

A-PRIORI BOUNDS FOR KDV EQUATION BELOW $H^{-\frac{3}{4}}$

BAOPING LIU

ABSTRACT. We consider the Korteweg-de Vries Equation (KdV) on the real line, and prove that the smooth solutions satisfy a-priori local in time H^s bound in terms of the H^s size of the initial data for $s \geq -\frac{4}{5}$.

CONTENTS

1. Introduction	1
2. Function spaces	5
3. Linear and bilinear estimate	8
4. Estimating the nonlinearity	13
5. Energy conservation	29
6. Local energy decay	36
7. Finishing the proof	39
References	40

1. INTRODUCTION

In this paper, we consider the Korteweg-de Vries (KdV) equation,

$$(1) \quad \begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) = 0, & u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases}$$

The equation is invariant respect to the scaling law

$$u(t, x) \rightarrow \lambda^2 u(\lambda^3 t, \lambda x),$$

which implies the scale invariance for initial data in $\dot{H}^{-\frac{3}{2}}(\mathbb{R})$. It has been shown to be locally well-posed (LWP) in H^s for $s > -\frac{3}{4}$ by Kenig, Ponce and Vega [18] using a bilinear estimate. They constructed solution on a time interval $[0, \delta]$, with δ depending on $\|u_0\|_{H^s(\mathbb{R})}$. Later, the result was extended to global well-posedness (GWP) for $s > -\frac{3}{4}$ by Colliander, Keel, Staffilani, Takaoka and Tao [7] using the I-operator and almost conserved quantities. See also the reference [2], [6], [17], [13], [20], [3], [21] for earlier results, and [4], [15], [28] for local and global results at the endpoint $s = -\frac{3}{4}$.

In [27], Nakanishi, Takaoka and Tsutsumi showed that the essential bilinear estimate fails if $s < -\frac{3}{4}$. In fact, Colliander, Christ and Tao [4] proved the illposedness of the \mathbb{R} -valued KdV equation for $s < -\frac{3}{4}$. They showed that the solution map fails to be uniform continuous. See [19] for the corresponding result for the \mathbb{C} -valued KdV equation.

On the other hand, the same question was posed in the periodic setting ($u : \mathbb{T} \times [0, T] \rightarrow \mathbb{R}$), where we have the result $s \geq -1/2$ LWP[18] and GWP[7]. Also, Kappeler and Topalov [16], using the inverse scattering method [12], proved GWP for initial data in $H^\beta(\mathbb{T})$, $\beta \geq -1$ in the sense that the solution map is C^0 globally in time. Their proof

2000 *Mathematics Subject Classification.* 35Q55.

The author was supported in part by NSF grant DMS0801261.

depends heavily on the complete integrability of the KdV equation. Interested readers are also referred to the work of Lax and Levermore [24], Deift and Zhou [9], [10]. There they used inverse scattering and Riemann-Hilbert methods to study the semiclassical limit of the completely integrable equations.

In addition, there is an interesting result by Molinet and Ribaud [25] on the initial-value problem for KdV-Burgers equation.

$$(2) \quad \begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^2) - \partial_x^2 u = 0, & t \in \mathbb{R}_+, x \in \mathbb{R} \text{ or } \mathbb{T}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases}$$

They showed that (2) is GWP in the space $H^s(\mathbb{R})$ for $s > -1$, and ill-posed when $s < -1$ in the sense that the corresponding solution map is not C^2 . This is a bit surprising since the initial-value problem for the Burgers equation

$$(3) \quad \begin{cases} \partial_t u + \partial_x(u^2) - \partial_x^2 u = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases}$$

is known to be LWP in the space $H^s(\mathbb{R})$ for $s \geq -\frac{1}{2}$, and is ill-posed in $H^s(\mathbb{R})$ for $s < -\frac{1}{2}$, see references [1] and [11]. Notice that the critical result for Burgers equation (3) agrees with prediction from usual scaling arguments. While KdV-Burgers equation (2) has no scaling invariance, the sharp result by Molinet and Ribaud $s = -1$ is lower than $s = -\frac{3}{4}$ for KdV, and $s = -\frac{1}{2}$ for Burgers equation.

From the results of Colliander-Keel-Staffilani-Takaoka-Tao [7], Colliander-Christ-Tao [4], Kappeler-Topalov [16] and Molinet-Ribaud [25], it seems reasonable to look for some well-posedness result of KdV equation (1) in $H^s(\mathbb{R})$, in the range $-1 \leq s < -\frac{3}{4}$, with some continuous but not uniform continuous dependence on the initial data. On the other hand, notice the recent work of Molinet [26], where he showed that the solution map can not be continuously extended in $H^s(\mathbb{R})$ when $s < -1$.

Another related topic is one dimensional cubic Nonlinear Schrödinger equation (NLS)

$$(4) \quad \begin{cases} i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, & u : \mathbb{R} \times [0, T] \rightarrow \mathbb{C}, \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases}$$

The NLS has scaling invariance for initial data in $\dot{H}^{-\frac{1}{2}}(\mathbb{R})$. It has GWP for initial data in $u_0 \in L^2$ and locally in time the solution has a uniform Lipschitz dependence on the initial data in balls. But below this scale, it has been shown that uniform dependence fails [4],[19]. Koch and Tataru [23] proved an a-priori local-in-time bounds for initial data in $H^s, s \geq -\frac{1}{4}$. Similar results were previously obtained by Koch and Tataru [22] for $s \geq -\frac{1}{6}$, and by Colliander, Christ and Tao [5] for $s > -\frac{1}{12}$. These a-priori estimates ensure that the equation is satisfied in the sense of distributions even for weak limits, and hence they also obtain existence of global weak solutions without uniqueness.

Inspired by the results above, we look at the KdV equation with initial data in H^s when $s < -\frac{3}{4}$, and prove that the solution satisfies a-priori local in time H^s bounds in terms of the H^s size of the initial data, for $s \geq -\frac{4}{5}$.

Theorem 1.1. (*A-priori bound*) *Let $s \geq -\frac{4}{5}$. For any $M > 0$ there exists time T and constant C , so that for any initial data in $H^{-\frac{3}{4}}$ satisfying*

$$\|u_0\|_{H^s} < M,$$

there exists a solution $u \in C([0, T], H^{-\frac{3}{4}})$ to the KdV equation which satisfies

$$(5) \quad \|u\|_{L_t^\infty H^s} \leq C \|u_0\|_{H^s}.$$

Using the uniform bound (5), together with the uniform bound on nonlinearity

$$\|\chi_{[-T,T]}u\|_{X^s \cap X_{le}^s} + \|\chi_{[-T,T]}\partial_x(u^2)\|_{X^s \cap X_{le}^s} \lesssim \|u_0\|_{H^s},$$

which come as a byproduct of our analysis in the previous theorem, one may also prove the existence of weak solution following a similar argument as in [5].

Theorem 1.2. (*Existence of weak solution*) *Let $s \geq -\frac{4}{5}$. For any $M > 0$ there exists time T and constant C , so that for any initial data in H^s satisfying*

$$\|u_0\|_{H^s} < M,$$

there exists a weak solution $u \in C([0, T], H^s) \cap (X^s \cap X_{le}^s)$ to the KdV equation which satisfies

$$\|u\|_{L_t^\infty H^s} + \|\chi_{[-T,T]}u\|_{X^s \cap X_{le}^s} + \|\chi_{[-T,T]}\partial_x(u^2)\|_{X^s \cap X_{le}^s} \leq C\|u\|_{H^s}.$$

Remark 1.3. We can always rescale the initial data and hence just need to prove the theorems in case $M \ll 1$.

We begin with a Littlewood-Paley frequency decomposition of the solution u ,

$$u = \sum_{\lambda \geq 1, \text{ dyadic}} u_\lambda$$

Here we put all frequencies smaller than 1 into one piece.

For each λ we also use a spatial partition of unity on the λ^{4s+5} scale

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x), \quad \chi_j^\lambda(x) = \chi(\lambda^{-4s-5}x - j),$$

with $\chi(x) \in C_0^\infty(-1, 1)$.

In order to prove the theorem, we need Banach spaces

- X^s and X_{le}^s to measure the regularity of the solution u . The first one measures dyadic pieces of the solution on a frequency dependent timescale, and the second one measures the spatially localized size of the solution on unit time scale. They are similar to the ones used by Koch and Tataru in [23].
- The corresponding Y^s and Y_{le}^s to measure the regularity of the nonlinear term.
- Energy spaces

$$\|u\|_{l_\lambda^2 L_t^\infty H^s}^2 = \sum_{\lambda \geq 1} \lambda^{2s} \|u_\lambda\|_{L_t^\infty L_x^2}^2,$$

and a local energy space

$$\|u\|_{l_\lambda^2 l_j^\infty L_t^2 H^{-s-\frac{3}{2}}}^2 = \sum_{\lambda \geq 1} \lambda^{-2s-5} \sup_j \|\chi_j^\lambda \partial_x u_\lambda\|_{L_{x,t}^2}^2.$$

With the spaces above, we will prove the following three propositions.

The first one is about the linear equation.

Proposition 1.4. *The following energy estimates hold for (1):*

$$(6) \quad \|u\|_{X^s} \lesssim \|u\|_{l_\lambda^2 L_t^\infty H^s} + \|(\partial_t + \partial_x^3)u\|_{Y^s},$$

$$(7) \quad \|u\|_{X_{le}^s} \lesssim \|u\|_{l_\lambda^2 l_j^\infty L_t^2 H^{-s-\frac{3}{2}}} + \|(\partial_t + \partial_x^3)u\|_{Y_{le}^s}.$$

The second one controls the nonlinearity.

Proposition 1.5. *Let $s \geq -1$ and $u \in X^s \cap X_{le}^s$ be a solution to equation (1), then*

$$(8) \quad \|\partial_x(u^2)\|_{Y^s \cap Y_{le}^s} \lesssim \|u\|_{X^s \cap X_{le}^s}^2 + \|u\|_{X^s \cap X_{le}^s}^3.$$

Finally, to close the argument we need to propagate the energy norms.

Proposition 1.6. *Let $s \geq -\frac{4}{5}$ and u be a solution to the (1) with*

$$\|u\|_{l_\lambda^2 L_t^\infty H^s} \ll 1.$$

Then we have the bound for energy norm

$$(9) \quad \|u\|_{l_\lambda^2 L_t^\infty H^s} \lesssim \|u_0\|_{H^s} + \sum_{k=3}^6 \|u\|_{X^s \cap X_{le}^s}^k,$$

and respectively the local energy norm

$$(10) \quad \|u\|_{l_{\lambda^j}^2 L_t^\infty H^{-s-\frac{3}{2}}} \lesssim \|u_0\|_{H^s} + \sum_{k=3}^6 \|u\|_{X^s \cap X_{le}^s}^k.$$

We organize our paper as follows: In section 2, we will define the spaces X^s, X_{le}^s , respectively Y^s, Y_{le}^s , and establish the linear mapping properties in Proposition 1.4. In section 3 we discuss the linear and bilinear Strichartz estimates for free solutions, and collect some useful estimates related to our spaces. In section 4 we control the nonlinearity as in Proposition 1.5. In sections 5, 6 we use a variation of the I-method to construct a quasi-conserved energy functional and compute its behavior along the flow, thus proving Proposition 1.6.

Now we end this section by showing that the three propositions imply Theorem 1.1.

Proof. Since $u_0 \in H^{-\frac{3}{4}}$, we can solve the equation iteratively to get a solution up to time 1, which implies that $u \in l_\lambda^2 L_t^\infty H^s$ and also that $u \in X^s \cap X_{le}^s$, because the space we use has the nesting property $X^{s_1} \subset X^{s_2}, s_1 < s_2$, same for $l_\lambda^2 L_t^\infty H^s$ and X_{le}^s .

Then we use a continuity argument. Suppose ϵ is a small constant and $\|u_0\|_{H^s(\mathbb{R})} < \epsilon$. Take a small δ , so that $\epsilon \ll \delta \ll 1$, denote

$$A = \{T \in [0, 1]; \|u\|_{l_\lambda^2 L_t^\infty H^s([0, T] \times \mathbb{R})} \leq 2\delta, \|u\|_{X^s \cap X_{le}^s([0, T] \times \mathbb{R})} \leq 2\delta\}$$

and we just need to prove $A = [0, 1]$. Clearly A is not empty and $0 \in A$. We need to prove that it is closed and open.

From definition in the next section, we can see that the norms used in A are continuous with respect to T , so A is closed.

Secondly, if $T \in A$, we have by proposition 1.6

$$\|u\|_{l_\lambda^2 L_t^\infty H^s([0, T] \times \mathbb{R})} \lesssim \epsilon + \delta^3,$$

and by proposition 1.4 and 1.5, we have

$$\|u\|_{X^s \cap X_{le}^s([0, T] \times \mathbb{R})} \lesssim \epsilon + \delta^2 + \delta^3.$$

So by taking ϵ and δ sufficiently small, we can conclude that

$$\|u\|_{l_\lambda^2 L_t^\infty H^s([0, T] \times \mathbb{R})} \leq \delta, \|u\|_{X^s \cap X_{le}^s([0, T] \times \mathbb{R})} \leq \delta.$$

Since the norms are continuous with respect to T , it follows that a neighborhood of T is in A . Hence we proved Theorem 1.1. \square

2. FUNCTION SPACES

The idea here follows the work of Koch and Tataru [22][23]. We begin with some heuristic argument: If the initial data in (1) has norm $\|u_0\|_{H^{-\frac{3}{4}}} \leq 1$, then the equation can be solved iteratively up to time 1. Now when taking the same problem with initial data $u_0 \in H^s$, $s < -\frac{3}{4}$, localized at frequency λ , the initial data will have norm $\|u\|_{H^{-\frac{3}{4}}} \leq \lambda^{-s-\frac{3}{4}}$. Now if we rescale it to have $H^{-\frac{3}{4}}$ norm 1, we see that the evolution will still be described by linear dynamics on time intervals of size λ^{4s+3} . So we decompose our solution into frequency pieces $u = \sum_{\lambda \geq 1} u_\lambda$ and measure each piece uniformly in size λ^{4s+3} time intervals.

Another important idea is to look at waves of frequency λ travelling with speed λ^2 , so for time λ^{4s+3} , it travels in spatial region of size λ^{4s+5} . So we also decompose the space into a grid of size λ^{4s+5} by using the partition of unity

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x).$$

$\chi_j^\lambda(x)$ is defined as before, and it's easy to see that the spatial scales increase with λ .

Bourgain's $X^{s,b}$ spaces are defined by

$$\|u\|_{X_{\tau=\xi^3}^{s,b}}^2 = \int |\hat{u}(\tau, \xi)|^2 (1 + |\xi|^2)^s (1 + |\tau - \xi^3|^2)^b d\tau d\xi.$$

We will use a modified version of it on frequency or modulation dyadic pieces.

We start with spatial Littlewood-Paley decomposition,

$$u = \sum_{\lambda \geq 1 \text{ dyadic}} P_\lambda u = \sum_{\lambda \geq 1 \text{ dyadic}} u_\lambda.$$

Also, we will use the decomposition with respect to modulation $|\tau - \xi^3|$

$$1 = \sum_{\lambda \geq 1 \text{ dyadic}} Q_\lambda.$$

Both decompositions are inhomogeneous, and uniformly bounded on $X^{s,b}$ spaces.

Denote $\eta_I(t)$ as sharp time cutoff with respect to any time interval I . Let I_λ be a time interval of size λ^{4s+3} , then we use $\eta_\lambda(t)$ or η_λ as a simplified notation for $\eta_{I_\lambda}(t)$. And $\chi^\lambda(x)$ is the smooth space cutoff with respect to spatial intervals of size λ^{4s+5} as before.

Define $|D|^\alpha$ to be the multiplier operator with Fourier multiplier $|\xi|^\alpha$. We use the convention that $f \in |D|^{-s} X \Leftrightarrow \|f\|^2 = \sum \lambda^{2s} \|f_\lambda\|_X^2 < \infty$ in our definitions.

Definition 2.1. The spaces we use contain the following elements:

- (i) Given an interval $I = [t_0, t_1]$, we define the space

$$\begin{aligned} \|\phi\|_{X^{0,1}[I]}^2 &= \|\phi(t_0)\|_{L^2}^2 + |I| \|(\partial_t + \partial_x^3)\phi\|_{L^2[I]}^2, \\ \|\phi\|_{X^1[I]}^2 &= \sum_{\lambda} \lambda^{2s} \|\phi_\lambda\|_{X^{0,1}[I]}^2. \end{aligned}$$

$X^1[I]$ is used to control the low modulation part of the solution in a classical space, which is extendable on the real line.

- (ii) We use sums of spaces, i.e. $\|u\|_{A+B} = \inf\{\|u_1\|_A + \|u_2\|_B, u = u_1 + u_2\}$ to define

$$Z = (X_{\tau=\xi^3}^{-3-4s, 2s+2} + |D|^{-2s-2} X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}) \cap |D| L_{t,x}^\infty.$$

Z will always be used for very high modulations ($\geq |\xi|^3$), i.e. in what are called the elliptic region.

(iii) The space S is defined by putting high and low modulation in different spaces.

$$\|u_\lambda\|_S = \lambda^{3s + \frac{3}{2s} + \frac{11}{2}} \|Q_{\sigma \leq \lambda^{4 + \frac{3}{2s}}} u_\lambda\|_{L^2_{t,x}} + \|Q_{\lambda^{4 + \frac{3}{2s}} \leq \sigma \leq \frac{1}{10} \lambda^3} u_\lambda\|_{X_{\tau=\xi^3}^{-s,1+s}} + \|Q_{\geq \frac{1}{10} \lambda^3} u_\lambda\|_Z.$$

The good thing here is space S is stable with respect to sharp time truncations, the L^2 structure deals with the tails when multiplying by a time-interval cutoff.

In particular, we have

$$\|\eta_\lambda(t)u_\lambda\|_S \lesssim \|u_\lambda\|_S.$$

(iv) Let $X_\lambda[I] = X^1[I] + S[I]$. Now we can define X^s norm in a time interval I by measuring the dyadic parts of u on small frequency-dependent time scales

$$\|u\|_{X^s[I]}^2 = \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset I} \|\eta_J(t)u_\lambda\|_{X_\lambda[J]}^2,$$

X_{le}^s measures the spatially localized size of the solution on the unit time scale

$$\|u\|_{X_{le}^s[I]}^2 = \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset I} \|\chi_j^\lambda(x)\eta_J(t)u_\lambda\|_{X_\lambda[J]}^2.$$

(v) Correspondingly, we have the space Y^s and Y_{le}^s

$$\|u\|_{Y^s[I]}^2 = \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset I} \|\eta_J(t)u_\lambda\|_{Y_\lambda[J]}^2,$$

$$\|u\|_{Y_{le}^s[I]}^2 = \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset I} \|\chi_j^\lambda(x)\eta_J(t)u_\lambda\|_{Y_\lambda[J]}^2.$$

Here

$$Y_\lambda[I] = |D_x|^{-s}|I|^{-\frac{1}{2}}L^2 + DS[I],$$

$DS = \{f = (\partial_t + \partial_x^3)u; u \in S\}$ with the induced norm and $DS[I] = \{f|_I, f \in DS\}$.

Through our paper, we will mostly drop the interval I in the notation if $I = [0, 1]$.

Remark 2.2. We look at each of the spaces in detail.

(1) $X^1[I]$ is not stable with respect to sharp time truncation as it would cause jumps at both ends. Also in order to talk about modulation, we need to extend functions so that they are defined on the real line. To fix the problem, we define

$$\begin{aligned} \|\phi\|_{X_I^{0,1}}^2 &= \|\phi(t_0)\|_{L^2}^2 + |I| \|(\partial_t + \partial_x^3)\phi\|_{L^2_{t,x}}^2, \\ \|\phi\|_{X_I^1}^2 &= \sum_{\lambda} \lambda^{2s} \|\phi_\lambda\|_{X_I^{0,1}}^2. \end{aligned}$$

Now take any function $u \in X^1[I]$, denote $u_E = \theta(t)\tilde{u}$, where \tilde{u} is the extension of u by free solutions with matching data at both ends and $\theta(t)$ is a smooth cutoff on a neighborhood of I . Clearly, $\|u\|_{X^1[I]} = \|u_E\|_{X_I^1}$, and when we talk about function $u \in X^1[I]$, we always mean u_E .

While $S[I]$ is stable with sharp time cutoff, $DS[I]$ is not. We can extend functions in $S[I]$ by 0 outside the interval. And from the definition, functions in $DS[I]$ always come from interval restriction of functions in DS , which are defined on the real line.

