',b')]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = [(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}}$.

If $[(a,b)]_{\mathbb{Q}} \neq 0_{\mathbb{Q}}$, then $[(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = [(b,a)]_{\mathbb{Q}}$, and we have

$$\begin{aligned}
(a',b') \in [(a,b)]_{\mathbb{Q}} \wedge [(a,b)]_{\mathbb{Q}} \neq 0_{\mathbb{Q}} &\Leftrightarrow (a,b) \equiv_{\mathbb{Q}} (a',b') \wedge a \cdot_{\mathbb{Z}} 1 \neq b \cdot_{\mathbb{Z}} 0 \\
&\Leftrightarrow a \cdot_{\mathbb{Z}} b' = b \cdot_{\mathbb{Z}} a' \wedge a \neq 0_{\mathbb{Z}} \\
&\Leftrightarrow a \cdot_{\mathbb{Z}} b' = b \cdot_{\mathbb{Z}} a' \wedge a' \neq 0_{\mathbb{Z}} \text{ (since } b, b' \neq 0_{\mathbb{Z}}) \\
&\Leftrightarrow b' \cdot_{\mathbb{Z}} a = a' \cdot_{\mathbb{Z}} b \wedge a' \cdot_{\mathbb{Z}} 1 \neq b' \cdot_{\mathbb{Z}} 0 \\
&\Leftrightarrow (b',a') \equiv_{\mathbb{Q}} (b,a) \wedge [(a',b')]_{\mathbb{Q}} \neq 0_{\mathbb{Q}} \\
&\Rightarrow [(a',b')]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = [(b',a']_{\mathbb{Q}} = [(b,a)]_{\mathbb{Q}} = [(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}}
\end{aligned}$$

If $[(a,b)]_{\mathbb{Q}} = 0_{\mathbb{Q}}$, then $[(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = 0_{\mathbb{Q}}$. We have already shown that $\equiv_{\mathbb{Q}}$ is transitive on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ (HW problem 3 above). Since $(a'b') \equiv_{\mathbb{Q}} (a,b)$, and $(a,b) \equiv_{\mathbb{Q}} (0,1)$, we must have $(a',b') \equiv_{\mathbb{Q}} (0,1)$ and thus $[(a',b')]_{\mathbb{Q}} = 0_{\mathbb{Q}}$. So in this case, $[(a',b')]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = 0_{\mathbb{Q}} = [(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}}$.

In either case, we have $[(a',b')]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}} = [(a,b)]_{\mathbb{Q}}^{-1}{}_{\mathbb{Q}}$. So ${}^{-1}{}_{\mathbb{Q}}$ is well-defined on $\mathbb{Q}$. □