(2) The space $X^1[I]$ is compatible with solutions to the homogeneous equation. Namely for any smooth time cutoff $\eta(t)$, we can prove

$$\|\eta(t)e^{t\partial_x^3}u_0\|_{X^1[I]} \lesssim \|u_0\|_{H^s},$$

It is also compatible with energy estimates

$$\|u\|_{L_t^\infty(I; H^s)} \lesssim \|u\|_{X^1[I]}.$$

- (3) In our paper, we will ignore the subscript notation $\tau = \xi^3$ in the $X_{\tau=\xi^3}^{s,b}$ space except for the special curve $\tau = \frac{1}{4}\xi^3$ which arises when two high frequency wave interact and generate an almost equally high frequency.
- (4) Since we are using sums of spaces, it is interesting to compare the norms of these spaces. We note the following facts by Bernstein inequality.

$$\left\{ \begin{array}{ll} \|u_\lambda\|_{X_\lambda[I_\lambda]} \approx \|u_\lambda\|_{X^1[I_\lambda]}, & \text{when } |\tau - \xi^3| \lesssim \lambda^{4+\frac{3}{2s}}, \\ \|u_\lambda\|_Z \approx \|u_\lambda\|_{|D|^{-2s-2}X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4},\frac{1}{4}} \cap |D|L_{t,x}^\infty}, & \text{when } |\tau - \frac{1}{4}\xi^3| \leq \frac{1}{10}\lambda^3, \\ \|u_\lambda\|_Z \approx \|u_\lambda\|_{X^{-3-4s,2s+2} \cap |D|L_{t,x}^\infty} \approx \|u_\lambda\|_{X^{-s,1+s}}, & \text{when } |\tau - \xi^3| \approx \frac{1}{10}\lambda^3. \end{array} \right.$$

The X^1 and S norm balance at modulation $|\tau - \xi^3| \approx \lambda^{4+\frac{3}{2s}}$, which is also where we split S into the L^2 structure and $X^{-s,1+s}$. Hence whenever we split into an X^1 and an S part, we always assume the S part have modulation larger than $\lambda^{4+\frac{3}{2s}}$ (which is larger than λ^2). The same applies for $|D|^{-s}|I|^{-\frac{1}{2}}L^2$ and DS .

The third equality is because when modulation is around $\frac{1}{10}\lambda^3$, the Z norm is in fact $X^{-3-4s,2s+2} \cap |D|L_{t,x}^\infty$. Using Bernstein, we can see that it matches with $X^{-s,1+s}$.

Now let us prove Proposition 1.4.

Proof. It suffices to prove the Proposition for a fixed dyadic frequency λ . We restrict our attention to time interval $J = [a, b]$ with size λ^{4s+3} , and we need to prove that

$$(11) \quad \|u_\lambda\|_{X_\lambda[J]} \lesssim \|u_\lambda\|_{L_t^\infty H^s} + \|f_\lambda\|_{Y_\lambda[J]}, \quad (\partial_t + \partial_x^3)u_\lambda = f_\lambda.$$

We now split f_λ into two components

$$f_\lambda = f_{1,\lambda} + f_{2,\lambda}, \quad f_{1,\lambda} \in L^2, \quad f_{2,\lambda} \in DS.$$

Pick v_λ such that $(\partial_t + \partial_x^3)v_\lambda = f_{2,\lambda}$, $\|f_{2,\lambda}\|_{DS} = \|v_\lambda\|_S$. (or $(v_\lambda^i)_1^\infty$ with $\|v_\lambda^i\|_S \rightarrow \|f_{2,\lambda}\|_{DS}$.)

Then we have $(\partial_t + \partial_x^3)(u_\lambda - v_\lambda) = f_{1,\lambda}$.

Notice the fact that, for any function ϕ and time interval $I = [t_0, t_1]$

$$\|\phi_\lambda\|_{X^1[I]} \approx \lambda^s |I|^{-\frac{1}{2}} \|\phi_\lambda\|_{L_{t,x}^2[I]} + \lambda^s |I|^{\frac{1}{2}} \|(\partial_t + \partial_x^3)\phi_\lambda\|_{L_{t,x}^2[I]}.$$

So we get

$$\begin{aligned} \|u_\lambda\|_{X_\lambda[J]} &\lesssim \|u_\lambda - v_\lambda\|_{X^1[J]} + \|v_\lambda\|_{S[J]} \\ &\lesssim \lambda^s |J|^{-\frac{1}{2}} \|u_\lambda - v_\lambda\|_{L_{t,x}^2[J]} + \|f_{1,\lambda}\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2[J]} + \|f_{2,\lambda}\|_{DS[J]} \\ &\lesssim \|u_\lambda\|_{L_t^\infty H^s} + \|f_{1,\lambda}\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2[J]} + \|f_{2,\lambda}\|_{DS[J]}. \end{aligned}$$

Here we used the fact

$$\lambda^s |J|^{-\frac{1}{2}} \|v_\lambda\|_{L_{t,x}^2[J]} \lesssim \|v_\lambda\|_{S[J]},$$

which can be checked easily.

For the second estimate about local energy space, we can still localize to fixed frequency, and need to show that

$$(12) \quad \sup_j \sum_{J \subset I}^{|J|=\lambda^{4s+3}} \|\chi_j^\lambda u_\lambda\|_{X_\lambda[J]}^2 \lesssim \sup_j \sum_{J \subset I}^{|J|=\lambda^{4s+3}} (\lambda^{-2s-5} \|\chi_j^\lambda \partial_x u_\lambda\|_{L_{t,x}^2[J]}^2 + \|\chi_j^\lambda f_\lambda\|_{Y_\lambda[J]}^2)$$

To prove the estimate, let us consider the inhomogeneous problem on interval $J = [a, b]$ of size $|J| = \lambda^{4s+3}$,

$$(\partial_t + \partial_x^3)u_\lambda^k = P_\lambda \chi_k^\lambda f_\lambda, \quad u_\lambda^k(a) = \chi_k^\lambda u_{0,\lambda}$$

and prove that

$$(13) \quad \|\chi_j^\lambda u_\lambda^k\|_{X_\lambda[J]} \lesssim \langle j-k \rangle^{-N} (\lambda^s |J|^{-\frac{1}{2}} \|\chi_k^\lambda u_\lambda^k\|_{L_{t,x}^2} + \|\chi_k^\lambda f_\lambda\|_{Y_\lambda[J]}).$$

When $j \approx k$, it is essentially the same as (11). Notice in the process of proving (11), we get

$$\|u_\lambda\|_{X_\lambda[J]} \lesssim \lambda^s |J|^{-\frac{1}{2}} \|u_\lambda\|_{L_{t,x}^2[J]} + \|f_\lambda\|_{Y_\lambda[J]}.$$

When $|j-k| \gg 1$, it follows from the rapid decay estimate on the kernel K_{jk} of $\chi_j^\lambda e^{t\partial_x^3} P_\lambda \chi_k^\lambda$:

$$|K_{jk}(t, x, y)| \lesssim \lambda^{-N} \langle j-k \rangle^{-N}, \quad |t| \leq \lambda^{4s+3}.$$

Since $u_\lambda = \sum_k u_\lambda^k$, so we sum up k in (13), and get

$$\|\chi_j^\lambda(x) u_\lambda\|_{X_\lambda[J]} \lesssim \sum_k \langle j-k \rangle^{-N} (\lambda^s |J|^{-\frac{1}{2}} \|\chi_k^\lambda u_\lambda\|_{L_{t,x}^2[J]} + \|\chi_k^\lambda(x) f_\lambda\|_{Y_\lambda[J]}),$$

which is equivalent to (12). □

3. LINEAR AND BILINEAR ESTIMATE

In this section, we look at solutions to the Airy equation,

$$(14) \quad \partial_t u + \partial_x^3 u = 0, \quad u(0, x) = u_0(x).$$

Solutions satisfy the following Strichartz and local smoothing estimate [20][30].

Proposition 3.1. *Let (q, r) be Strichartz pair*

$$(15) \quad \frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 4 \leq q \leq \infty.$$

Then the solution of the Airy equation satisfies

$$\|u\|_{L_t^q L_x^r} \lesssim \| |D|^{-\frac{1}{q}} u_0 \|_{L^2}.$$

Proposition 3.2. *The solution of the Airy equation satisfies the local smoothing estimate*

$$\|\partial_x u\|_{L_x^\infty L_t^2} \lesssim \|u_0\|_{L^2}.$$

Proposition 3.3. *The solution of the Airy equation satisfies maximal function estimate*

$$\|\partial_x^{-\frac{1}{4}} u\|_{L_x^4 L_t^\infty} \lesssim \|u_0\|_{L^2}.$$

Once we have estimates for linear equation, we can extend it to X^1 .

Corollary 3.4. *Let (q, r) be a Strichartz pair as in relation (15). Then we have*

$$(16) \quad \|\eta_I(t) u_\lambda\|_{L_t^q L_x^r} \lesssim \lambda^{-\frac{1}{q}-s} \|u_\lambda\|_{X^1[I]},$$

Also, the following smoothing estimate and maximal function estimate hold

$$(17) \quad \|\eta_I(t) u_\lambda\|_{L_x^\infty L_t^2} \lesssim \lambda^{-1-s} \|u_\lambda\|_{X^1[I]},$$

$$(18) \quad \|\eta_I(t) u_\lambda\|_{L_x^4 L_t^\infty} \lesssim \lambda^{\frac{1}{4}-s} \|u_\lambda\|_{X^1[I]},$$

Proof. The results follow by expanding u_λ via Duhamel's formula.

If $(\partial_t + \partial_x^3)u_\lambda = f$, then

$$u_\lambda = e^{-t\partial_x^3}u_\lambda(t_0) + \int_{t_0}^t e^{-(t-s)\partial_x^3}f(s)ds.$$

From Strichartz estimate, and its dual form - the inhomogeneous Strichartz estimate, see Theorem 2.3 in Tao [30] section 2.3, and we get

$$\begin{aligned} \|\eta_I(t)u_\lambda\|_{L_t^q L_x^r} &\lesssim \|\eta_I(t)e^{-t\partial_x^3}u_\lambda(t_0)\|_{L_t^q L_x^r} + \lambda^{-\frac{1}{q}}\|\eta_I(t)f\|_{L_t^1 L_x^2} \\ &\lesssim \lambda^{-\frac{1}{q}-s}\|u_\lambda\|_{X^1[I]}. \end{aligned}$$

We can prove the local smoothing and maximal estimate in the same way. \square

We will also need the bilinear estimate as in [14].

Proposition 3.5. *Let I_\pm^s be defined by its Fourier transform in the space variable:*

$$\mathcal{F}_x I_\pm^s(f, g)(\xi) := \int_{\xi_1 + \xi_2 = \xi} |\xi_1 \pm \xi_2|^s \hat{f}(\xi_1) \hat{g}(\xi_2) d\xi_1.$$

Assume u, v be two solutions to the Airy equation with initial data u_0, v_0 . Then we have the bilinear estimate

$$(19) \quad \|I_+^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L_{x,t}^2} \lesssim \|u_0\|_{L_x^2} \|v_0\|_{L_x^2}.$$

Proof. For a solution to the Airy equation, we can write down its Fourier transform,

$$\tilde{u} = \delta(\tau - \xi^3) \hat{u}_0, \quad \tilde{v} = \delta(\tau - \xi^3) \hat{v}_0.$$

Then

$$\widetilde{I_+^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)}(\tau, \xi) = \int_{\xi_1 + \xi_2 = \xi} |\xi_1 + \xi_2|^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}} \hat{u}_0(\xi_1) \hat{v}_0(\xi_2) \delta(\tau - \xi_1^3 - \xi_2^3) d\xi_1.$$

Let us make change of variable $\xi_1 + \xi_2 = \xi, \tau - \xi_1^3 - \xi_2^3 = \eta$.

With τ, ξ fixed, we have

$$d\xi_1 = \frac{1}{3|\xi_1 + \xi_2||\xi_1 - \xi_2|} d\eta,$$

hence we get

$$\widetilde{I_+^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)}(\tau, \xi) = \frac{1}{3|\xi_1 + \xi_2|^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}}} \hat{u}_0(\xi_1) \hat{v}_0(\xi_2).$$

Now ξ_1, ξ_2 are solutions to

$$\xi_1 + \xi_2 = \xi, \quad \xi_1^3 + \xi_2^3 = \tau,$$

So we have

$$d\tau d\xi = 3|\xi_1^2 - \xi_2^2| d\xi_1 d\xi_2,$$

and it follows

$$\|I_+^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L_{x,t}^2} \lesssim \|u_0\|_{L_x^2} \|v_0\|_{L_x^2}.$$

\square

Remark 3.6. Propostion 3.5 gives us the usual L^2 estimate on product of two free solutions whenever they have frequency separation, i.e. $|\xi_1 \pm \xi_2| \neq 0$, $\xi_1 \in \text{supp } \hat{u}$, $\xi_2 \in \text{supp } \hat{v}$.

It is very useful especially when we localize the solutions into dyadic frequency pieces, then the operators I_{\pm}^s can be simply replaced by scalar multiplication. We have the following cases:

When $|\xi_1| \approx \mu, |\xi_2| \approx \lambda, \mu \gg \lambda$, then we get

$$(20) \quad \|uv\|_{L_{t,x}^2} \lesssim \mu^{-1} \|u_0\|_{L^2} \|v_0\|_{L^2}.$$

When $|\xi_1| \approx |\xi_2| \approx \lambda$, and ξ_1, ξ_2 have opposite sign, so the output has frequency $|\xi_1 + \xi_2| \approx \alpha \lesssim \lambda$, then we get

$$(21) \quad \|uv\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \|u_0\|_{L^2} \|v_0\|_{L^2}.$$

In case $|\xi_1| \approx |\xi_2| \approx \lambda$, but ξ_1, ξ_2 have same sign, the output lies close to a new curve $\tau = \frac{1}{4}\xi^3$. Following the idea in [18], we have the following Proposition.

Proposition 3.7. *Assume u, v are two smooth solutions to the Airy equation with initial data u_0, v_0 , localized at frequencies about the comparable size and also the same sign, and I be an interval of size less than 1, then we have the following estimate*

$$(22) \quad \|\eta_I(t)uv\|_{X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}} \lesssim \|u_0\|_{L_x^2} \|v_0\|_{L_x^2}.$$

Proof. The proof is essentially the same as Proposition 3.5. There we first take two frequency really close, but have small separation, i.e. $|\xi_1 - \xi_2| \geq \epsilon$, so that all the calculation are still true, and we get the estimate (19). Notice that

$$\xi_1 + \xi_2 = \xi, \quad \xi_1^3 + \xi_2^3 = \tau.$$

So we solve for ξ_1, ξ_2 and get $|(\tau - \frac{1}{4}\xi^3)\xi|^{\frac{1}{2}} = \frac{3}{4}|(\xi_1 + \xi_2)(\xi_1 - \xi_2)|$, which is exactly the multiplier we have in the space $\dot{X}_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}$. Then we take the limit as $\epsilon \rightarrow 0$, and the norm converges as long as we are considering smooth functions. So we get

$$(23) \quad \|uv\|_{\dot{X}_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}} \lesssim \|u_0\|_{L_x^2} \|v_0\|_{L_x^2}.$$

To pass to nonhomogeneous space, notice the following estimate

$$\|\eta_I(t)f\|_{L_{t,x}^2} \lesssim |I|^{\frac{1}{4}} \|f\|_{L_t^4 L_x^2} \lesssim \|f\|_{\dot{X}_{\tau=\frac{1}{4}\xi^3}^{0, \frac{1}{4}}}.$$

The last inequality is by Sobolev embedding. \square

In Proposition 3.9, we will extend these estimates (20) (21) from free solutions to functions in X^1 .

Now we list some L^p estimates, which are mostly straightforward.

Proposition 3.8. *When $-1 \leq s \leq -\frac{3}{4}$, we have the following estimates.*

$$(24) \quad \|\eta_I(t)Q_{\sigma}u_{\lambda}\|_{L_{x,t}^2} \lesssim \sigma^{-1} \lambda^{-s} |I|^{-\frac{1}{2}} \|u_{\lambda}\|_{X^1},$$

$$(25) \quad \|Q_{\lambda^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \lambda^3} u_{\lambda}\|_{L_{x,t}^2} \lesssim \lambda^{-3s-\frac{3}{2s}-\frac{11}{2}} \|u_{\lambda}\|_{X^{-s,1+s}} \lesssim \lambda^{-2-s} \|u_{\lambda}\|_{X^{-s,1+s}},$$

$$(26) \quad \|Q_{\lambda^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \lambda^3} u_{\lambda}\|_{L_{x,t}^{\infty}} \lesssim \lambda^{-2s-1} \|u_{\lambda}\|_{X^{-s,1+s}},$$

$$(27) \quad \|u_{\lambda}\|_{L_{x,t}^3} \lesssim \lambda^{-\frac{1}{3}-2s-2} \|u_{\lambda}\|_{|D|^{-2s-2} X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}},$$

$$(28) \quad \|Q_{\gtrsim\lambda^3} u_\lambda\|_{L_{t,x}^2} \lesssim \lambda^{-2s-3} \|u_\lambda\|_{X^{-3-4s, 2s+2}},$$

$$(29) \quad \|Q_{\gtrsim\lambda^3} u_\lambda\|_{L_{t,x}^q} \lesssim \lambda^{1-\frac{4(s+2)}{q}} \|u_\lambda\|_{X^{-3-4s, 2s+2} \cap |D| L_{t,x}^\infty}, \quad 2 \leq q \leq p,$$

$$(30) \quad \|Q_{\gtrsim\lambda^3} u_\lambda\|_{L_{t,x}^3} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}} \|u_\lambda\|_Z,$$

$$(31) \quad \|Q_{\gtrsim\lambda^3} u_\lambda\|_{L_{t,x}^6} \lesssim \lambda^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\lambda\|_Z.$$

Proof. The proofs are mostly simple. (24) is by definition combined with the size of the interval. (25) (26)(28) are consequences of Bernstein inequality. (29) is by interpolating the L^2 estimate with L^p .

The only nontrivial one is (27), Similar to [31], we look at the operator $S(\sigma)$ defined by multiplier $e^{\sigma^2} \Gamma(\sigma) \frac{1}{(\tau - \frac{1}{4}\xi^3 \pm i0)^\sigma}$, where $\Gamma(\sigma)$ is the complex valued Gamma-function. We claim that

$$\begin{aligned} S(0 + iy)P_\lambda &: L_{t,x}^2 \rightarrow L_{x,t}^2, \\ S\left(\frac{3}{2} + iy\right)P_\lambda &: L_{x,t}^1 \rightarrow L_{t,x}^\infty. \end{aligned}$$

Let us prove the second one by computing its Fourier inversion,

$$\mathcal{F}^{-1} \frac{\theta_\lambda(\xi)}{(\tau - \frac{1}{4}\xi^3 \pm i0)^{\frac{3}{2}+iy}} = \mathcal{F}^{-1}(\tau \pm i0)^{-\frac{3}{2}-iy} \cdot \mathcal{F}^{-1}[\theta_\lambda(\xi)\delta_{\tau=\frac{1}{4}\xi^3}].$$

$\theta_\lambda(\xi)$ is some smooth bump function around $\xi = \lambda$, which we used to define P_λ .

From direct computation, we have

$$\|e^{(\frac{3}{2}+iy)^2} \Gamma\left(\frac{3}{2} + iy\right) \mathcal{F}^{-1}(\tau \pm i0)^{-\frac{3}{2}-iy}\|_{L^\infty} \lesssim t^{\frac{1}{2}},$$

and by stationary phase we get

$$\|\mathcal{F}^{-1}[\theta_\lambda(\xi)\delta_{\tau=\frac{1}{4}\xi^3}]\|_{L^\infty} = \left\| \int \theta_\lambda(\xi) e^{ix\xi + i\frac{1}{4}t\xi^3} d\xi \right\|_{L^\infty} \lesssim (t\lambda)^{-\frac{1}{2}}.$$

Combining them together, we get

$$\|S\left(\frac{3}{2} + iy\right)P_\lambda\|_{L_{x,t}^1 \rightarrow L_{t,x}^\infty} \lesssim \lambda^{-\frac{1}{2}}.$$

Also notice the trivial bound

$$\|S(0 + iy)P_\lambda\|_{L_{t,x}^2 \rightarrow L_{x,t}^2} \lesssim C.$$

We interpolate to get

$$\|S\left(\frac{1}{2} + iy\right)P_\lambda\|_{L_{x,t}^{\frac{3}{2}} \rightarrow L_{x,t}^3} \lesssim \lambda^{-\frac{1}{6}}.$$

Define the operator T by multiplier $\frac{1}{(\tau - \frac{1}{4}\xi^3 \pm i0)^{\frac{1}{4}}}$, and $S(\frac{1}{2}) = cTT^*$, $c = e^{\frac{1}{4}}\Gamma(\frac{1}{2})$. So by

the TT^* argument [29] [30], we have $\|TP_\lambda\|_{L_{t,x}^2 \rightarrow L_{x,t}^3} \lesssim \lambda^{-\frac{1}{12}}$.

Hence we get

$$\|u_\lambda\|_{L_{x,t}^3} \lesssim \lambda^{-\frac{1}{3}-2s-2} \|u_\lambda\|_{|D|^{-2s-2} X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}}.$$

If we take $q = 3$ in (29), combining with (27) we get (30).

If we take $q = 6$ in (29), also compare with

$$\|Q_{\sigma \approx \lambda^3} u_\lambda\|_{L_{t,x}^6} \lesssim (\lambda\sigma)^{\frac{1}{6}} \|u_\lambda\|_{L_{t,x}^3} \lesssim \lambda^{-2s-2+\frac{1}{3}} \|u_\lambda\|_{|D|^{-2s-2} X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4}, \frac{1}{4}}},$$

we get (31). From Remark 2.2(4), we only put pieces in $|D|^{-2s-2}X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4},\frac{1}{4}}$ norm when it lies close to the special curve, and in that case its modulation is close to λ^3 . \square

Also, let us collect some bilinear estimates that will be very useful in the next section.

Proposition 3.9. *For $\mu \gg \lambda \geq \alpha$, as before $\eta_\lambda(t)$ is the sharp cutoff on time interval I_λ of size $|I_\lambda| = \lambda^{4s+3}$. We have the following estimates:*

$$(32) \quad \|\eta_\mu(t)u_\mu v_\lambda\|_{L_{t,x}^2} \lesssim \mu^{-1-s}\lambda^{-s}\|u_\mu\|_{X^1[I_\mu]}\|v_\lambda\|_{X^1[I_\lambda]},$$

$$(33) \quad \|\eta_\mu(t)P_{\approx\lambda}(u_\mu v_\mu)\|_{L_{t,x}^2} \lesssim \mu^{-\frac{1}{2}-2s}\lambda^{-\frac{1}{2}}\|u_\mu\|_{X^1[I_\mu]}\|v_\lambda\|_{X^1[I_\mu]},$$

$$(34) \quad \|\eta_\lambda(t)u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \max\{\lambda^{-\frac{1}{3}-2s-2}\alpha^{-\frac{1}{6}-s}, \lambda^{-2-s}\alpha^{\frac{1}{2}-s}\}\|u_\lambda\|_{S[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]}.$$

Proof. For (32) and (33), we expand u, v via Duhamel's formula, and apply the bilinear estimates (20) (21) repeatedly. See [8] Lemma 3.4 for a similar proof.

For (34), we still break u_λ by the size of modulation, and see that the worst estimate comes when $u_\lambda \in |D|^{-2s-2}X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4},\frac{1}{4}} \cap |D|L_{t,x}^\infty$. Then we use L^3 for u_λ , and L^6 for v_α .

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \|\eta_\lambda u_\lambda\|_{L_{t,x}^2}\|\eta_\alpha v_\alpha\|_{L_{t,x}^\infty} \lesssim \lambda^{-2s-3}\alpha^{\frac{1}{2}-s}\|u_\lambda\|_{X^{-3-4s,2s+2}[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]},$$

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \|\eta_\lambda u_\lambda\|_{L_{t,x}^3}\|\eta_\alpha v_\alpha\|_{L_{t,x}^6} \lesssim \lambda^{-\frac{1}{3}-2s-2}\alpha^{-\frac{1}{6}-s}\|u_\lambda\|_{|D|^{-2s-2}X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4},\frac{1}{4}}[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]}.$$

By comparing the coefficients in the estimates above, we get

$$(35) \quad \|\eta_\lambda(t)u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{3}-2s-2}\alpha^{-\frac{1}{6}-s}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]}.$$

If we also consider the case of $u_\lambda \in X^{-s,1+s}$,

$$(36) \quad \|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \|\eta_\lambda u_\lambda\|_{L_{t,x}^2}\|\eta_\alpha v_\alpha\|_{L_{t,x}^\infty} \lesssim \lambda^{-2-s}\alpha^{\frac{1}{2}-s}\|u_\lambda\|_{X^{-s,1+s}[I_\lambda]}\|v_\alpha\|_{X^1[I_\alpha]}$$

and compare the coefficients, we get (34). \square

Remark 3.10. We don't have a good L^2 estimate on the product of two pieces both in S . But we will still list here some of the cases, which are manageable.

When $u_\lambda, v_\alpha \in X^{-s,1+s}$, bound u_λ in $L_{t,x}^2$, and v_α in $L_{t,x}^\infty$.

$$(37) \quad \|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-2-s}\alpha^{-2s-1}\|u_\lambda\|_{X^{-s,1+s}[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.$$

When $u_\lambda, v_\alpha \in Z$, bound u_λ in L^3 , and v_α in L^6 , we get

$$(38) \quad \|\eta_\lambda(t)u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{Z[I_\alpha]}.$$

When $u_\lambda \in Z, v_\alpha \in X^{-s,1+s}$, bound u_λ in L^3 , and v_α in L^6 which comes from Bernstein together with L^2 bound, we get

$$(39) \quad \|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\max\{\alpha^{-s-1}, \alpha^{-\frac{5}{3}-2s}\}\|u_\lambda\|_{Z[I_\lambda]}\|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.$$

The above three inequalities imply that

$$(40) \quad \|\eta_\lambda(t)u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{4}{3}(s+1)-\frac{1}{3}}\alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}}\|u_\lambda\|_{S[I_\lambda]}\|v_\alpha\|_{S[I_\alpha]}.$$

is true except for the case $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$, which corresponds to case the high-frequency low modulation interacting with low-frequency high modulation.

To estimate the case $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$, use L^2 on u_λ , L^∞ on v_α , and we get

$$(41) \quad \|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \|\eta_\lambda u_\lambda\|_{L_{t,x}^2}\|\eta_\lambda v_\alpha\|_{L_{t,x}^\infty}$$

$$\lesssim \lambda^{-2-s} \alpha \|u_\lambda\|_{X^{-s,1+s}[I_\lambda]} \|v_\alpha\|_{Z[I_\alpha]}.$$

The bound here is worse than the one in (40).

4. ESTIMATING THE NONLINEARITY

The goal of this part is to estimate the nonlinearity as in Proposition 1.5. Since functions in $X^s \cap X_{I_e}^s$ have different piece measured differently, we show that the estimate

$$(42) \quad \|\partial_x(uv)\|_{Y^s \cap Y_{I_e}^s} \lesssim \|u\|_{X^s \cap X_{I_e}^s} \|v\|_{X^s \cap X_{I_e}^s}$$

is almost true except for one special case.

Let us expand the estimate (42), the energy norm takes the form

$$\begin{aligned} & \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\eta_J(t) P_\lambda(\partial_x(uv))\|_{Y_\lambda[J]}^2 \\ & \lesssim \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\eta_J(t) \lambda \sum_{\alpha \ll \lambda} P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]}^2 \\ & + \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\eta_J(t) \lambda \sum_{\mu \gtrsim \lambda} P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]}^2. \end{aligned}$$

We can do same expansion for the local energy norm.

In the case of high-low frequency interaction, our goal would be to prove

$$(43) \quad \|\lambda \eta_J(t) P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]} \lesssim C \|u_\lambda\|_{X_\lambda[J]} \|v_\alpha\|_{X_\alpha[K]}.$$

Here $C = C(\lambda, \alpha) \lesssim 1$, and K is a time interval with size α^{4s+3} , so that $J \subset K$.

Now given (43), we get bound for energy norm in the case of high-low interaction

$$\begin{aligned} & \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\eta_J(t) \lambda \sum_{\alpha \ll \lambda} P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]}^2 \\ & \lesssim \sum_{\lambda \geq 1} C(\lambda, \alpha)^2 \sup_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|u_\lambda\|_{X_\lambda[J]}^2 \sum_{\alpha \ll \lambda} \|v_\alpha\|_{X^s}^2 \\ & \lesssim \|u\|_{X^s}^2 \|v\|_{X^s}^2. \end{aligned}$$

And we can prove a spatial localized version of (43) in exactly the same way.

$$(44) \quad \|\lambda \chi_j^\lambda(x) \eta_J(t) P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]} \lesssim C \|\chi_j^\lambda(x) u_\lambda\|_{X_\lambda[J]} \|v_\alpha\|_{X_\alpha[K]}.$$

Then we also get bound for local energy norm in the case of high-low interaction

$$\begin{aligned} & \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\chi_j^\lambda(x) \eta_J(t) \lambda \sum_{\alpha \ll \lambda} P_\lambda(u_\lambda v_\alpha)\|_{Y_\lambda[J]}^2 \\ & \lesssim \sum_{\lambda \geq 1} C(\lambda, \alpha)^2 \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\chi_j^\lambda(x) u_\lambda\|_{X_\lambda[J]}^2 \sum_{\alpha \ll \lambda} \|v_\alpha\|_{X^s}^2 \\ & \lesssim \|u\|_{X_{I_e}^s}^2 \|v\|_{X^s}^2. \end{aligned}$$

One remark here is that we secretly turn the summation of α from l^1 to l^2 summation, which is not true in general. Luckily, in our proof for (43), the bound $C(\lambda, \alpha)$ mostly involves negative power of α or λ , which makes the summation valid. The only case worth attention is case 1.1(b), where we illuminate the α summation in detail.

In the case of high-high frequency interaction, we need to measure each u_μ on smaller time interval $I_\mu \subset J$, of size $|I_\mu| = \mu^{4s+3}$.

We will prove the estimate

$$(45) \quad \|\lambda \eta_J P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]} \lesssim C \sup_{I_\mu \subset J} \|u_\mu\|_{X_\mu[I_\mu]} \|v_\mu\|_{X^s \cap X_{I_e}^s},$$

and its corresponding spatial localized version

$$(46) \quad \|\lambda \eta_J \chi_j^\lambda(x) P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]} \lesssim C \sup_{I_\mu \subset J} \|\chi_{k(j)}^\mu u_\mu\|_{X_\mu[I_\mu]} \|v_\mu\|_{X^s \cap X_{I_e}^s}.$$

Here $\chi_{k(j)}^\mu(x)$ is a chosen spatial cutoff so that $\chi_j^\lambda(x) \leq \chi_{k(j)}^\mu(x)$ (we might need two adjacent spatial cutoffs), $C = C(\lambda, \mu) \lesssim 1$.

Given (45), we get the bound for energy norm in the case of high-high interaction

$$\begin{aligned} & \sum_{\lambda \geq 1} \sup_{|J|=\lambda^{4s+3}} \sup_{J \subset [0,1]} \|\lambda \eta_J(t) \sum_{\mu \gtrsim \lambda} P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]}^2 \\ & \lesssim \sum_{\lambda \geq 1} C(\lambda, \mu)^2 \left(\sum_{\mu \gtrsim \lambda} \sup_{|I_\mu|=\mu^{4s+3}} \sup_{I_\mu \subset [0,1]} \|u_\mu\|_{X_\mu[I_\mu]}^2 \right) \sum_{\mu \gtrsim \lambda} \|v_\mu\|_{X^s \cap X_{I_e}^s}^2 \\ & \lesssim \|u_\mu\|_{X^s}^2 \|v_\mu\|_{X^s \cap X_{I_e}^s}^2. \end{aligned}$$

And with (46), we can bound the local energy norm in the case of high-high interaction

$$\begin{aligned} & \sum_{\lambda \geq 1} \sup_j \sum_{|J|=\lambda^{4s+3}, J \subset [0,1]} \|\chi_j^\lambda(x) \eta_J(t) \lambda \sum_{\mu \gtrsim \lambda} P_\lambda(u_\mu v_\mu)\|_{Y_\lambda[J]}^2 \\ & \lesssim \sum_{\lambda \geq 1} C(\lambda, \mu)^2 \left(\sum_{\mu \gtrsim \lambda} \sup_{k(j)} \sum_{|I_\mu|=\mu^{4s+3}, I_\mu \subset [0,1]} \|\chi_{k(j)}^\mu(x) u_\mu\|_{X_\mu[I_\mu]}^2 \right) \sum_{\mu \gtrsim \lambda} \|v_\mu\|_{X^s \cap X_{I_e}^s}^2 \\ & \lesssim \|u_\mu\|_{X_{I_e}^s}^2 \|v_\mu\|_{X^s \cap X_{I_e}^s}^2. \end{aligned}$$

In both of the estimates, we need change the order of λ, μ summation. Luckily the bound $C(\lambda, \mu)$ in (45) (46) will help us to perform the λ summation.

Need to check it with Case 1.2(a). The only case where $C = 1$. Maybe also 3.1.1(c)

Since the proofs for (44) (46) are essentially the same as (43) (45). We will discard the spatial cutoff in our proofs unless needed.

Remark 4.1. To be more precise, for spatial localization, instead of writing a function as $u_\lambda = \sum_j \chi_j^\lambda(x) u_\lambda$, we need to decompose each function as

$$u_\lambda = \sum_j u_{\lambda,j}, \quad u_{\lambda,j} = P_\lambda(\chi_j^\lambda u_\lambda).$$

In this way, we preserve the frequency localization while blurring the spatial localization. But thanks to the fast decay property of the kernel of $\chi_k^\lambda(x) P_\lambda \chi_j^\lambda(x)$, we have

$$|\chi_k^\lambda u_{\lambda,j}| \lesssim |k-j|^{-N} \lambda^{-N} \|\chi_j^\lambda u_\lambda\|_{L_t^\infty L_x^2}, \quad |k-j| \gg 1.$$

So the difference of the two decompositions is really negligible. Similar reasoning applies when we interchange the modulation localization and time localization.

Before getting into detail, notice that $\widetilde{uv}(\tau, \xi) = \widetilde{u}(\tau_1, \xi_1) * \widetilde{v}(\tau_2, \xi_2)$, so we have

$$\tau = \tau_1 + \tau_2, \quad \xi = \xi_1 + \xi_2,$$

and the resonance identity

$$(47) \quad \tau - \xi^3 = (\tau_1 - \xi_1^3) + (\tau_2 - \xi_2^3) - 3\xi\xi_1\xi_2.$$

Also, the following high modulation relation is quite useful in our proof.

$$(48) \quad \sigma_m = \max(|\tau - \xi^3|, |\tau_1 - \xi_1^3|, |\tau_2 - \xi_2^3|) \gtrsim |\xi \xi_1 \xi_2|.$$

This relation forces high modulation either on the input or on output, which gives a gain.

4.1. Estimate for $X^1 \times X^1$. When $u, v \in X^1$, we break them into dyadic pieces and discuss the problem in different cases. As pointed out in Remark 2.2(1), for function $u_\lambda \in X^1[I_\lambda]$, $|I_\lambda| = \lambda^{4s+3}$, we think of it as its extension $u_{\lambda,E}$, which is defined on the whole real time line and still supported on neighborhood of I_λ .

Case 1.1: High-Low frequency interaction. Suppose $\lambda \gg \alpha$, then the output frequency is λ . From (48), let $M = \lambda^2 \alpha$, then

$$\lambda \eta_\lambda u_\alpha v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim M}, Q_{\ll M}\}} \lambda \eta_\lambda Q_1 [Q_2 u_\alpha Q_3 v_\lambda].$$

Clearly in each term, at least one of Q_i must be $Q_{\gtrsim M}$.

Case 1.1(a): When high modulation comes from input, simply bound that piece in L^2 and the other in L^∞ . Combining with Bernstein inequality, we get

$$\begin{aligned} \|\lambda \eta_\lambda Q_1 [Q_2 u_\alpha Q_3 v_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \|\lambda \eta_\lambda Q_1 [Q_2 u_\alpha Q_3 v_\lambda]\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2[I_\lambda]} \\ &\lesssim \lambda \alpha^{-s} M^{-1} (\alpha^{\frac{1}{2}} + \lambda^{\frac{1}{2}}) \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]} \\ &\lesssim \lambda^{-\frac{1}{2}} \alpha^{-1-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]}. \end{aligned}$$

For $s \geq -1$ we can sum up with respect to α then λ .

Case 1.1(b): If none of Q_2, Q_3 have high modulation, this forces $Q_1 = Q_{\approx M}$. Depends on the size of M , we bound the output in different spaces ($|D|^{-s}|I|^{-\frac{1}{2}}L^2$ or $X^{-s,s}$). Using the bilinear estimate (32), we have

$$\begin{aligned} \|\lambda \eta_\lambda Q_{\approx M \lesssim \lambda^{4+\frac{3}{2s}}} [Q_2 u_\alpha Q_3 u_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \lambda^{-1-s} \alpha^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]} \\ &\lesssim \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]}, \\ \|\lambda \eta_\lambda Q_{\approx M \geq \lambda^{4+\frac{3}{2s}}} [Q_2 u_\alpha Q_3 u_\lambda]\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1-s} M^s \lambda^{-1-s} \alpha^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]} \\ &\lesssim \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^1[I_\lambda]}. \end{aligned}$$

Remark 4.2. We need to be careful with α summation in above estimates. For the first one we use factor α^s to turn l^1 summation to l^2 . A careful way of doing the second one is to write the modulation as a multiple of $\lambda^2 \alpha$, and use the l^2 summability of modulation.

$$\begin{aligned} \left\| \sum_{\alpha \ll \lambda} \sum_{\theta} \lambda \eta_\lambda Q_{(\lambda^2 \alpha) \theta} (u_\lambda v_\alpha) \right\|_{Y_\lambda}^2 &\lesssim \left(\sum_{\theta} \left\| \sum_{\alpha \ll \lambda} \lambda \eta_\lambda Q_{(\lambda^2 \alpha) \theta} (u_\lambda v_\alpha) \right\|_{DS} \right)^2 \\ &\lesssim \left\{ \sum_{\theta} \left(\sum_{\alpha \ll \lambda} \|\lambda \eta_\lambda Q_{(\lambda^2 \alpha) \theta} (u_\lambda v_\alpha)\|_{DS}^2 \right)^{1/2} \right\}^2 \\ &\lesssim \left\{ \sum_{\theta} \left(\sum_{\alpha \ll \lambda} \theta^{2s} \|u_\lambda\|_{X^1[I_\lambda]}^2 \|v_\alpha\|_{X^1[I_\alpha]}^2 \right)^{1/2} \right\}^2 \\ &\lesssim \left(\sum_{\theta} \theta^s \right)^{\frac{1}{2}} \|u_\lambda\|_{X^1[I_\lambda]}^2 \|v\|_{X^s}^2. \end{aligned}$$

In second inequality, since the modulation is different, we do have the l^2 summation.

Case 1.2: High-High frequency interaction with low frequency output, $\lambda \ll \mu$. Here we need to cut the interval I_λ into finer scale so that u_μ is measured on smaller intervals I_μ .

$$u_\mu = \sum_i u_\mu^i, \quad u_\mu^i \in X^1[I_\mu^i], \quad \cup I_\mu^i = I_\lambda.$$

Then the output has the expression

$$\lambda \eta_\lambda u_\mu v_\mu = \sum_i \sum_{Q_j \in \{Q_{\gtrsim \lambda \mu^2}, Q_{\ll \lambda \mu^2}\}} \lambda Q_1[Q_2 u_\mu^i Q_3 v_\mu^j].$$

Case 1.2(a): When $Q_1 = Q_{\gtrsim \lambda \mu^2}$, we place the output in $DZ[I_\lambda]$, by using (33)

$$\|Q_2 u_\mu^i Q_3 v_\mu^j\|_{L_{t,x}^2} \lesssim \lambda^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \mu^{-2s} \|u_\mu\|_{X^1[I_\mu^i]} \|v_\mu\|_{X^1[I_\mu^j]}$$

and the almost orthogonality of the product $\lambda Q_\sigma(u_\mu^i v_\mu^j)$ with $\lambda Q_\sigma(u_\mu^j v_\mu^i)$, we get

$$\begin{aligned} \left\| \sum_i \lambda Q_{\sigma \gtrsim \lambda \mu^2} [Q_2 u_\mu^i Q_3 v_\mu^i] \right\|_{X^{-3-4s, 2s+1}[I_\lambda]} &\lesssim \lambda^{2-4s} (\sigma_{\gtrsim \lambda \mu^2})^{2s+1} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} \|u_\mu^i v_\mu^i\|_{L_{t,x}^2} \\ &\lesssim \sup_{I_\mu^i \subset I_\lambda} \|u_\mu\|_{X^1[I_\mu^i]}^2 \|v_\mu\|_{X^s}^2. \end{aligned}$$

The DZ norm also has the L^p component. Here because the modulation is high, we can interchange interval and modulation cutoff and have l^p summation of the intervals. Using Strichartz estimates (16) and Bernstein inequality on the product, we get

$$\begin{aligned} &\left\| \sum_i \lambda Q_{\sigma \gtrsim \lambda \mu^2} [Q_2 u_\mu^i Q_3 v_\mu^i] \right\|_{|D_t - D_x^3| |D| L_{t,x}^\infty [I_\lambda]} \\ &\lesssim \sup_{I_\mu^i} \left\| \lambda Q_{\sigma \gtrsim \lambda \mu^2} [Q_2 u_\mu^i Q_3 v_\mu^i] \right\|_{|D_t - D_x^3| |D| L_{t,x}^\infty [I_\mu^i]} \\ &\lesssim \sup_{I_\mu^i} \frac{\lambda}{\lambda \mu^2} \|u_\mu\|_{L_t^\infty L_x^2 [I_\mu^i]} \|v_\mu\|_{L_t^\infty L_x^2 [I_\mu^i]} \\ &\lesssim \mu^{-2(s+1)} \sup_{I_\mu \subset I_\lambda} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

We need to be careful with the summation on λ , since the bound is 1. There is a chance we might only get $s > -1$ because of this.

Case 1.2(b): When input has high modulation, we use the local energy space to get good control of the interval summation.

Before that, let us state a useful lemma:

Lemma 4.3. *Suppose $-1 \leq s \leq -\frac{3}{4}$, $0 \leq k \leq \frac{1}{2}$, then we have*

$$\begin{aligned} \|Q_{\sigma \gtrsim \lambda^3} f_\lambda\|_{DZ[I_\lambda]} &\lesssim \sup_{\sigma \gtrsim \lambda^3} \lambda^{2s+3k} \sigma^{-k} \|Q_\sigma f_\lambda\|_{L_{t,x}^2 [I_\lambda]}, \\ \|f_\lambda\|_{Y_\lambda [I_\lambda]} &\lesssim \sup_\sigma \lambda^{3s+\frac{3}{2}+3k} \sigma^{-k} \|Q_\sigma f_\lambda\|_{L_{t,x}^2 [I_\lambda]}. \end{aligned}$$

Proof. From the definition of $Y_\lambda [I_\lambda]$, we just need to bound different modulation in suitable spaces, and compare the bounds with the ones in our lemma. The DZ norm also has L^p component, we use Bernstein to turn L^p into L^2 norm. \square

Remark 4.4. These estimates are very crude. When applying on the nonlinearity, we might need to do modulation analysis, or use better interval summation in some cases, e.g. case 1.2(a). But when one of the inputs has high modulation, a simple L^2 estimate saves us from tedious case by case calculation.

Let us first bound the spatial localized output in L^2 .

$$\begin{aligned}
& \|\lambda\eta_\lambda\chi_j^\lambda(x)Q_\sigma[\sum_i(Q_{\gtrsim\lambda\mu^2}u_\mu^i)(Q_3v_\mu^i)]\|_{L_{t,x}^2[I_\lambda]}^2 \\
& \lesssim \sigma\|\lambda\eta_\lambda\chi_j^\lambda(x)[\sum_i(Q_{\gtrsim\lambda\mu^2}u_\mu^i)(Q_3v_\mu^i)]\|_{L_x^2L_t^1}^2 \\
& \lesssim \lambda^2\sigma\sum_i\|\chi_j^\lambda(x)Q_{\gtrsim\lambda\mu^2}u_\mu^i\|_{L_{t,x}^2[I_\mu^i]}^2\sum_i\|\chi_j^\lambda(x)Q_3v_\mu^i\|_{L_x^\infty L_t^2[I_\mu^i]}^2 \\
& \lesssim \lambda^2\sigma|\frac{I_\lambda}{I_\mu}| \sup_i\|\chi_j^\lambda(x)Q_{\gtrsim\lambda\mu^2}u_\mu^i\|_{L_{t,x}^2[I_\mu^i]}^2\sup_j\sum_i\|\chi_j^\lambda(x)Q_3v_\mu^i\|_{L_x^\infty L_t^2[I_\mu^i]}^2 \\
& \lesssim \sigma\lambda^{4s+3}\mu^{-12s-12}\sup_{I_\mu}\|\chi_{k(j)}^\mu(x)u_\mu\|_{X^1[I_\mu]}^2\|v_\mu\|_{X_{le}^s}^2.
\end{aligned}$$

To get same estimate without the spatial localization, we need to sum up j

$$\begin{aligned}
& \sum_j\|\lambda\eta_\lambda\chi_j^\lambda(x)Q_\sigma[\sum_i(Q_{\gtrsim\lambda\mu^2}u_\mu^i)(Q_3v_\mu^i)]\|_{L_{t,x}^2[I_\lambda]}^2 \\
& \lesssim \lambda^2\sigma\sum_{i,j}\|\chi_j^\lambda(x)Q_{\gtrsim\lambda\mu^2}u_\mu^i\|_{L_{t,x}^2[I_\mu^i]}^2\sup_j\sum_i\|\chi_j^\lambda(x)Q_3v_\mu^i\|_{L_x^\infty L_t^2[I_\mu^i]}^2 \\
& \lesssim \lambda^2\sigma|\frac{I_\lambda}{I_\mu}| \sup_i\|Q_{\gtrsim\lambda\mu^2}u_\mu^i\|_{L_{t,x}^2[I_\mu^i]}^2\sup_j\sum_i\|\chi_j^\lambda(x)Q_3v_\mu^i\|_{L_x^\infty L_t^2[I_\mu^i]}^2 \\
& \lesssim \sigma\lambda^{4s+3}\mu^{-12s-12}\sup_{I_\mu}\|u_\mu\|_{X^1[I_\mu]}^2\|v_\mu\|_{X_{le}^s}^2.
\end{aligned}$$

By Lemma 4.3 $k = \frac{1}{2}$, we have the following estimate with or without spatial localization.

$$\|\lambda\eta_\lambda[\sum_iQ_{\gtrsim\lambda\mu^2}u_\mu^iQ_3v_\mu^i]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{5s+9/2}\mu^{-6s-6}\|u_\mu\|_{X^1[I_\mu]}\|v_\mu\|_{X_{le}^s}.$$

We can sum up frequency λ and μ when $-1 \leq s \leq -\frac{3}{4}$.

Remark 4.5. The estimates above demonstrate how we can use local energy norm to get good interval summations, especially in the case when time truncation blur the output modulation too much.

Case 1.3: High-High frequency interaction giving out the output of the same size. Now high modulation (48) means λ^3 .

$$\lambda\eta_\lambda u_\lambda v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim\lambda^3}, Q_{\ll\lambda^3}\}} \lambda\eta_\lambda Q_1[Q_2 u_\lambda Q_3 v_\lambda].$$

Case 1.3(a): When high modulation comes from input, we estimate the output in $|D|^{-s}|I|^{-\frac{1}{2}}L^2$

$$\begin{aligned}
\|\lambda\eta_\lambda Q_1[Q_{\sigma \gtrsim \lambda^3} u_\lambda Q_3 v_\lambda]\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} & \lesssim \lambda^{1+s}|I_\lambda|^{\frac{1}{2}}\|Q_{\sigma \gtrsim \lambda^3}(\eta_\lambda u_\lambda)\|_{L_{t,x}^2}\|\eta_\lambda v_\lambda\|_{L_{t,x}^\infty} \\
& \lesssim \lambda^{-\frac{3}{2}-s}\|u_\lambda\|_{X^1[I_\lambda]}\|v_\lambda\|_{X^1[I_\lambda]}.
\end{aligned}$$

Case 1.3(b): When inputs have low modulation, this forces the output to have modulation approximately λ^3 . In fact, the output has Fourier support lying closer to another curve $\tau = \frac{1}{4}\xi^3$. To give a good bound in this case, we want to prove

$$(49) \quad \|\lambda P_\lambda(u_\lambda v_\lambda)\|_{|\partial_t + \partial_x^3|^{-1}|D|^{-2s-2}X^{\frac{1}{4}, \frac{1}{4}}_{\tau=\frac{1}{4}\xi^3}}} \lesssim \|u_\lambda\|_{X^1[I_\lambda]}\|v_\lambda\|_{X^1[I_\lambda]}.$$

To do this, let us use another space [22]

$$\|u\|_{\dot{X}^{s, \frac{1}{2}, 1}} = \sum_{\vartheta} \left(\int_{|\tau - \xi^3| = \vartheta} |\tilde{u}(\tau, \xi)|^2 \xi^{2s} |\tau - \xi^3| d\xi d\tau \right)^{\frac{1}{2}},$$

and claim the embedding inequality

$$(50) \quad \|u_\lambda\|_{\dot{X}^{s, \frac{1}{2}, 1}} \lesssim \|u_\lambda\|_{X^1[I_\lambda]},$$

which is proved by looking at the extension $u_{\lambda, E}$, and definitions of both norms.

Now for functions in $\dot{X}^{s, \frac{1}{2}, 1}$, we use foliation. The idea is same as in Chapter 2.6 Lemma 2.9 in Tao [30]. From Fourier inversion, we have

$$u_\lambda(t, x) = \frac{1}{(2\pi)^2} \int \int \tilde{u}_\lambda(\tau, \xi) e^{it\tau + ix\xi} d\tau d\xi.$$

Then if we write $\tau_0 = \tau - \xi^3$, we will have the foliation

$$u_\lambda(t, x) = \frac{1}{2\pi} \int e^{it\tau_0} e^{t\partial_x^3} f_{\tau_0} d\tau_0,$$

where

$$e^{t\partial_x^3} f_{\tau_0} = \frac{1}{2\pi} \int \tilde{u}_\lambda(\tau_0 + \xi^3, \xi) e^{it\xi^3 + ix\xi} d\xi,$$

and f_{τ_0} has frequency about size λ , modulation about size τ_0 .

Similarly we write down $v_\lambda = u_\lambda(t, x) = \frac{1}{2\pi} \int e^{it\tau_0'} e^{t\partial_x^3} g_{\tau_0'} d\tau_0'$.

Now using (23) and Minkowski inequality

$$\begin{aligned} \|\lambda P_\lambda(u_\lambda v_\lambda)\|_{|(\partial_t + \partial_x^3)|^{-1}|D|^{-2s-2} \dot{X}^{\frac{1}{4}, \frac{1}{4}}_{\tau = \frac{1}{4}\xi^3}} &\lesssim \frac{\lambda^{2s+3}}{\lambda^3} \sum_{\tau_0, \tau_0'} \iint \|e^{t\partial_x^3} f_{\tau_0} e^{t\partial_x^3} g_{\tau_0'}\|_{\dot{X}^{\frac{1}{4}, \frac{1}{4}}_{\tau = \frac{1}{4}\xi^3}} d\tau_0 d\tau_0' \\ &\lesssim \lambda^{2s} \sum_{\tau_0, \tau_0'} \iint \|f_{\tau_0}\|_{L_x^2} \|g_{\tau_0'}\|_{L_x^2} d\tau_0 d\tau_0' \\ &\lesssim \|u_\lambda\|_{\dot{X}^{s, \frac{1}{2}, 1}} \|v_\lambda\|_{\dot{X}^{s, \frac{1}{2}, 1}}. \end{aligned}$$

With the time cutoff we can pass to nonhomogeneous space, as in Proposition 3.7. Combining with the embedding (50), we proved (49).

4.2. Estimate for $S \times S$. When $u, v \in S$, we still need to consider different frequency interaction. Notice that because of Remark 2.2(4), we only consider pieces that have relatively high modulation: $|\tau - \xi^3| \gtrsim |\xi|^{4+\frac{3}{2s}}$

Case 2.1: High low frequency interaction. The nonlinearity look like $\lambda\eta_\lambda u_\lambda v_\alpha$, $\lambda \gg \alpha$. As discussed in Remark 3.10, we don't have a good bilinear estimate, but (40) breaks down only for one case.

Case 2.1.1: If $u_\lambda, v_\alpha \in X^{-s, 1+s}$, or $u_\lambda \in Z, v_\alpha \in X^{-s, 1+s}$, or $u_\lambda, v_\alpha \in Z$, we can still use the L^2 estimate (40) and Lemma 4.3 with $k = 0$ to get

$$\begin{aligned} \|\lambda\eta_\lambda u_\lambda v_\alpha\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{3s+\frac{3}{2}} \lambda^{1-\frac{4}{3}(s+1)-\frac{1}{3}} \alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{Z[I_\alpha]} \\ &\lesssim \lambda^{\frac{1}{6}+s+\frac{2}{3}(s+1)} \alpha^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{Z[I_\alpha]}. \end{aligned}$$

Notice that the exponents add up to $-\frac{3}{2} - s < 0$, we can still sum up frequencies.

Case 2.1.2: Now if $u_\lambda \in X^{-s, 1+s}, v_\alpha \in Z$, where (40) failed. We use L^2 on u_λ and $L_t^3 L_x^\infty$ on v_α , still by Bernstein,

$$\|\lambda Q_\sigma(\eta_\lambda u_\lambda v_\alpha)\|_{L_{t,x}^2} \lesssim \lambda \alpha^{\frac{1}{3}} \sigma^{\frac{1}{3}} \lambda^{-2-s} \alpha^{-\frac{4}{3}(s+1)-\frac{1}{3}} \|u_\lambda\|_{X^{-s, 1+s}[I_\lambda]} \|v_\alpha\|_{Z[I_\alpha]},$$

so we from Lemma 4.3, we get

$$\|\lambda\eta_\lambda u_\lambda v_\alpha\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{2s+\frac{3}{2}} \alpha^{-\frac{4}{3}(s+1)} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{X^{-s,1+s}[I_\alpha]}.$$

And we can still sum up the frequencies.

Case 2.2: High-high frequency interaction giving out equal or lower frequency, $\lambda \lesssim \mu$. When $\lambda \ll \mu$, we cut up intervals as in case (1.2). When $\lambda \approx \mu$, this procedure degenerate.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \lambda u_\mu^i v_\mu^i, \quad u_\mu^i, v_\mu^i \in X_\mu[I_\mu^i]$$

Here we don't have a good L^2 bound on the product, so we need to do modulation analysis again to get better control. Also, all the estimates here have the corresponding version with spatial localization, the proofs are exactly the same.

Case 2.2.1: $X^{-s,1+s} \times X^{-s,1+s}$, both u_μ^i, v_μ^i have modulation $\mu^{4+\frac{3}{2s}} \lesssim \sigma \lesssim \mu^3$. we use Bernstein inequality for frequency on product, for modulation on any one of input. And we have l^2 summation of the small intervals.

$$\begin{aligned} \left\| \sum_i \lambda(Q_\sigma u_\mu^i) v_\mu^i \right\|_{Y_\lambda[I_\lambda]} &\lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} \sup_i \|(Q_\sigma u_\mu^i) v_\mu^i\|_{L_{t,x}^2[I_\mu^i]} \\ &\lesssim \lambda^{1+s} |I_\lambda| |I_\mu|^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \sigma^{\frac{1}{2}} \sup_i \|u_\mu^i\|_{L_{t,x}^2} \|v_\mu^i\|_{L_{t,x}^2} \\ &\lesssim \lambda^{5s+\frac{9}{2}} \mu^{-5s-5} \sup_{I_\mu} \|u_\mu\|_{S[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

Case 2.2.2: $X^{-s,1+s} \times Z$, suppose v_μ has modulation $\sigma_m \gtrsim \mu^3$. By modulation analysis (48), this forces another high modulation on the output.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \lambda Q_{\approx\sigma_m} [u_\mu^i (Q_{\sigma_m} v_\mu^i)]$$

We comment that when $\sigma_m \approx \mu^3$, there is chance high modulation can also fall on u_μ . But in that case, from Prop 2.2(4), the norm Z and $X^{-s,1+s}$ match with each other. So it is essentially the same as in the following case 2.2.3.

We use L^2 (25) on u_μ , and L^p for v_μ , together with Bernstein.

$$\begin{aligned} &\left\| \sum_i \lambda Q_{\sigma_m} [u_\mu^i (Q_{\sigma_m} v_\mu^i)] \right\|_{X^{-3-4s,2s+1}[I_\lambda]} \\ &\lesssim \lambda^{-2-4s} \sigma_m^{2s+1} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} \sup_i \|u_\mu^i (Q_{\sigma_m} v_\mu^i)\|_{L_{t,x}^2[I_\mu]} \\ &\lesssim \lambda^{-2-4s} \sigma_m^{2s+1} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} (\lambda\sigma_m)^{\frac{1}{p}} \sup_i \|u_\mu^i\|_{L_{t,x}^2[I_\mu]} \|Q_{\sigma_m} v_\mu^i\|_{L_{t,x}^p[I_\mu]} \\ &\lesssim \lambda^{-2s-\frac{1}{2}+\frac{1}{p}} \mu^{3s+\frac{1}{2}+\frac{4s+3}{p}} \sup_{I_\mu} \|u_\mu\|_{S[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

We also need to bound the L^p component, here we exchange the interval cutoff with modulation factor and have l^p summation.

$$\begin{aligned} &\left\| \sum_i \lambda Q_{\approx\sigma} (u_\mu^i Q_\sigma v_\mu^i) \right\|_{|D_t-D_x^3||D|L_{t,x}^\infty[I_\lambda]} \\ &\lesssim \sigma^{-1} \sup_i \|u_\mu^i\|_{L_{t,x}^\infty} \|Q_\sigma v_\mu^i\|_{L_{x,t}^p[I_\mu]} \\ &\lesssim \sigma^{-1} \mu^{-2s-1} \mu \sup_i \|u_\mu^i\|_{X^{-s,1+s}[I_\mu]} \|v_\mu^i\|_{|D|L_{t,x}^\infty[I_\mu]} \\ &\lesssim \mu^{-2s-3} \sup_{I_\mu} \|u_\mu\|_{S[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

In both case, we can sum up frequency when $-1 \leq s \leq -\frac{3}{4}$.

Case 2.2.3: $Z \times Z$. When u_μ, v_μ both have high modulation, we put them in L^3 (30).

We begin with the L^2 estimate

$$\begin{aligned} \left\| \sum_i \lambda Q_\sigma(u_\mu^i v_\mu^i) \right\|_{L^2_{t,x}[I_\lambda]} &\lesssim \lambda(\lambda\sigma)^{\frac{1}{6}} \left\| \sum_i Q_\sigma(u_\mu^i v_\mu^i) \right\|_{L^{\frac{3}{2}}_{t,x}[I_\lambda]} \\ &\lesssim \lambda(\lambda\sigma)^{\frac{1}{6}} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{2}{3}} \sup_i \|u_\mu^i\|_{L^3_{t,x}[I_\mu^i]} \|v_\mu^i\|_{L^3_{t,x}[I_\mu^i]} \\ &\lesssim \sigma^{\frac{1}{6}} \lambda^{\frac{19}{6} + \frac{8s}{3}} \mu^{-\frac{16}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{Z[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

Here notice we used $l^{\frac{3}{2}}$ summation of the intervals.

From Lemma 4.3, we get

$$\left\| \sum_i \lambda u_\mu^i v_\mu^i \right\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{-\frac{1}{2} + \frac{17(s+1)}{3}} \mu^{-\frac{16}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{Z[I_\mu]} \|v_\mu\|_{X^s}.$$

To see we can sum up frequency, notice exponent for μ is negative and all the exponents add up to $-\frac{1}{2} + \frac{1}{3}(s+1) < 0$.

4.3. Estimate for $X^1 \times S$. Suppose $u \in X^1, v \in S$. This includes the most dedicate case, i.e. low frequency high modulation piece interact with high frequency low modulation, where we can not prove the bilinear estimate (42). Instead we have to reiterate the equation and turn the bilinear estimate to trilinear. Let us work on high-high frequency interaction first.

Case 3.1: High-high frequency interaction giving out equal or lower frequency, $\lambda \lesssim \mu$. Same as before, we need to cut into smaller intervals if $\lambda \ll \mu$, and this procedure degenerate if $\lambda \approx \mu$.

Case 3.1.1: $X^1 \times X^{-s, 1+s}$, by (48) we must have modulation $\sigma \gtrsim \lambda\mu^2$ in some term.

$$\lambda\eta_\lambda u_\mu v_\mu = \sum_i \sum_{Q_j \in \{Q_{\gtrsim \lambda\mu^2}, Q_{\ll \lambda\mu^2}\}} \lambda Q_1[(Q_2 u_\mu^i)(Q_3 v_\mu^i)].$$

Case 3.1.1(a): When high modulation is on output, i.e. $Q_1 = Q_{\sigma \gtrsim \lambda\mu^2}$. Using $L_t^\infty L_x^2$ on $u_\mu^i, L^2_{t,x}$ on v_μ^i , together with Bernstein on the product, we get,

$$\begin{aligned} \left\| \lambda \sum_i Q_\sigma[(Q_2 u_\mu^i)(Q_3 v_\mu^i)] \right\|_{L^2_{t,x}[I_\lambda]} &\lesssim \lambda^{\frac{3}{2}} \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} \sup_i \|u_\mu^i\|_{L_t^\infty L_x^2[I_\mu^i]} \|v_\mu^i\|_{L^2_{t,x}[I_\mu^i]} \\ &\lesssim \lambda^{2s+3} \mu^{-4s-\frac{7}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}. \end{aligned}$$

Using the fact that output has high modulation and Lemma 4.3 with $k = \frac{1}{2}$, we get

$$\left\| \lambda \sum_i Q_{\sigma \gtrsim \lambda\mu^2}(u_\mu^i v_\mu^i) \right\|_{DZ[I_\lambda]} \lesssim \lambda^{4s+4} \mu^{-4s-\frac{9}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.$$

Case 3.1.1(b): High modulation on u_μ , $Q_2 = Q_{\gtrsim\lambda\mu^2}$. We put them both in $L_{t,x}^2$.

$$\begin{aligned}
& \left\| \sum_i \lambda Q_\sigma [(Q_{\gtrsim\lambda\mu^2} u_\mu^i) Q_3 v_\mu^i] \right\|_{L_{t,x}^2[I_\lambda]} \\
& \lesssim \lambda (\lambda\sigma)^{\frac{1}{2}} \left\| \sum_i Q_\sigma [(Q_{\gtrsim\lambda\mu^2} u_\mu^i) Q_3 v_\mu^i] \right\|_{L_{t,x}^1[I_\lambda]} \\
& \lesssim \lambda (\lambda\sigma)^{\frac{1}{2}} \left| \frac{I_\lambda}{I_\mu} \right| \sup_i \|Q_{\gtrsim\lambda\mu^2} u_\mu^i\|_{L_{t,x}^2[I_\mu^i]} \|Q_3 v_\mu^i\|_{L_{t,x}^2[I_\mu^i]} \\
& \lesssim \sigma^{\frac{1}{2}} \lambda^{4s+\frac{7}{2}} \mu^{-8s-\frac{17}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.
\end{aligned}$$

hence we have

$$\left\| \sum_i \lambda (Q_{\gtrsim\lambda\mu^2} u_\mu^i) Q_3 v_\mu^i \right\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{7s+\frac{13}{2}} \mu^{-8s-\frac{17}{2}} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.$$

Case 3.1.1(c): High modulation comes from input $Q_3 = Q_{\gtrsim\lambda\mu^2}$. We use local smoothing (17) on u_μ , and $L_{t,x}^2$ on v_μ .

$$\begin{aligned}
& \left\| \sum_i \lambda Q_\sigma [Q_2 u_\mu^i (Q_{\gtrsim\lambda\mu^2} v_\mu^i)] \right\|_{L_{t,x}^2[I_\lambda]} \\
& \lesssim \lambda \sigma^{\frac{1}{2}} \left\| \sum_i Q_\sigma [Q_2 u_\mu^i (Q_{\gtrsim\lambda\mu^2} v_\mu^i)] \right\|_{L_x^2 L_t^1[I_\lambda]} \\
& \lesssim \lambda \sigma^{\frac{1}{2}} \left| \frac{I_\lambda}{I_\mu} \right| \sup_i \|Q_2 u_\mu^i\|_{L_x^\infty L_t^2[I_\mu^i]} \|Q_{\gtrsim\lambda\mu^2} v_\mu^i\|_{L_{t,x}^2[I_\mu^i]} \\
& \lesssim \sigma^{\frac{1}{2}} \lambda^{3s+3} \mu^{-6s-6} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.
\end{aligned}$$

Hence we have

$$\left\| \sum_i \lambda Q_2 u_\mu^i (Q_{\gtrsim\lambda\mu^2} v_\mu^i) \right\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{6s+6} \mu^{-6s-6} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{S[I_\mu]}.$$

Case 3.1.2: $X^1 \times Z$. This forces high modulation $\sigma_m \gtrsim \mu^3$ also on the output.

$$\lambda \eta_\lambda u_\mu v_\mu = \sum_i \lambda Q_{\approx\sigma_m} [u_\mu^i (Q_{\sigma_m} v_\mu^i)]$$

We still bound the output in L^2 by using L^6 on u_μ , L^3 (30) on v_μ .

$$\begin{aligned}
& \left\| \sum_i \lambda Q_\sigma [u_\mu^i (Q_{\sigma_m} v_\mu^i)] \right\|_{L_{t,x}^2[I_\lambda]} \\
& \lesssim \lambda \left| \frac{I_\lambda}{I_\mu} \right|^{\frac{1}{2}} \sup_i \|u_\mu^i\|_{L_{t,x}^6[I_\mu^i]} \|Q_{\sigma_m} v_\mu^i\|_{L_{t,x}^3[I_\mu^i]} \\
& \lesssim \lambda^{2s+\frac{5}{2}} \mu^{-2s-\frac{3}{2}} \mu^{-\frac{1}{6}-s} \mu^{-\frac{4}{3}(s+1)-\frac{1}{3}} \sup_i \|u_\mu^i\|_{X^1[I_\mu^i]} \|v_\mu^i\|_{S[I_\mu^i]} \\
& \lesssim \lambda^{2s+\frac{5}{2}} \mu^{-5s-4+\frac{2}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.
\end{aligned}$$

From Lemma 4.3 with $k = \frac{1}{2}$, we get

$$\left\| \sum_i \lambda Q_{\sigma_m} [u_\mu^i (Q_{\sigma_m} v_\mu^i)] \right\|_{DZ[I_\lambda]} \lesssim \lambda^{4s+4} \mu^{-5s-\frac{11}{2}+\frac{2}{3}(s+1)} \sup_{I_\mu} \|u_\mu\|_{X^1[I_\mu]} \|v_\mu\|_{X^s}.$$

Case 3.2: High low frequency interaction. $u_\alpha \in X^1, v_\lambda \in S, \lambda \gg \alpha$. The bilinear estimate (34) is not good enough, so we have to break into more cases.

Case 3.2.1: $u_\alpha \in X^1, v_\lambda \in X^{-s, 1+s}$. Because of high modulation relation (48), we have

$$\lambda \eta_\lambda u_\alpha v_\lambda = \sum_{Q_i \in \{Q_{\gtrsim \lambda^2 \alpha}, Q_{\ll \lambda^2 \alpha}\}} \lambda \eta_\lambda Q_1[(Q_2 u_\alpha)(Q_3 v_\lambda)].$$

Case 3.2.1(a): High modulation on u_α . $Q_2 = Q_{\sigma \gtrsim \lambda^2 \alpha}$. Put u_α in L^2 , v_λ in L^∞ (26).

$$\|\lambda \eta_\lambda Q_1[(Q_{\sigma \gtrsim \lambda^2 \alpha} u_\alpha)(Q_3 v_\lambda)]\|_{L_{t,x}^2[I_\lambda]} \lesssim \lambda^{-4s - \frac{7}{2}} \alpha^{-1-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]},$$

so from Lemma 4.3, we get

$$\|\lambda \eta_\lambda [(Q_{\sigma \gtrsim \lambda^2 \alpha} u_\alpha)(Q_3 v_\lambda)]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{-s-2} \alpha^{-1-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]}.$$

Case 3.2.1(b): High modulation on v_λ . $Q_3 = Q_{\sigma \gtrsim \lambda^2 \alpha}$. Put u_α in L^∞ , v_λ in L^2 .

$$\|\lambda \eta_\lambda Q_1[(Q_2 u_\alpha)(Q_{\sigma \gtrsim \lambda^2 \alpha} v_\lambda)]\|_{L_{t,x}^2[I_\lambda]} \lesssim \lambda^{-1-s} \alpha^{-\frac{1}{2}-2s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]},$$

so we get

$$\|\lambda \eta_\lambda [(Q_2 u_\alpha)(Q_{\sigma \gtrsim \lambda^2 \alpha} v_\lambda)]\|_{Y_\lambda[I_\lambda]} \lesssim \lambda^{2s + \frac{1}{2}} \alpha^{-\frac{1}{2}-2s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]}.$$

Case 3.2.1(c): When none of u_α, v_λ have high modulation, this forces the output to be approximately $\lambda^2 \alpha$. $Q_1 = Q_{\sigma \approx \lambda^2 \alpha}$, put u_α in L^∞ , v_λ in L^2 .

When $\lambda^2 \alpha \lesssim \lambda^{4 + \frac{3}{2s}}$, i.e. $\alpha \lesssim \lambda^{2 + \frac{3}{2s}}$ we have

$$\begin{aligned} & \|\lambda \eta_\lambda Q_{\sigma \approx \lambda^2 \alpha} [(Q_2 u_\alpha)(Q_3 v_\lambda)]\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2} \\ & \lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \alpha^{\frac{1}{2}-s} \lambda^{-2-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]} \\ & \lesssim \alpha^{\frac{1}{2}-s} \lambda^{2s + \frac{1}{2}} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]}, \end{aligned}$$

notice we have $\alpha^{\frac{1}{2}-s} \lambda^{2s + \frac{1}{2}} \lesssim \lambda^{\frac{3}{4s}}$, which is good for summation.

When $\lambda^2 \alpha \gtrsim \lambda^{4 + \frac{3}{2s}}$, we have

$$\begin{aligned} & \|\lambda \eta_\lambda Q_{\sigma \approx \lambda^2 \alpha} [(Q_2 u_\alpha)(Q_3 v_\lambda)]\|_{X^{-s, s}} \\ & \lesssim \lambda^{1-s} \sigma^s \alpha^{\frac{1}{2}-s} \lambda^{-2-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]} \\ & \lesssim \alpha^{\frac{1}{2}} \lambda^{-1} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\lambda\|_{X^{-s, 1+s}[I_\lambda]}. \end{aligned}$$

Case 3.2.2: $u_\alpha \in X^1, v_\lambda \in Z$. Here the bilinear estimate (35) is good enough.

$$\begin{aligned} \|\lambda \eta_\lambda u_\alpha v_\lambda\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2[I_\lambda]} & \lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \|\eta_\lambda u_\alpha v_\lambda\|_{L_{t,x}^2[I_\lambda]} \\ & \lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \lambda^{-\frac{1}{3}-2s-2} \alpha^{-\frac{1}{6}-s} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]} \\ & \lesssim \lambda^{\frac{1}{6}+s} \alpha^{-\frac{1}{6}-s} \|u_\lambda\|_{Z[I_\lambda]} \|v_\alpha\|_{X^1[I_\alpha]}. \end{aligned}$$

Case 3.3: High low frequency interaction. $u_\lambda \in X^1, v_\alpha \in S, \lambda \gg \alpha$.

Case 3.3.1: $u_\lambda \in X^1, v_\alpha \in X^{-s, 1+s}$. Without going into modulation analysis, we use $L_x^\infty L_t^2$ on u_λ , and $L_x^2 L_t^\infty$ on v_α , together with Bernstein and notice the modulation on v_α is small.

$$\begin{aligned} \|\lambda \eta_\lambda u_\lambda v_\alpha\|_{|D|^{-s}|I|^{-\frac{1}{2}} L^2} & \lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \|v_\lambda\|_{L_x^\infty L_t^2[I_\lambda]} \|v_\alpha\|_{L_x^2 L_t^\infty[I_\alpha]} \\ & \lesssim \lambda^{1+s} |I_\lambda|^{\frac{1}{2}} \lambda^{-1-s} \alpha^{-\frac{3}{2}-2s} \|u_\lambda\|_{X^1[I_\lambda]} \|v_\alpha\|_{X^{-s, 1+s}[I_\alpha]} \\ & \lesssim \lambda^{2s + \frac{3}{2}} \alpha^{-\frac{3}{2}-2s} \|u_\lambda\|_{X^1[I_\lambda]} \|v_\alpha\|_{X^{-s, 1+s}[I_\alpha]}. \end{aligned}$$

Case 3.3.2: $u_\lambda \in X^1, v_\alpha \in Z$. Here we can not prove any bilinear estimate if high modulation fall on v_α , so we need the following lemma to reiterate the equation.

Lemma 4.6. (Reiterate the equation) Let w be a solution to KdV equation (1). Then we can write its high modulation part as

$$Q_{\sigma \gtrsim \alpha^3} w_\alpha = M_1 + M_2 + R,$$

where M_1, M_2, R are as follows:

- M_1 is the output of two higher frequency-low modulation interaction,

$$M_1 = \sum_{\alpha \lesssim \beta_1 \approx \beta_2} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(w_{\beta_1} w_{\beta_2}), \quad w_{\beta_1}, w_{\beta_2} \in X^1$$

where w_{β_1}, w_{β_2} all have very low modulation $|\tau - \xi^3| \lesssim |\xi|^{4 + \frac{3}{2s}}$.

- M_2 is the output of the high frequency-low modulation piece interact with low frequency-high modulation piece.

$$M_2 = \sum_{\sigma \gtrsim \alpha^3, \gamma \ll \beta \approx \alpha} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(w_\beta w_\gamma), \quad w_\beta \in X^1, w_\gamma \in Z.$$

w_β has modulation $|\tau - \xi^3| \lesssim |\xi|^{4 + \frac{3}{2s}}$, w_γ has high modulation $|\tau - \xi^3| \gtrsim |\xi|^3$.

- R is the remainder, which comes from interaction of all other cases

$$R = \sum_{\sigma \gtrsim \alpha^3, \beta, \gamma} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(w_\beta w_\gamma).$$

For R , we have the estimate

$$(51) \quad \|\eta_\alpha(t) R_\alpha\|_{\alpha^{-2s - \frac{3}{2}} L_x^2 L_t^\infty} \lesssim \|w\|_{X^s \cap X_{le}^s[I_\alpha]}^2.$$

The decomposition above is true modulo \pm sign on each term.

Proof. : If we apply frequency and modulation projection on the equation, we get

$$(\partial_t + \partial_x^3) P_\alpha Q_\sigma w = -P_\alpha Q_\sigma \partial_x(w^2).$$

Hence modulo \pm sign we have

$$P_\alpha Q_\sigma w = (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(w^2).$$

Here we decompose w into dyadic pieces, $P_\alpha Q_\sigma w = (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(w_\beta w_\gamma)$. Now we first break each w_λ into sum of functions supported on time scale $|\lambda|^{4s+3}$. Next, for each $w_\lambda \in X^s \cap X_{le}^s[I_\lambda]$, let us decompose it as $w_\lambda = w_{\lambda,1} + w_{\lambda,2}$, $w_{\lambda,1} \in X^1$, $w_{\lambda,2} \in S$. Then we can just take u_β, v_γ to represent $w_{\beta,i}, w_{\gamma,j}$, $i, j \in \{1, 2\}$.

We will prove that except for the two cases in M_1 and M_2 , we have the estimate (51). We list the estimates of all cases below, which are similar to what we have done before. Notice the modulation is always larger than α^3 in the summation.

Case 1: $\beta \approx \alpha \gg \gamma$.

(1) $u_\alpha, v_\gamma \in X^1$, use Bernstein and bilinear estimate (32)

$$\begin{aligned} \|\eta_\alpha \sum_{\sigma} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(u_\alpha v_\gamma)\|_{\alpha^{2s + \frac{3}{2}} L_x^2 L_t^\infty} &\lesssim \frac{\alpha^{2s + \frac{5}{2}}}{\sigma^{\frac{1}{2}}} \alpha^{-1-s} \gamma^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]} \\ &\lesssim \alpha^{-1+s} \gamma^{-s} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]}. \end{aligned}$$

(2) $u_\alpha \in X^1, v_\gamma \in S$ we only deal with $v_\gamma \in X^{-s,1+s}$. And leave $v_\gamma \in Z$ term into M_2 . Notice here u_α must have high modulation σ (48). Put L^2 on u_α , L^∞ on v_γ

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma((Q_\sigma u_\alpha)v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+\frac{5}{2}} \sigma^{-\frac{1}{2}} \sigma^{-1} |I_\alpha|^{-\frac{1}{2}} \alpha^{-s} \gamma^{-2s-1} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]} \\ & \lesssim \alpha^{-s-\frac{7}{2}} \gamma^{-2s-1} \|u_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}. \end{aligned}$$

(3) $u_\alpha \in S, v_\gamma \in X^1$, use the bilinear estimate (34)

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(u_\alpha v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+\frac{5}{2}} \sigma^{-\frac{1}{2}} \max\{\alpha^{-\frac{1}{3}-2s-2} \gamma^{-\frac{1}{6}-s}, \alpha^{-2-s} \gamma^{\frac{1}{2}-s}\} \|u_\alpha\|_{S[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]} \\ & \lesssim \max\{\alpha^{-\frac{4}{3}} \gamma^{-\frac{1}{6}-s}, \alpha^{s-1} \gamma^{\frac{1}{2}-s}\} \|u_\alpha\|_{S[I_\alpha]} \|v_\gamma\|_{X^1[I_\gamma]}. \end{aligned}$$

(4) $u_\alpha, v_\gamma \in S$, we consider several cases:

If $u_\alpha, v_\gamma \in X^{-s,1+s}$, or $u_\alpha \in Z, v_\gamma \in X^{-s,1+s}$, or $u_\alpha, v_\gamma \in Z$, then we have the bilinear estimate (40). So we have

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma((Q_\sigma u_\alpha)v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \frac{\alpha^{2s+\frac{5}{2}}}{\sigma^{\frac{1}{2}}} \alpha^{-\frac{4}{3}(s+1)-\frac{1}{3}} \gamma^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]} \\ & \lesssim \alpha^{-\frac{4}{3}+\frac{2}{3}(s+1)} \gamma^{-\frac{2}{3}(s+1)+\frac{1}{3}} \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}. \end{aligned}$$

Notice the exponents add up to -1 .

If $u_\alpha \in X^{-s,1+s}, v_\gamma \in Z$, use L^2 on u_α , L^p on v_γ

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma(u_\alpha v_\gamma)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+\frac{5}{2}} \sigma^{-1} \sigma^{\frac{1}{2}} \alpha^{-2-s} \gamma^1 \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{Z[I_\gamma]} \\ & \lesssim \alpha^{s-1} \gamma \|u_\alpha\|_{X^{-s,1+s}[I_\alpha]} \|v_\gamma\|_{X^{-s,1+s}[I_\gamma]}. \end{aligned}$$

The exponents add up to $s < 0$.

Case 2: $\beta \approx \gamma \gtrsim \alpha$. This part is every similar to the estimates in Case 1.2, 2.2 and 3.1. We still need to decompose u_β into sums of functions that are supported on the μ^{4s+3} time scale. $u_\beta = \sum_i u_\beta^i$, $u_\beta^i \in X_\beta[I_\beta^i]$

(1) $u_\beta, v_\beta \in X^1$, when one of input e.g. u_β has high modulation $Q_{\gtrsim \alpha \beta^2}$, estimate u_β in L^2 , and v_β in $L_x^\infty L_t^2$. Here because we want to use Bernstein, but also want to have better summation of time intervals. So we need to use local energy space X_{le}^s similarly as in

case 1.2(b).

$$\begin{aligned}
& \sum_j \|\chi_j^\alpha(x) \eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma((Q_{\sigma_m} u_\beta) v_\beta)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty}^2 \\
\lesssim & \sum_j \alpha^{4s+5} \|\chi_j^\alpha(x) \sum_\sigma P_\alpha Q_\sigma((Q_{\sigma_m} u_\beta) v_\beta)\|_{L_x^2 L_t^1[I_\alpha]}^2 \\
\lesssim & \sum_j \alpha^{4s+5} \|\chi_j^\alpha(x) Q_{\sigma_m} u_\beta\|_{L_{t,x}^2[I_\alpha]}^2 \|\chi_j^\alpha(x) v_\beta\|_{L_x^\infty L_t^2[I_\alpha]}^2 \\
\lesssim & \alpha^{4s+5} \sum_{i,j} \|\chi_j^\alpha(x) Q_{\sigma_m} u_\beta^i\|_{L_{t,x}^2[I_\beta^i]}^2 \sup_j \sum_i \|\chi_j^\alpha(x) v_\beta^i\|_{L_x^\infty L_t^2[I_\beta^i]}^2 \\
\lesssim & \alpha^{4s+5} \left| \frac{I_\alpha}{I_\beta} \right| \sup_i \|Q_{\sigma_m} u_\beta^i\|_{L_{t,x}^2[I_\beta^i]}^2 \sup_j \sum_i \|\chi_j^\alpha(x) v_\beta^i\|_{L_x^\infty L_t^2[I_\beta^i]}^2 \\
\lesssim & \alpha^{8s+6} \beta^{-12s-12} \|u_\beta\|_{X_\beta^s[I_\beta]}^2 \|v_\beta\|_{X_{te}^s}^2.
\end{aligned}$$

Remark 4.7. In these estimates, we need to sum up all the modulations larger than α^3 . It is fine as long as there is a negative factor of σ through the estimate. But in the one above, we need be more careful. Split the problem into $\sigma \approx \alpha^3$, and $\sigma \gg \alpha$.

When $\sigma \approx \alpha^3$, we can sum up modulation easily.

When $\sigma \gg \alpha^3$, we prove $(\partial_t + \partial_x^3)^{-1} : L_x^2 L_t^1 \rightarrow L_x^2 L_t^\infty$ is bounded operator which is done by looking at the symbol $\frac{1}{\tau - \xi^3} = \frac{1}{\tau} + \frac{\xi^3}{\tau(\tau - \xi^3)} \approx \frac{1}{\tau}$. And $\partial_t^{-1} : L_x^2 L_t^1 \rightarrow L_x^2 L_t^\infty$ is bounded if it acts on functions which vanish at ∞ .

(2) $u_\beta \in X^1, v_\beta \in S$, we also split it into two cases:

(a) When $u_\beta \in X^1, v_\beta \in X^{-s, 1+s}$. Now if the output modulation $\sigma \gtrsim \alpha\beta^2$, use L^∞ on u_β , and L^2 on v_β ,

$$\begin{aligned}
& \|\eta_\alpha \sum_{\sigma \gtrsim \alpha\beta^2} (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\
\lesssim & \frac{\alpha^{2s+\frac{5}{2}}}{\sigma^{\frac{1}{2}}} \left| \frac{I_\alpha}{I_\beta} \right|^{\frac{1}{2}} \beta^{\frac{1}{2}-s} \beta^{-2-s} \sup_i \|u_\beta^i\|_{X^1[I_\beta]^i} \|v_\beta^i\|_{X^{-s, 1+s}[I_\beta^i]} \\
\lesssim & \alpha^{4s+\frac{7}{2}} \beta^{-4s-4} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}
\end{aligned}$$

And if $\sigma \ll \alpha\beta^2$, we use $L_x^\infty L_t^2$ on u_β , L^2 on v_β . We still play the trick: using local energy space to get l^2 summation of the intervals.

$$\begin{aligned}
& \|\eta_\alpha \sum_{\sigma \ll \alpha\beta^2} (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\
\lesssim & \alpha^{2s+\frac{5}{2}} \left| \frac{I_\alpha}{I_\beta} \right|^{\frac{1}{2}} \beta^{-1-s} \beta^{-2-s} \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{X_{te}^s} \\
\lesssim & \alpha^{4s+4} \beta^{-4s-\frac{9}{2}} \|u_\beta\|_{X^s} \|v_\beta\|_{X_{te}^s}.
\end{aligned}$$

The point here is we can sum up the modulation $\alpha^3 \lesssim \sigma \ll \alpha\beta^2$, which give us at most $\log \beta$ loss. But we are fine because of the negative power on β . We will do a similar thing whenever we want to be careful with modulation summation, hence we will ignore it.

(b) When $u_\beta \in X^1, v_\beta \in Z$. This force high modulation $\sigma_m \gtrsim \beta^3$ on u_β , or on output.

When σ_m is on u_β , use L^2 on u_β , L^∞ on v_β .

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \frac{\alpha^{2s+\frac{5}{2}}}{\sigma} \sigma^{\frac{1}{2}} \left| \frac{I_\alpha}{I_\beta} \right|^{\frac{1}{2}} \sigma_m^{-1} |I_\beta|^{-\frac{1}{2}} \beta^{-s} \beta \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{Z[I_\beta^i]} \\ & \lesssim \alpha^{4s+\frac{5}{2}} \beta^{-5s-5} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

When σ_m is on output, simply put L^6 on u_β , and L^3 on v_β .

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i Q_{\sigma_m} v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \frac{\alpha^{2s+\frac{5}{2}}}{\sigma_m} \sigma_m^{\frac{1}{2}} \left| \frac{I_\alpha}{I_\beta} \right|^{\frac{1}{2}} \beta^{-\frac{1}{6}-s} \beta^{-\frac{4}{3}(s+1)-\frac{1}{3}} \sup_i \|u_\beta^i\|_{X^1[I_\beta^i]} \|v_\beta^i\|_{Z[I_\beta^i]} \\ & \lesssim \alpha^{4s+\frac{7}{2}} \beta^{-5s-5-\frac{2}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(3) $u_\beta, v_\beta \in S$. We still break into cases.

(a) $u_\beta, v_\beta \in X^{-s,1+s}$, use L^2 on both, and l^1 summation of interval is good enough.

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+3} \left| \frac{I_\alpha}{I_\beta} \right| \sup_i \|u_\beta^i\|_{L_{t,x}^2[I_\beta^i]} \|v_\beta^i\|_{L_{t,x}^2[I_\beta^i]} \\ & \lesssim \alpha^{6s+6} \beta^{-6s-7} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(b) $u_\beta \in X^{-s,1+s}, v_\beta \in Z$, L^2 on u_β , L^3 on v_β , with a l^1 summation of interval.

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+\frac{5}{2}+\frac{1}{3}} \sigma^{-\frac{1}{6}} \left| \frac{I_\alpha}{I_\beta} \right| \sup_i \|u_\beta^i\|_{L_{t,x}^2[I_\beta^i]} \|v_\beta^i\|_{L_{t,x}^3[I_\beta^i]} \\ & \lesssim \alpha^{6s+\frac{16}{3}} \beta^{-7s-7-\frac{1}{3}+\frac{2}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

(c) $u_\beta, v_\beta \in Z$, Here we are a bit careful about interval cut off, using the $l^{\frac{3}{2}}$ summation.

$$\begin{aligned} & \|\eta_\alpha \sum_\sigma (\partial_t + \partial_x^3)^{-1} \alpha \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i)\|_{\alpha^{2s+\frac{3}{2}} L_x^2 L_t^\infty} \\ & \lesssim \alpha^{2s+\frac{5}{2}+\frac{1}{6}} \sigma^{-\frac{1}{3}} \left\| \sum_i P_\alpha Q_\sigma(u_\beta^i v_\beta^i) \right\|_{L_{t,x}^{\frac{3}{2}}[I_\alpha]} \\ & \lesssim \alpha^{2s+\frac{5}{2}+\frac{1}{6}} \sigma^{-\frac{1}{3}} \left| \frac{I_\alpha}{I_\beta} \right|^{\frac{2}{3}} \sup_i \|u_\beta^i\|_{L_{t,x}^3[I_\beta^i]} \|v_\beta^i\|_{L_{t,x}^3[I_\beta^i]} \\ & \lesssim \alpha^{2s+1+\frac{8s+8}{3}} \beta^{-\frac{16}{3}(s+1)} \|u_\beta\|_{X^s} \|v_\beta\|_{X^s}. \end{aligned}$$

□

Now we use this lemma to finish our estimate of Case 3.3, $u_\lambda \in X^1, v_\alpha \in Z$.

$$\lambda u_\lambda v_\alpha = \lambda u_\lambda (M_{1\alpha} + M_{2\alpha} + R_\alpha).$$

Step 1: Let us do R_α first, using the estimate for R_α in the lemma.

$$\begin{aligned}
\|\lambda\eta_\lambda u_\lambda R_\alpha\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} &\lesssim \lambda^{3s+\frac{5}{2}}\|\eta_\lambda u_\lambda\|_{L_x^\infty L_t^2}\|\eta_\lambda R_\alpha\|_{L_x^2 L_t^\infty} \\
&\lesssim \lambda^{3s+\frac{5}{2}}\lambda^{-1-s}\alpha^{-2s-\frac{3}{2}}\|\eta_\lambda u_\lambda\|_{X^1}\|\eta_\lambda R_\alpha\|_{\alpha^{2s+\frac{3}{2}}L_x^2 L_t^\infty} \\
&\lesssim \left(\frac{\alpha}{\lambda}\right)^{-2s-\frac{3}{2}}\|u_\lambda\|_{X^s[I_\lambda]}\|v\|_{X^s\cap X_{le}^s}.
\end{aligned}$$

Step 2: Feed M_1 into the bilinear term, we divide it into two terms.

$$\lambda u_\lambda \sum_{\sigma \approx \alpha\beta^2, \alpha \lesssim \beta \ll \lambda} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha\beta^2}(v_\beta v_\beta) + \lambda u_\lambda \sum_{\sigma \approx \alpha\beta^2, \alpha \lesssim \lambda \lesssim \beta} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha\beta^2}(v_\beta v_\beta)$$

The first term, we will bilinear estimate for $u_\beta v_\lambda$, also here for fixed β , $P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)$ is almost orthogonal to each other, so we can sum up α

$$\begin{aligned}
&\|\lambda\eta_\lambda u_\lambda \sum_{\sigma \approx \alpha\beta^2, \alpha \lesssim \beta \ll \lambda} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha\beta^2}(v_\beta v_\beta)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \\
&\lesssim \lambda^{3s+\frac{5}{2}} \sum_{\alpha \lesssim \beta \ll \lambda} \frac{1}{\beta^2} \|\eta_\lambda u_\lambda P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)\|_{L_{t,x}^2} \\
&\lesssim \lambda^{3s+\frac{5}{2}} \sum_{\alpha \lesssim \beta \ll \lambda} \frac{1}{\beta^2} \|\eta_\lambda u_\lambda u_\beta\|_{L_{t,x}^2} \|\eta_\lambda v_\beta\|_{L_{t,x}^\infty} \\
&\lesssim \lambda^{3s+\frac{5}{2}} \sum_{\alpha \lesssim \beta \ll \lambda} \frac{1}{\beta^2} \lambda^{-1-s} \alpha^{\frac{1}{2}} \beta^{-2s} \|u_\lambda\|_{X^1[I_\lambda]} \|v_\beta\|_{X^1[I_\beta]} \|v_\beta\|_{X^1[I_\beta]} \\
&\lesssim \left(\frac{\lambda}{\beta}\right)^{2s+\frac{3}{2}} \|u_\lambda\|_{X^s[I_\lambda]} \|v_\beta\|_{X^s[I_\beta]} \|v_\beta\|_{X^s[I_\beta]}.
\end{aligned}$$

Here we actually used the fact that, when fix α , the two v_β 's can be decomposed to functions with \hat{v}_β supported on size α interval, so we used bernstein to get

$$\|v_\beta\|_{L_{t,x}^\infty} \lesssim \alpha^{\frac{1}{2}} \|v_\beta\|_{L_t^\infty L_x^2}.$$

So for $s \leq -\frac{3}{4}$, we can sum up β .

For the second term, we will use at least l^4 interval summation (or better if we use local energy space). The good thing is that for β fixed, then $P_\alpha Q_{\alpha\beta^2}(v_\beta v_\beta)$ are almost orthogonal to each other in both space and time, so we can sum up α and then ignore it.

Also because u_β is measured on the smallest time scale, we still need to cut the interval.

$$\begin{aligned}
& \|\lambda \eta_\lambda u_\lambda \sum_{\sigma \approx \alpha \beta^2, \alpha \lesssim \lambda \lesssim \beta} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_{\sigma \approx \alpha \beta^2} (v_\beta v_\beta)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \\
& \lesssim \lambda^{3s+\frac{5}{2}} \|\eta_\lambda u_\lambda\|_{L_t^4 L_x^\infty} \|\eta_\lambda \sum_{\alpha \lesssim \lambda \lesssim \beta} \frac{1}{\beta^2} P_\alpha Q_{\alpha \beta^2} (v_\beta v_\beta)\|_{L_t^4 L_x^2} \\
& \lesssim \lambda^{3s+\frac{5}{2}-\frac{1}{4}-s} \|\eta_\lambda u_\lambda\|_{X^1} \sum_{\lambda \lesssim \beta} \|\eta_\lambda \sum_{\alpha \lesssim \lambda} P_\alpha Q_{\alpha \beta^2} (v_\beta v_\beta)\|_{L_t^4 L_x^2} \\
& \lesssim \lambda^{2s+\frac{9}{4}} \|u_\lambda\|_{X^s} \sum_{\lambda \lesssim \beta} \left| \frac{I_\lambda}{I_\beta} \right|^{\frac{1}{4}} \|\eta_\beta (v_\beta v_\beta)\|_{L_t^4 L_x^2} \\
& \lesssim \lambda^{2s+\frac{9}{4}} \|u_\lambda\|_{X^s} \sum_{\lambda \lesssim \beta} \left| \frac{I_\lambda}{I_\beta} \right|^{\frac{1}{4}} \|\eta_\beta v_\beta\|_{L_t^8 L_x^4} \|\eta_\beta v_\beta\|_{L_t^8 L_x^4} \\
& \lesssim \left(\frac{\lambda}{\beta}\right)^{3s+3} \|u_\lambda\|_{X^s} \|v_\beta\|_{X^s}^2
\end{aligned}$$

So we combine the two cases together and get

$$\|\lambda u_\lambda \sum_{\alpha \ll \lambda} M_{1\alpha}\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \lesssim \|u_\lambda\|_{X^s} \|v\|_{X^s}^2.$$

Step 3: Now we feed in the term M_2 , We want to use local energy norm, so let us cut up the space using $\chi_j^\alpha(x)$.

$$\begin{aligned}
& \|\lambda \eta_\lambda \chi_j^\alpha(x) u_\lambda \sum_{\gamma \ll \alpha \lesssim \lambda, \sigma \gtrsim \alpha^3} (\partial_t + \partial_x^3)^{-1} \alpha P_\alpha Q_\sigma (v_\alpha v_\gamma)\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \\
& \lesssim \lambda^{3s+\frac{5}{2}} \sum_{\gamma \ll \alpha \lesssim \lambda, \sigma \gtrsim \alpha^3} \frac{\alpha}{\sigma} \|\eta_\lambda u_\lambda\|_{L_x^\infty L_t^2} \|\eta_\lambda \chi_j^\alpha(x) P_\alpha Q_\sigma (v_\alpha v_\gamma)\|_{L_x^2 L_t^\infty} \\
& \lesssim \lambda^{3s+\frac{5}{2}} \lambda^{-1-s} \|u_\lambda\|_{X^s} \sum_{\gamma \ll \alpha \lesssim \lambda, \sigma \gtrsim \alpha^3} \frac{\alpha}{\sigma} \|\eta_\lambda \chi_j^\alpha(x) P_\alpha Q_\sigma (v_\alpha v_\gamma)\|_{L_x^2 L_t^\infty} \\
& \lesssim \sum_{\gamma \ll \alpha \lesssim \lambda, \sigma \gtrsim \alpha^3} \lambda^{2s+\frac{3}{2}} \|u_\lambda\|_{X^s} \alpha \sigma^{-1} \|\eta_\lambda \chi_j^\alpha(x) v_\alpha\|_{L_x^2 L_t^\infty} \|\eta_\lambda \chi^\alpha(x) v_\gamma\|_{L_{t,x}^\infty} \\
& \lesssim \sum_{\gamma \ll \alpha \lesssim \lambda, \sigma \gtrsim \alpha^3} \lambda^{2s+\frac{3}{2}} \alpha^{s+\frac{5}{2}} \sigma^{-1} \gamma \|u_\lambda\|_{X^s} \|\chi_j^\alpha(x) v_\alpha\|_{X^1[I_\alpha]} \|v_\gamma\|_{X^s} \\
& \lesssim \lambda^{2s+\frac{3}{2}} \alpha^{s+\frac{1}{2}} \|u_\lambda\|_{X^s} \|\chi_j^\alpha(x) v_\alpha\|_{X^1[I_\alpha]} \|v\|_{X^s}
\end{aligned}$$

We can also square sum up the spatial cutoff in the estimate above, and get

$$\|\lambda \eta_\lambda u_\lambda M_2\|_{|D|^{-s}|I|^{-\frac{1}{2}}L^2} \lesssim \|u_\lambda\|_{X^s} \|v\|_{X^s \cap X_{I_e}^s}.$$

In the proof we used the estimate

$$\|\eta_\lambda \chi_j^\alpha(x) v_\alpha\|_{L_x^2 L_t^\infty} \lesssim \|\chi_j^\alpha(x)\|_{L_x^4} \|\eta_\lambda \chi_j^\alpha(x) v_\alpha\|_{L_x^4 L_t^\infty} \lesssim \alpha^{s+\frac{3}{2}} \|u_\alpha\|_{X^1[I_\alpha]}.$$

Actually we also have $L_x^2 L_t^\infty$ maximal function estimate [20] on small time interval.

We end this section with two bilinear estimates, as a companion to Proposition 3.9. The proof is essentially repeating what we did previously.

Proposition 4.8. *For $\lambda \gg \alpha$ we have the following estimates*

$$(52) \quad \|\eta_\lambda u_\lambda (Q_{\sigma \gtrsim \alpha^3} v_\alpha)\|_{L_{t,x}^2} \lesssim \lambda^{-3s-\frac{5}{2}} \|u_\lambda\|_{X^1[I_\lambda]} \|v\|_{X^s \cap X_{I_e}^s},$$

$$(53) \quad \|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \max(\lambda^{-1-s}\alpha^{-s}, \lambda^{-3s-\frac{5}{2}}) \|u_\lambda\|_{X^s \cap X_{l_e}^s} (\|v\|_{X^s \cap X_{l_e}^s} + \|v\|_{X^s \cap X_{l_e}^s}^2),$$

$$(54) \quad \|\eta_\lambda(Q_{\sigma \gtrsim \lambda^3} u_\lambda) v_\alpha\|_{L_{t,x}^2} \lesssim \lambda^{-3s-\frac{5}{2}} \|u_\lambda\|_{X^s \cap X_{l_e}^s} \|v_\alpha\|_{X^s \cap X_{l_e}^s}.$$

Proof. For (52), we reiterate the equation, and notice in all the proofs we did, we are proving a L^2 estimate of the product, with weight $\lambda^{3s+\frac{5}{2}}$.

For (53), we compare the estimate in the following cases

If $u_\lambda, v_\alpha \in X^1$, we have (32); If $u_\lambda \in S, v_\alpha \in X^1$, we have (34).

If $u_\lambda \in X^1, v_\alpha \in S$, we have (52) and

$$\|\eta_\lambda u_\lambda v_\alpha\|_{L_{t,x}^2} \lesssim \|\eta_\lambda u_\lambda\|_{L_x^\infty L_t^2} \|\eta_\lambda v_\alpha\|_{L_x^2 L_t^\infty} \lesssim \lambda^{-1-s} \alpha^{-2s-\frac{3}{2}} \|u_\lambda\|_{X^1[l_\lambda]} \|v_\alpha\|_{X^{-s,1+s}[l_\alpha]}.$$

If $u_\lambda, v_\alpha \in S$, we have (40) except for $u_\lambda \in X^{-s,1+s}, v_\alpha \in Z$. But notice that the estimate (41) is larger than $\lambda^{-1-s}\alpha^{-s}$.

Hence we can sum up the estimates to get (53).

The proof of (54) is carried out in the same way as all the detailed analysis before. We discuss cases of $u_\lambda \in X^1$ or $X^{-s,1+s}$ or Z , and be a bit careful when $Q_{\sigma \gtrsim \lambda^3} u_\lambda \in X^{-s,1+s}$ or $|D|^{-2s-2} X_{\tau=\frac{1}{4}\xi^3}^{\frac{1}{4},\frac{1}{4}}$. \square

5. ENERGY CONSERVATION

In this section, we aim to study the conservation of H^s energy, this part of calculation follows similar as in [7] and [22].

Given a positive multiplier a , we set

$$E_2(u) = \langle a(D)u, u \rangle.$$

We want to take the symbol $a(\xi) = (1 + \xi^2)^s$, but as in [22], [23], we will allow a slightly larger class of symbols.

Definition 5.1. a) Let $s \in \mathbb{R}, \epsilon > 0$. Then S_ϵ^s is the class of spherically symmetric symbols with the following properties:

(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(1 + \xi^2)^{-\frac{\alpha}{2}}.$$

(ii) decay at infinity,

$$s \leq \frac{\ln a(\xi)}{\ln(1 + \xi^2)} \leq s + \epsilon, \quad s - \epsilon \leq \frac{\ln a(\xi)}{\ln(1 + \xi^2)} \leq s + \epsilon.$$

b) If a satisfies (i) and (ii) then we say that d is dominated by a , written as $d \in S(a)$, if

$$|\partial^\alpha d| \lesssim a(\xi)(1 + \xi^2)^{-\frac{\alpha}{2}},$$

with constant depending only on a .

Definition 5.2. (a) A k -multiplier generates a k -linear functional or k -form acting on k functions u_1, \dots, u_k

$$\Lambda_k(m; u_1, \dots, u_k) = \int_{\xi_1 + \dots + \xi_k = 0} m(\xi_1, \dots, \xi_k) \widehat{u}_1(\xi_1) \cdots \widehat{u}_k(\xi_k).$$

We will write $\Lambda_k(m)$ for $\Lambda_k(m; u, \dots, u)$.

(b) The symmetrization of a k -multiplier m is the multiplier

$$[m]_{sym}(\xi) = \frac{1}{n!} \sum_{\sigma \in S_k} m(\sigma(\xi)).$$

We have the following computation [7].

Proposition 5.3. *Suppose u satisfies the KdV equation (1) and m is a symmetric k -multiplier. Then*

$$\frac{d}{dt}\Lambda_k(m) = \Lambda_k(m\Delta_k) - i\frac{k}{2}\Lambda_{k+1}(m(\xi_1, \dots, \dots, \xi_{k-1}, \xi_k + \xi_{k+1})(\xi_k + \xi_{k+1})),$$

where

$$\Delta_k = i(\xi_1^3 + \dots + \xi_k^3).$$

5.1. Symbol calculation of modified energy. Here we construct modified energy, following the calculation in [7].

We first compute the derivative of E_2 along the flow

$$\frac{d}{dt}E_2(u) = \Lambda_3(M_3).$$

Easy to see that $M_3 = c \sum_{i=1}^3 (a(\xi_i)\xi_i)$, we will ignore the constant. Now we form modified energy

$$E_3(u) = E_2(u) + \Lambda_3(\sigma_3),$$

and we aim to choose the symmetric 3-multiplier σ_3 to achieve a cancellation.

$$\frac{d}{dt}E_3(u) = \Lambda_3(M_3) + \Lambda_3(\sigma_3\Delta_3) + \Lambda_4(-i\frac{3}{2}\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)).$$

So if we take

$$\sigma_3 = -\frac{M_3}{\Delta_3},$$

we get

$$\frac{d}{dt}E_3(u) = \Lambda_4(M_4), \quad M_4 = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}.$$

Similarly, we can define $E_4(u) = E_3(u) + \Lambda_4(\sigma_4)$, $\sigma_4 = -\frac{M_4}{\Delta_4}$,

$$\frac{d}{dt}E_4(u) = \Lambda_5(M_5),$$

then we have

$$M_5 = -2i[\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)]_{sym}.$$

This process can be continued to have further corrections, but we will stop here, since higher corrections are harder to estimate.

5.2. Bounds for multipliers. In order to estimate the derivative of modified energy, we need to have good bounds for M_i and σ_i . Also now M_i is defined only on the diagonal $\xi_1 + \dots + \xi_k = 0$, but in order to separate variables, we want to extend it off diagonal, this is useful when we prove local energy decay later on.

Proposition 5.4. *Assume that $a \in S_\epsilon^s$ and $d \in S(a)$, then there exist functions b and c such that*

$$\sum_{i=1}^3 a(\xi_i)\xi_i = b(\xi_1, \xi_2, \xi_3)(\xi_1^3 + \xi_2^3 + \xi_3^3) + c(\xi_1, \xi_2, \xi_3)(\xi_1 + \xi_2 + \xi_3).$$

And on each dyadic region $\{\xi_1 \sim \alpha, \xi_2 \sim \lambda, \xi_3 \sim \mu, \alpha \leq \lambda \leq \mu\}$, we have the regularity conditions

$$\begin{aligned} \partial_1^{s_1} \partial_2^{s_2} \partial_3^{s_3} b(\xi_1, \xi_2, \xi_3) &\lesssim a(\alpha) \lambda^{-1} \mu^{-1} \alpha^{-s_1} \lambda^{-s_2} \mu^{-s_3}, \\ \partial_1^{s_1} \partial_2^{s_2} \partial_3^{s_3} c(\xi_1, \xi_2, \xi_3) &\lesssim a(\alpha) \lambda^{-1} \mu \alpha^{-s_1} \lambda^{-s_2} \mu^{-s_3}. \end{aligned}$$

Proof. Since

$$\xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3 + (\xi_1 + \xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 - \xi_1\xi_3).$$

Let's construct

$$b = \frac{\sum_{i=1}^3 a(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3},$$

$$c = -b(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_2\xi_3 - \xi_1\xi_3).$$

Notice that $a(x)x$ is a decreasing function for x , then the estimates are straightforward. \square

5.2.1. *Bound for M_3 and σ_3 .* We have $M_3 = \sum_{i=1}^3 a(\xi_i)\xi_i$, $\sigma_3 = \frac{M_3}{\Delta_3}$ modulo a constant.

Proposition 5.5. *On the set*

$$\Omega = \{\xi_1 + \xi_2 + \xi_3 = 0\} \cap \{\xi_1 \sim \alpha, \xi_2 \sim \xi_3 \approx \lambda \geq \alpha\}$$

we have

$$\begin{aligned} |M_3(\xi_1, \xi_2, \xi_3)| &\lesssim a(\alpha)\alpha, \\ |\sigma_3(\xi_1, \xi_2, \xi_3)| &\lesssim \frac{a(\alpha)}{\lambda^2}. \end{aligned}$$

Proof. If $\alpha \approx \lambda$, no need to do any proof. In case $\alpha \ll \lambda$, using the fact a is spherical symmetric,

$$\sum_{i=1}^3 a(\xi_i)\xi_i = a(\xi_1)\xi_1 - a(\xi_2)\xi_1 - a(\xi_2)\xi_3 + a(\xi_3)\xi_3$$

and we have $|a(\xi_3)\xi_3 - a(\xi_2)\xi_3| \lesssim |a'(\xi_3)\xi_1\xi_3| \lesssim |a(\xi_3)\xi_1|$. So the estimate for M_3 become obvious. Using the fact that $\Delta_3 = 3\xi_1\xi_2\xi_3$ on set Ω , we get bounds for σ_3 . \square

From this we can prove that $E_3(u)$ is bounded by $E_2(u)$.

Proposition 5.6. *We have the fact that*

$$(55) \quad |\Lambda_3(\sigma_3)| \lesssim |E_2(u)|^{\frac{3}{2}}.$$

Proof. We can expand the trilinear expression in dyadic frequency band $\{\lambda, \lambda, \alpha \leq \lambda\}$. Then using the estimate for σ_3 , we can bound $|\Lambda_3(\sigma_3)|$ by

$$\begin{aligned} a(\alpha)\lambda^{-2} \int u_\lambda u_\lambda u_\alpha dx &\lesssim a(\alpha)\lambda^{-2}\alpha^{\frac{1}{2}}\|u_\lambda\|_{L^2}\|u_\lambda\|_{L^2}\|u_\alpha\|_{L^2} \\ &\lesssim (a(\alpha)\alpha)^{\frac{1}{2}}(a(\lambda)\lambda^2)^{-1}E_2(u_\lambda)E_2(u_\alpha)^{\frac{1}{2}}. \end{aligned}$$

We can sum up the frequencies and get (55). \square

5.2.2. *Bound for M_4 and σ_4 .* Recall that

$$M_4 = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}$$

We adopt the calculation done in [7] (Notice, our $a(\xi)$ corresponds to $m^2(\xi)$, Δ_k corresponds to α_k in their paper), we have the following formula for M_4

$$(56) \quad \begin{aligned} M_4(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{-1}{108} \frac{\Delta_4}{\xi_1\xi_2\xi_3\xi_4} [a(\xi_1) + \dots + a(\xi_4) - a(\xi_{12}) - a(\xi_{13}) - a(\xi_{14})] \\ &\quad + \frac{1}{36} \left[\frac{a(\xi_1)}{\xi_1} + \dots + \frac{a(\xi_4)}{\xi_4} \right]. \end{aligned}$$

Here we used the notation $\xi_{jk} = \xi_j + \xi_k$, and

$$(57) \quad \Delta_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = 3(\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4) = 3\xi_{12}\xi_{13}\xi_{14}.$$

Proposition 5.7. *We have the estimate for M_4*

$$(58) \quad |M_4| \lesssim \frac{\Delta_4 a(\min(|\xi_i|, |\xi_{jk}|))}{|\xi_1\xi_2\xi_3\xi_4|}.$$

Proof. The proof repeats the argument of Lemma 4.4 in [7]. We can also deduce it from our next proposition. \square

We have bounds on σ_4 immediately from Proposition 5.7. But in order to do correction, we need improve it slightly.

Proposition 5.8.

$$(59) \quad |\sigma_4| \lesssim \frac{a(\min(|\xi_i|, |\xi_{jk}|))}{|\xi_1\xi_2\xi_3\xi_4|}, \quad |\Lambda_4(\sigma_4)| \lesssim |E_2(u)|^2.$$

Proof. We look at $\Lambda_4(\sigma_4)$, expand it into dyadic frequency components, since ξ_i are symmetric, we can assume $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$

(1) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \lambda\}, \mu \gg \lambda$. Then we have $\min(\xi_i, \xi_{ij}) = \xi_{12} \lesssim \lambda$ and $|\sigma_4| \lesssim \frac{a(\xi_{12})}{\lambda^2\mu^2}$. In this case, we can bound $\Lambda_4(\sigma_4)$ by

$$\begin{aligned} & a(\xi_{12})\lambda^{-2}\mu^{-2} \int u_\mu u_\mu u_\lambda u_\lambda dx \\ & \lesssim a(\xi_{12})|\xi_{34}|\lambda^{-2}\mu^{-2} \|u_\mu\|_{L^2} \|u_\mu\|_{L^2} \|u_\lambda\|_{L^2} \|u_\lambda\|_{L^2} \\ & \lesssim a(\xi_{12})|\xi_{12}|(a(\mu)\mu^2)^{-1}(a(\lambda)\lambda^2)^{-1} E_2(u_\mu)E_2(u_\lambda). \end{aligned}$$

Here notice that $a(x)x$ is bounded and we can sum up the frequencies.

(2) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \alpha\}, \mu \gg \lambda \gg \alpha$. In this case, we have $\min(\xi_i, \xi_{ij}) = \xi_4$, but we need attention with the estimate here. In fact, with the expression for M_4 (56), we can separate the expression of σ_4 into two parts.

One term looks like

$$-\frac{1}{108} \frac{1}{\xi_1\xi_2\xi_3\xi_4} [a(\xi_1) + a(\xi_2) - a(\xi_{13}) - a(\xi_{14})] + \frac{1}{36\Delta_4} \left[\frac{a(\xi_1)}{\xi_1} + \frac{a(\xi_2)}{\xi_2} \right]$$

and it is bounded by $\frac{a(\mu)}{\alpha\lambda\mu^2}$.

And the other term looks like(if we ignore the constant $-\frac{1}{108}$),

$$\begin{aligned} & \frac{a(\xi_3) + a(\xi_4) - a(\xi_{12})}{\xi_1\xi_2\xi_3\xi_4} - \frac{1}{\xi_{12}\xi_{13}\xi_{14}} \left[\frac{a(\xi_3)}{\xi_3} + \frac{a(\xi_4)}{\xi_4} \right] \\ = & \frac{a(\xi_3)\xi_3\xi_4\xi_{12} + a(\xi_3)\xi_1\xi_2\xi_3 + a(\xi_4)\xi_3\xi_4\xi_{12} + a(\xi_4)\xi_1\xi_2\xi_4 - a(\xi_{12})\xi_{12}\xi_{13}\xi_{14}}{\xi_1\xi_2\xi_3\xi_4\xi_{12}\xi_{13}\xi_{14}} \end{aligned}$$

So it is bounded by $\frac{a(\alpha)}{\lambda^2\mu^2}$. Now we can bound $\Lambda_4(\sigma_4)$ by

$$\begin{aligned} & a(\alpha)\lambda^{-2}\mu^{-2} \int u_\mu u_\mu u_\lambda u_\alpha dx \\ & \lesssim a(\alpha)\lambda^{-2}\mu^{-2}\lambda^{\frac{1}{2}}\alpha^{\frac{1}{2}} \|u_\mu\|_{L^2} \|u_\mu\|_{L^2} \|u_\lambda\|_{L^2} \|u_\alpha\|_{L^2} \\ & \lesssim (a(\alpha)\alpha)^{\frac{1}{2}}(a(\lambda)\lambda^3)^{-\frac{1}{2}}(a(\mu)\mu^2)^{-1} E_2(u_\mu)E_2(u_\lambda)^{\frac{1}{2}} E_2(u_\alpha)^{\frac{1}{2}}. \end{aligned}$$

(3) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \lambda\}$, $\mu \gg \lambda$. Here $\min(\xi_i, \xi_{ij}) = \lambda$, we can do same estimate as in previous case and get $|\sigma_4| \lesssim \frac{a(\lambda)}{\mu^4}$, we need bound the expression

$$\begin{aligned} & a(\lambda)\mu^{-4} \int u_\mu u_\mu u_\mu u_\lambda dx \\ & \lesssim a(\lambda)\mu^{-4} \lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} \|u_\mu\|_{L^2} \|u_\mu\|_{L^2} \|u_\lambda\|_{L^2} \|u_\alpha\|_{L^2} \\ & \lesssim (a(\lambda)\lambda)^{\frac{1}{2}} (a(\mu)\mu^2)^{-1} (a(\mu)\mu^3)^{-\frac{1}{2}} E_2(u_\mu)^{\frac{3}{2}} E_2(u_\lambda)^{\frac{1}{2}}. \end{aligned}$$

(4) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \mu\}$, $\min(\xi_i, \xi_{ij}) = \xi_{ij}$. For convenience, suppose it is ξ_{12} , then we have $|\sigma_4| \lesssim \frac{a(\xi_{12})}{\mu^4}$. And we can bound $\Lambda_4(\sigma_4)$ by

$$a(\xi_{12})\mu^{-4} \int u_\mu u_\mu u_\mu u_\mu dx \lesssim a(\xi_{12})|\xi_{12}|(a(\mu)\mu^2)^{-2} E_2(u_\mu)^2.$$

In all the cases above, we can sum up the frequency and get (59). \square

Remark 5.9. From the estimate in the proof, we see that actually we have slightly better bound for M_4 than Proposition 5.7 in the following two cases

- (1) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \alpha\}$, $\alpha \ll \lambda \ll \mu$, $|M_4| \lesssim \frac{a(\alpha)}{\lambda}$,
- (2) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \lambda\}$, $\lambda \ll \mu$, $|M_4| \lesssim \frac{a(\lambda)}{\mu}$.

Proposition 5.10. *We have the error estimate when $s \geq -\frac{4}{5}$*

$$\left| \int_0^1 \Lambda_4(M_4) dt \right| \lesssim \|u\|_{X^s \cap X_{le}^s}^4 (1 + \|u\|_{X^s \cap X_{le}^s} + \|u\|_{X^s \cap X_{le}^s}^2)$$

Proof. As before, we expand the error term $\Lambda_4(M_4)$ in the dyadic frequency component and discuss in each cases. Since $u \in X^s \cap X_{le}^s$, we still decompose each piece as $u_\lambda = u_{\lambda,1} + u_{\lambda,2}$, $u_{\lambda,1} \in X^1[I_\lambda]$, $u_{\lambda,2} \in S[I_\lambda]$. We abuse the notation and still use u_λ to represent any of them. We assume $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$.

One thing to notice the the high modulation relation. Since

$$\int_0^1 \Lambda_4(M_4) dt = \int_\Sigma M_4 \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{u}_4 d\xi d\tau.$$

$$\Sigma = \{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0\}.$$

We have

$$(60) \quad (\tau_1 - \xi_1^3) + (\tau_2 - \xi_2^3) + (\tau_3 - \xi_3^3) + (\tau_4 - \xi_4^3) = -\Delta_4 = -3\xi_{12}\xi_{13}\xi_{14}.$$

Hence we get the high modulation

$$(61) \quad \sigma_M = \max\{|\tau_1 - \xi_1^3|, |\tau_2 - \xi_2^3|, |\tau_3 - \xi_3^3|, |\tau_4 - \xi_4^3|\} \gtrsim |\xi_{12}\xi_{13}\xi_{14}|.$$

(1) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \lambda\}$, $\mu \gg \lambda$. Then we have $\min(\xi_i, \xi_{ij}) = \xi_{12} \lesssim \lambda$ and $|M_4| \lesssim \left| \frac{a(\xi_{12})\xi_{12}}{\lambda^2} \right|$, also notice function $a(x)x$ is bounded. Let us use the crude bilinear

estimate (53), and also we need cut the time interval $[0, 1]$ into smaller scale of size μ^{4s+3} .

$$\begin{aligned} & \int_0^1 \Lambda_4(M_4) dt \\ & \lesssim \left| \frac{a(\xi_{12})\xi_{12}}{\lambda^2} \right| (\max\{\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}}\})^2 \mu^{-4s-3} \|u_\mu\|_{X^s \cap X_{I_e}^s}^2 (\|u\|_{X^s \cap X_{I_e}^s} + \|u\|_{X^s \cap X_{I_e}^s}^2)^2 \\ & \lesssim \max\{\mu^{-6s-5}\lambda^{-2s-2}, \mu^{-10s-8}\lambda^{-2}\} \|u_\mu\|_{X^s \cap X_{I_e}^s}^2 \sum_{k=2}^4 \|u\|_{X^s \cap X_{I_e}^s}^k. \end{aligned}$$

It is summable when $s \geq -\frac{4}{5}$.

(2) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \lambda, \alpha\}$, $\mu \gg \lambda \gg \alpha$. $|M_4| \lesssim \frac{a(\alpha)}{\lambda}$. We estimate it in exactly the same way as (1).

$$\begin{aligned} & \int_0^1 \Lambda_4(M_4) dt \\ & \lesssim \frac{a(\alpha)}{\lambda} \max\{\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}}\} \max\{\mu^{-1-s}\alpha^{-s}, \mu^{-3s-\frac{5}{2}}\} \mu^{-4s-3} \\ & \quad \times \|u_\mu\|_{X^s \cap X_{I_e}^s}^2 (\|u\|_{X^s \cap X_{I_e}^s} + \|u\|_{X^s \cap X_{I_e}^s}^2)^2. \end{aligned}$$

By computing the exponents, we can sum up the frequencies when when $s \geq -\frac{4}{5}$.

(3) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \lambda\}$, $\mu \gg \lambda$, here $\min(\xi_i, \xi_{ij}) = \lambda$, $\sigma_m \gtrsim \mu^3$.

Case 1 When at least one of u_μ have high modulation, here we cut the interval to size μ^{4s+3} and use bilinear on $(Q_{\sigma_m} u_\mu)u_\mu$ (54) and $u_\mu u_\lambda$, we see that we get the bound

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}} \frac{a(\lambda)}{\mu} (Q_{\sigma_m} u_\mu) u_\mu u_\mu u_\lambda dx dt \\ & \lesssim \frac{a(\lambda)}{\mu} \mu^{-3s-\frac{5}{2}} \max(\mu^{-1-s}\lambda^{-s}, \mu^{-3s-\frac{5}{2}}) \mu^{-4s-3} \|u_\mu\|_{X^s \cap X_{I_e}^s}^3 (\|u\|_{X^s \cap X_{I_e}^s} + \|u\|_{X^s \cap X_{I_e}^s}^2) \\ & \lesssim \mu^{-10s-9} \|u_\mu\|_{X^s \cap X_{I_e}^s}^3 (\|u\|_{X^s \cap X_{I_e}^s} + \|u\|_{X^s \cap X_{I_e}^s}^2). \end{aligned}$$

So it is summable for $s \geq -\frac{9}{10}$

Case 2 When the high modulation fall on u_λ , this is the hard case, we use the L^2 on $Q_{\sigma_m} u_\lambda$, and L^2 on the product $u_\mu u_\mu u_\mu$.

$$(62) \quad \|\eta_\lambda Q_{\sigma \gtrsim \mu^3} u_\lambda\|_{L_{t,x}^2} \lesssim \lambda^{-3s-\frac{3}{2}} \mu^{-3} \|u_\lambda\|_{X_\lambda[I_\lambda]},$$

$$(63) \quad \|\eta_\lambda Q_{\sigma \gtrsim \mu^3} u_\lambda\|_{L_{t,x}^2} \lesssim \lambda^{3+4s} \mu^{-6s-6} \|u_\lambda\|_{X^{-3-4s, 2s+2}[I_\lambda]},$$

$$(64) \quad \|\eta_\mu u_\mu u_\mu u_\mu\|_{L_{t,x}^2} \lesssim \mu^{-\frac{1}{2}-3s} \|u_\mu\|_{X^s}^3.$$

The third one is proved by discussing $u_\mu \in X^1$ or S , and notice that none of them has high modulation. Then we get

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}} \frac{a(\lambda)}{\mu} u_\mu u_\mu u_\mu Q_{\sigma \gtrsim \mu^3} u_\lambda dt dx \\ & \lesssim \frac{a(\lambda)}{\mu} \max\{\lambda^{-3s-\frac{3}{2}} \mu^{-3}, \lambda^{3+4s} \mu^{-6s-6}\} \mu^{-\frac{1}{2}-3s} \mu^{-4s-3} \|u_\mu\|_{X^s \cap X_{I_e}^s}^3 \|u_\lambda\|_{X^s \cap X_{I_e}^s} \end{aligned}$$

And we can sum up frequencies when $s \geq -\frac{21}{26}$.

(4) $\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{\mu, \mu, \mu, \mu\}$ here we need to discuss the size of ξ_{ij} .

$$\xi_{12} + \xi_{13} + \xi_{14} = 2\xi_1$$

so at least one of them is of size μ

Case 1: When $\xi_{ij} \gtrsim \mu$, then we have $|M_4| \lesssim \frac{a(\mu)}{\mu}$, and we have the high modulation factor $\sigma_m \gtrsim \mu^3$, so we use bilinear on $(Q_{\sigma_m} u_\mu)u_\mu$, and L^2 for each of $u_\mu u_\mu$.

Notice the (8,4) is Strichartz pair and using the size of interval we get

$$(65) \quad \|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{\frac{1}{2}-s} \|u_\mu\|_{X^1_\mu[I_\mu]}^2.$$

From (34) we have

$$(66) \quad \|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-\frac{3}{2}-2s} \|u_\mu\|_{X^1_\mu[I_\mu]} \|u_\mu\|_{S[I_\mu]}.$$

From (40) and (41) we get

$$(67) \quad \|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-1-s} \|u_\mu\|_{S[I_\mu]} \|u_\mu\|_{S[I_\mu]}.$$

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}} \frac{a(\mu)}{\mu} (Q_{\sigma_m} u_\mu) u_\mu u_\mu u_\mu dx dt \\ & \lesssim \frac{a(\mu)}{\mu} \mu^{-3s-\frac{5}{2}} \mu^{\frac{1}{2}-s} \mu^{-4s-3} \|u_\mu\|_{X^s}^4 \\ & \lesssim a(\mu) \mu^{-8s-6} \|u_\mu\|_{X^s}^4 \end{aligned}$$

so it is summable when $s \geq -1$.

Case 2: When two of ξ_{ij} is big, one is small, let's assume $\xi_{13} \ll \mu, \xi_{12}, \xi_{14} \gtrsim \mu$, we have $|M_4| \lesssim |\frac{a(\xi_{13})\xi_{13}}{\mu^2}|$. Then we can easily calculate that

$$(\xi_1 - \xi_2) + (\xi_1 - \xi_4) - (\xi_1 + \xi_3) = 2\xi_1$$

since $\xi_{13} \ll \mu$, we must have at least one of $\xi_1 - \xi_2$ or $\xi_1 - \xi_4$ be of size μ , with out loss of generality, we assume $|\xi_1 - \xi_2| \gtrsim \mu$, so we have separation of frequency, i.e

$$|\xi_1 - \xi_2| \approx \mu, |\xi_1 + \xi_2| \approx \mu$$

and we can also prove that

$$|\xi_3 + \xi_4| = |\xi_1 + \xi_2| \approx \mu, |\xi_3 - \xi_4| = |\xi_3 + \xi_1 - (\xi_1 + \xi_4)| \approx \mu$$

Now we have the bilinear estimate of two u_μ 's which have frequency separation.

$$(68) \quad \|\eta_\mu u_\mu u_\mu\|_{L^2_{t,x}} \lesssim \mu^{-1-2s} \|u_\mu\|_{X^1[I_\mu]}.$$

Together with (66) and (67), we get

$$\begin{aligned} & \int_0^1 \int_{\mathbb{R}} \left| \frac{a(\xi_{12})\xi_{12}}{\mu^2} \right| u_\mu u_\mu u_\mu u_\mu dx dt \\ & \lesssim \left| \frac{a(\xi_{12})\xi_{12}}{\mu^2} \right| (\mu^{-1-2s})^2 \mu^{-4s-3} \|u_\mu\|_{X^s}^4 \\ & \lesssim \mu^{-8s-7} \|u_\mu\|_{X^s}^4. \end{aligned}$$

so we can sum up for $s \geq -\frac{7}{8}$.

Case 3: When one of ξ_{ij} is big, the other two small. We can assume $\xi_{12} \leq \xi_{13} \ll \mu, \xi_{14} \gtrsim \mu$. In this case, we don't have frequency separation. $|M_4| \lesssim |\frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3}|$.

But we still have (33), so together with (66) and (67), we get

$$\begin{aligned} & \int_0^1 \int \mathbb{R} \left| \frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3} \right| u_\mu u_\mu u_\mu u_\mu dx dt \\ & \lesssim \left| \frac{a(\xi_{12})\xi_{12}\xi_{13}}{\mu^3} \right| (|\xi_{13}|^{-\frac{1}{2}} \mu^{-\frac{1}{2}})^2 \mu^{-4s} \mu^{-4s-3} \|u_\mu\|_{X^s}^4 \\ & \lesssim \mu^{-8s-7} \|u_\mu\|_{X^s}^4 \end{aligned}$$

so it is summable when $s \geq -\frac{7}{8}$. \square

6. LOCAL ENERGY DECAY

Let $\chi(x)$ be a positive, rapidly decaying function, with Fourier transform supported in $[-1, 1]$. Let a be as in the previous section. We define the indefinite quadratic form

$$(69) \quad \tilde{E}_2(u) = \sum_\lambda \frac{1}{2} \int (\phi_\lambda a(D) + a(D)\phi_\lambda) u_\lambda u_\lambda dx.$$

Here ϕ_λ is an odd smooth function whose derivative has the form $\phi'_\lambda(x) = \psi_\lambda(x)^2$, $\psi_\lambda(x) = \lambda^{-2s-\frac{5}{2}} \chi(\frac{x}{\lambda^{4s+5}})$. We will abuse the notation a bit, and (69)

$$(70) \quad \tilde{E}_2(u) = \frac{1}{2} \int (\phi a(D) + a(D)\phi) u u dx,$$

with the understanding that it is really defined on each dyadic pieces, and $\phi = \phi_\lambda$ on each piece.

Then we have the calculation

$$(71) \quad \frac{d}{dt} \tilde{E}_2(u) = \tilde{R}_2(u) + \tilde{R}_3(u),$$

where

$$\begin{aligned} \tilde{R}_2(u) &= \langle (a(D)\phi_x + \phi_x a(D))u_x, u_x \rangle + \langle (a(D)\phi_{xxx} + \phi_{xxx}a(D))u, u \rangle, \\ \tilde{R}_3(u) &= c \operatorname{Re} \langle (a(D)\phi + \phi a(D))u, (u^2)_x \rangle. \end{aligned}$$

We will see in the following propositions that \tilde{R}_2 can be used to measure local energy.

Proposition 6.1. *Let $a \in S_\epsilon^s$, ϕ defined as above, then we have the fixed time bound*

$$|\tilde{E}_2(u)| \lesssim E_2(u),$$

$$|\langle (a(D)\phi_{xxx} + \phi_{xxx}a(D))u, u \rangle| \lesssim E_2(u).$$

Proof. Since ϕ and ϕ_{xxx} are bounded and its fourier transform has compact support,

$$|\langle a(D)\phi u, u \rangle| = |\langle (a(D)^{1/2}\phi a(D)^{-1/2})a(D)^{1/2}u, a(D)^{1/2}u \rangle| \lesssim E_2(u).$$

Other terms are proved similarly. \square

Proposition 6.2. *We can use R_2 to bound the local energy*

$$(72) \quad \|\psi a(D)^{\frac{1}{2}} Du\|_{L_x^2}^2 \lesssim \tilde{R}_2(u) + cE_2(u).$$

Proof.

$$\langle (a(D)\phi_x + \phi_x a(D))u_x, u_x \rangle = 2\|\psi(a(D)^{\frac{1}{2}}D)u\|_{L_x^2}^2 + \langle r^w(x, D)u, u \rangle.$$

Here

$$r^w(x, D) = [a(D)^{1/2}, [a(D)^{1/2}, \psi^2]],$$

so its symbol r satisfy the estimate

$$\partial_x^\alpha \partial_\xi^\beta r(x, \xi) \lesssim \langle x \rangle^{-N} (1 + \xi)^{-\frac{\beta}{2}} a(\xi).$$

Hence

$$|\langle r^w(x, D)u, u \rangle| \lesssim E_2(u).$$

Combine with previous proposition and the formula for \tilde{R}_2 , we get estimate (72). \square

Integrating (71) and (72) on time interval $[0, 1]$, together with Proposition 6.1 we get

$$(73) \quad \int_0^1 \|\psi a(D)^{\frac{1}{2}} Du\|_{L_x^2}^2 dt \lesssim \|u\|_{L_t^\infty H^s}^2 + \left| \int_0^1 \tilde{R}_3(u) dt \right|.$$

Next, we can rewrite \tilde{R}_3 in the Fourier space. Notice that original definition of (69) is on dyadic pieces, so \tilde{R}_3 takes the following form

$$\begin{aligned} \tilde{R}_3(u) &= 2 \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} (a(\xi_1 - \xi) + a(\xi_1)) \chi(\xi) (\xi_{23}) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx, \\ P_\xi &= \{\xi_1 + \xi_2 + \xi_3 = \xi\}. \end{aligned}$$

Here ϕ is actually ϕ_λ , $\chi(\xi)$ is the multiplier used to define projection P_λ .

Now we can symmetrize it, using the notation $A(\xi_i) = (a(\xi_i - \xi) + a(\xi_i)) \chi(\xi_i)$

$$\begin{aligned} \tilde{R}_3(u) &= \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} \left(\sum_{i=1}^3 A(\xi_i) \xi_i \right) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx \\ &\quad - \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} \left(\sum_{i=1}^3 A(\xi_i) \xi_i \right) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx. \end{aligned}$$

To better estimate it, we use proposition 5.4, and rewrite

$$(74) \quad \sum_{i=1}^3 A(\xi_i) \xi_i = B(\xi_1, \xi_2, \xi_3) (\xi_1^3 + \xi_2^3 + \xi_3^3) + C(\xi_1, \xi_2, \xi_3) (\xi_1 + \xi_2 + \xi_3).$$

So we split \tilde{R}_3 into

$$\tilde{R}_3(u) = \tilde{R}_{good,3} + \tilde{R}_{bad,3},$$

where $\tilde{R}_{good,3}$ and $\tilde{R}_{bad,3}$ take the following form,

$$\begin{aligned} \tilde{R}_{good,3} &= \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} \left(\sum_{i=1}^3 A(\xi_i) - C \right) \xi_i \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx, \\ \tilde{R}_{bad,3} &= - \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} (B(\xi_1, \xi_2, \xi_3) (\xi_1^3 + \xi_2^3 + \xi_3^3)) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx. \end{aligned}$$

Proposition 6.3. *Let a, ϕ as before, then we have the estimate*

$$\left| \int_0^1 \tilde{R}_{good,3}(u) dt \right| \lesssim \sum_{k=3,4} \|u\|_{X^s \cap X_{le}^s}^k.$$

Proof. As in proposition 5.4, we have

$$C = - \frac{\sum A(\xi_i) \xi_i}{3\xi_1 \xi_2 \xi_3} (\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1 \xi_2 - \xi_1 \xi_3 - \xi_2 \xi_3).$$

Let's look at one term of $\sum A(\xi) - C$,

$$\begin{aligned} & A(\xi_1) + \frac{A(\xi_1)\xi_1}{3\xi_1\xi_2\xi_3}(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_1\xi_2 - \xi_1\xi_3 - \xi_2\xi_3) \\ &= \frac{A(\xi_1)}{3\xi_2\xi_3}[(\xi_1 + \xi_2 + \xi_3)^2 - 3\xi_1(\xi_1 + \xi_2 + \xi_3) + 3\xi_1^2]. \end{aligned}$$

So on P_ξ , we have

$$\begin{aligned} & \sum A(\xi) - C \\ &= \frac{\sum A(\xi_i)\xi_i\xi^2 - 3\sum A(\xi_i)\xi_i^2\xi + 3\sum A(\xi_i)\xi_i^3}{3\xi_1\xi_2\xi_3}. \end{aligned}$$

When we feed it to the integral, we can do integration by parts to trade ξ for derivative of ϕ

$$\begin{aligned} \tilde{R}_{good,3} &= -i \int_{\mathbb{R}} \phi_{xxx}(x) e^{ix\xi} \int_{P_\xi} \frac{\sum A(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_i d\xi dx \\ &+ \int_{\mathbb{R}} \phi_{xx}(x) e^{ix\xi} \int_{P_\xi} \frac{\sum A(\xi_i)\xi_i^2}{\xi_1\xi_2\xi_3} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_i d\xi dx \\ &+ i \int_{\mathbb{R}} \phi_x(x) e^{ix\xi} \int_{P_\xi} \frac{\sum A(\xi_i)\xi_i^3}{\xi_1\xi_2\xi_3} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_i d\xi dx. \end{aligned}$$

Let's decompose the region into dyadic region $\{\alpha, \lambda, \lambda\}$, $\alpha \leq \lambda$ and we can estimate the symbols, using the fact $a \in S_\epsilon^s$, the the proof is similar to proposition 5.5.

$$\left| \frac{\sum A(\xi_i)\xi_i}{3\xi_1\xi_2\xi_3} \right| \lesssim \frac{a(\alpha)}{\lambda^2}, \quad \left| \frac{\sum A(\xi_i)\xi_i^2}{3\xi_1\xi_2\xi_3} \right| \lesssim \frac{a(\lambda)}{\alpha}, \quad \left| \frac{\sum A(\xi_i)\xi_i^3}{3\xi_1\xi_2\xi_3} \right| \lesssim \frac{a(\lambda)\lambda}{\alpha}.$$

The three terms in $\tilde{R}_{good,3}$ are similar, so we only do the third term, since that has the worst bound. Denote it as III

$$\left| \int_0^1 III \right| \lesssim \frac{a(\lambda)\lambda}{\alpha} \int_0^1 \int_{\mathbb{R}} \phi_x(x) u_\lambda u_\lambda u_\alpha dx dt.$$

Case 1. $\alpha \ll \lambda$, put L^2 on one of u_λ , and bilinear estimate on $u_\lambda u_\alpha$ (53), also notice ϕ_x is fast decaying on spatial scale λ^{4s+5} , so we can use local energy norm to avoid interval summation. (Based on our computation below, we can even perform interval summation with no difficulty.)

$$\begin{aligned} \left| \int_0^1 III \right| &\lesssim \frac{a(\lambda)\lambda}{\alpha} \sum_{I_\lambda} \int_{I_\lambda} \int_{\mathbb{R}} \phi_x(x) u_\lambda u_\lambda u_\alpha dx dt \\ &\lesssim \sum_{I_\lambda} \frac{a(\lambda)\lambda}{\alpha} \lambda^{-4s-5} \|\eta_\lambda \chi^\lambda(x) u_\lambda\|_{L^2_{t,x}} \|\eta_\lambda \chi^\lambda(x) u_\lambda u_\alpha\|_{L^2_{t,x}} \\ &\lesssim \frac{a(\lambda)\lambda^{-4s-4}}{\alpha} \lambda^{\frac{3}{2}+s} \max\{\lambda^{-3s-\frac{5}{2}}, \lambda^{-1-s}\alpha^{-s}\} \sum_{I_\lambda} \|\eta_\lambda \chi^\lambda(x) u_\lambda\|_{X^\lambda}^2 (\|u\|_{X^s} + \|u\|_{X^s}^2) \\ &\lesssim \lambda^{-4s-5} \max\{\alpha^{-1}, \lambda^{2s+\frac{3}{2}}\alpha^{-1-s}\} \|u_\lambda\|_{X^s_{le}}^2 (\|u\|_{X^s} + \|u\|_{X^s}^2). \end{aligned}$$

Case 2. $\alpha \approx \lambda$, notice we have high modulation $\sigma_m \gtrsim \lambda^3$. Then bound $(Q_{\sigma_m} u_\lambda)u_\lambda$ in L^2 (54), and the other one in L^2 .

$$\begin{aligned} \left| \int_0^1 III \right| &\lesssim a(\lambda) \sum_{I_\lambda} \int_{I_\lambda} \int_{\mathbb{R}} \phi_x(x) (Q_{\sigma_m} u_\lambda) u_\lambda u_\lambda dx dt \\ &\lesssim a(\lambda) \lambda^{-3s - \frac{5}{2}} \lambda^{\frac{3}{2} + s} \lambda^{-4s - 5} \|u_\lambda\|_{X^s \cap X_{le}^s}^3 \\ &\lesssim \lambda^{-4s - 6} \|u_\lambda\|_{X^s \cap X_{le}^s}^3. \end{aligned}$$

□

For the part $\tilde{R}_{bad,3}$ we can not estimate it directly, so we will add some correction as we did before. Take

$$\begin{aligned} \tilde{E}_3(u) &= \tilde{E}_2(u) + \Lambda_B(u), \\ \Lambda_B(u) &= -i \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} B(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_i d\xi dx. \end{aligned}$$

Notice (74), then we have

$$\frac{d}{dt} \tilde{E}_3(u) = \tilde{R}_2(u) + \tilde{R}_{good,3} + \tilde{R}_4(u).$$

Here

$$\tilde{R}_4(u) = - \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} [B(\xi_1, \xi_2, \xi_3) \xi_{34}]_{sym} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) d\xi_i d\xi dx.$$

Here we need to do two things, show $|\tilde{E}_3(u)| \lesssim E_2^{\frac{3}{2}}(u)$, which is same as proposition 6.1 and 5.6 and 6.2. And estimate the error, which repeats the proof of proposition 5.10 using the fact that ϕ is bounded, $\hat{\phi}$ has compact support, so they does no change to the proof, in fact, since we have the spatial localization, we have the privilege of omitting the interval summation by control them in local energy space.

Proposition 6.4. *With a, ϕ as before, when $s \geq -\frac{4}{5}$, we have the error estimate*

$$\left| \int_0^1 \tilde{R}_4(u) dt \right| \lesssim \|u\|_{X^s \cap X_{le}^s}^4 (1 + \|u\|_{X^s \cap X_{le}^s} + \|u\|_{X^s \cap X_{le}^s}^2).$$

Combining all the propositions in this section, we get the local energy bound.

Lemma 6.5. *The solution to the KdV equation (1) satisfy the following bound*

$$\begin{aligned} &\sum_{\lambda} \lambda^{-2s-5} \sup_j \|\chi_j^\lambda \partial_x u_\lambda\|_{L_{x,t}^2}^2 \\ &\lesssim \sup_t \|u(t)\|_{H^s}^2 (1 + \|u(t)\|_{H^s}) + \|u\|_{X^s \cap X_{le}^s}^3 + \|u\|_{X^s \cap X_{le}^s}^4 (1 + \|u\|_{X^s \cap X_{le}^s} + \|u\|_{X^s \cap X_{le}^s}^2). \end{aligned}$$

7. FINISHING THE PROOF

To finish the whole argument, we need to pick suitable symbol $a(\xi)$ in the previous two sections. As in [22], we pick slow varying sequence.

$$\begin{aligned} \beta_\lambda^0 &= \frac{\lambda^{2s} \|u_{0\lambda}\|_{L^2}^2}{\|u_0\|_{H^s}^2}, \\ \beta_\lambda &= \sum_{\mu} 2^{-\frac{\epsilon}{2} |\log \lambda - \log \mu|} \beta_\mu^0. \end{aligned}$$

These β_λ satisfy the following property

$$(i) \lambda^{2s} \|u_{0\lambda}\|_{L^2}^2 \lesssim \beta_\lambda \|u_0\|_{H^s}^2,$$

$$(ii) \sum \beta_\lambda \lesssim 1,$$

(iii) β_λ is slow varying in the sense that

$$(75) \quad |\log_2 \beta_\lambda - \log_2 \beta_\mu| \lesssim \frac{\epsilon}{2} |\log_2 \lambda - \log_2 \mu|.$$

Now if we take $a_\lambda = \lambda^{2s} \max(1, \beta_{\lambda_0}^{-1} 2^{-\epsilon |\log_2 \lambda - \log_2 \lambda_0|})$, and correspondingly we take

$$a(\xi) \approx a_\lambda, \quad |\xi| \approx \lambda$$

Then from the slow varying property (75), we get

$$\sum_\lambda a_\lambda \|u_{0\lambda}\|_{L_x^2}^2 \lesssim \sum_\lambda \lambda^{2s} \|u_{0\lambda}\|_{L_x^2}^2 + 2^{-\epsilon |\log_2 \lambda - \log_2 \lambda_0|} \lambda^{2s} \beta_{\lambda_0}^{-1} \|u_{0\lambda}\|_{L_x^2}^2 \lesssim \|u_0\|_{H^s}^2.$$

Assume that $\|u\|_{L_t^\infty L_x^s} \ll 1$, which implies $\sup_t E_2(u(t)) \ll 1$. Recall that

$$\frac{d}{dt}(E_2(u) + \Lambda_3(\sigma_3)) = \Lambda_4(M_4),$$

so from Proposition 5.6 and 5.10, we get

$$\left(\sum_\lambda a(\lambda) \|u_\lambda(t)\|_{L_x^2}^2\right)^{\frac{1}{2}} \lesssim \|u_0\|_{H^s} + \|u\|_{X^s \cap X_{le}^s}^4 (1 + \|u\|_{X^s \cap X_{le}^s} + \|u\|_{X^s \cap X_{le}^s}^2).$$

At fixed frequency $\lambda = \lambda_0$, we get

$$\sup_t \lambda_0^s \|u_{\lambda_0}(t)\|_{L^2} \lesssim \beta_{\lambda_0}^{\frac{1}{2}} (\|u_0\|_{H^s} + \|u\|_{X^s \cap X_{le}^s}^4 (1 + \|u\|_{X^s \cap X_{le}^s} + \|u\|_{X^s \cap X_{le}^s}^2)).$$

From the property of β_λ , we can sum up λ_0 , and get (9).

Together with the previous section, we can prove the local energy bound in exactly the same way, so we conclude the proof of proposition 1.6.

Acknowledgement. The author would like to thank his advisor, Daniel Tataru, for suggesting the problem and for all the guidance and encouragement along the way.

REFERENCES

- [1] D. Bekiranov, The initial-value problem for the generalized Burgers equation, *Differential Integral Equations* 9 (1996) 1253C1265.
- [2] J. L. Bona and R. Smith. The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287):555C601, 1975.
- [3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II. *Geom. Funct. Anal.*, 3:107-156, 209-262, 1993.
- [4] M. Christ, J. Colliander, and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, *Amer. J. Math.* 125 (2003)
- [5] M. Christ, J. Colliander, and T. Tao, A priori bounds and weak solutions for the nonlinear Schrodinger equation in Sobolev spaces of negative order. *Journal of Functional Analysis* Volume 254, Issue 2, 15 January 2008, Pages 368-395
- [6] A. Cohen. Existence and regularity for solutions of the Korteweg-deVries equation. *Arch. Rational Mech. Anal.*, 71(2):143C175, 1979.
- [7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T} , *J. Amer. Math. Soc.* 16 (2003), 705C749
- [8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrodinger equation in \mathbb{R}^3 . *Ann. of Math.* 167 (2008), 767C865. MR2415387

- [9] P. Deift, S. Venakides, and X. Zhou. An extension of the steepest descent method for Riemann-Hilbert problems: the small dispersion limit of the Korteweg-de Vries (KdV) equation. Proc. Natl. Acad. Sci. USA, 95(2):450454 (electronic), 1998.
- [10] P. Deift and X. Zhou. A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. Ann. of Math. (2), 137(2):295368, 1993.
- [11] D. B. Dix, Nonuniqueness and uniqueness in the initial-value problem for Burgers equation, SIAM J. Math. Anal. 27 (1996), no. 3, 708C724.
- [12] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. 19, 1095C1097 (1967).
- [13] J. Ginibre, Y. Tsutsumi, and G. Velo. Existence and uniqueness of solutions for the generalized Korteweg de Vries equation. Math. Z., 203(1):9C36, 1990.
- [14] A. Grünrock. A bilinear Airy-estimate with application to gKdV-3. Differential and Integral Equations 18, No. 12, 1333-1339 (2005)
- [15] Zihua Guo. Global well-posedness of Korteweg-de Vries equation in $H^{-\frac{3}{4}}(\mathbb{R})$. J. Math. Pures Appl. (9), 91(6):583C597, 2009.
- [16] T. Kappeler and P. Topalov, Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$. Duke Math. J. 135(2), 327C 360 (2006)
- [17] T. Kato. The Cauchy problem for the Korteweg-de Vries equation. In Nonlinear partial differential equations and their applications. College de France Seminar, Vol. I (Paris, 1978/1979), pages 293C307. Pitman, Boston, Mass., 1981.
- [18] C. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equations, J. Amer.Math.Soc. 9(1996), 573-603.
- [19] C. Kenig, G. Ponce, L. Vega. On the ill-posedness of some canonical dispersive equations. Duke Math. J., 106(3):617C633, 2001.
- [20] C. Kenig, G. Ponce, L. Vega. Well-posedness of the initial value problem for the Korteweg-de Vries, J. Amer. Math. Soc 4 (1991), 323347. MR 92c:35106
- [21] C. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Communications on Pure and Applied Mathematics 46 (1993), no. 4, 527-620.
- [22] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces. Int. Math. Res. Not. IMRN, 16:Art. ID rnm053, 36, (2007)
- [23] H. Koch and D. Tataru, Energy and local energy bounds for the 1-D cubic NLS equation in $H^{-1/4}$, preprint
- [24] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. I, II, III. Comm. Pure Appl. Math., 36(3, 5, 6):253290, 571593, 809829, 1983.
- [25] L. Molinet, F. Ribaud, On the low regularity of the KortewegCde VriesCBurgers equation, Int. Math. Res. Notices 37 (2002) 1979C2005.
- [26] L. Molinet, A note on ill posedness for the KdV equation. Differential Integral Equations 24 (2011), no. 7-8, 759C765
- [27] K. Nakanishi, H. Takaoka, and Y. Tsutsumi. Counterexamples to bilinear estimates related to the KdV equation and the nonlinear Schrödinger equation. Methods and Applications of Analysis Vol. 8, No. 4, pp.569-578,
- [28] N. Kishimoto, Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity, preprint.
- [29] P. Tomas, A restriction theorem for the Fourier transform, Bull. Amer. Math. Soc. 81 (1975), 477-478.
- [30] T. Tao, Nonlinear dispersive equations:local and global analysis, CBMS regional conference series in mathematics, 2006
- [31] D.Tataru, The X_{θ}^2 spaces and unique continuation for solutions to the semilinear wave equation. Commun.In Partial Differential Equations, 21(5&6), 841-887(1996)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY
E-mail address: baoping@math.berkeley.edu