

1. (2 points) Integrate

$$\int_{-2}^1 (5 - x^2 - 4x)^{3/2} dx$$

The integrand is

$$\left(\sqrt{5 - x^2 - 4x}\right)^3,$$

suggesting that some trigonometric substitution may be involved. But to be sure, we have to rewrite the terms under the radical to be the sum or difference of two squares. We'll do this by completing the square:

$$5 - x^2 - 4x = 5 - (x^2 + 4x + 4 - 4) = 3^2 - (x + 2)^2$$

This looks like $a^2 - x^2$, so we'd like to substitute in $a \sin(u)$. Except, we have to substitute $a \sin(u)$ for whatever is being squared in the second term, i.e. we need to set $x + 2 = 3 \sin(u)$. Then our square root becomes:

$$\sqrt{5 - x^2 - 4x} = \sqrt{3^2 - (x + 2)^2} = \sqrt{3^2 - 3^2 \sin^2(u)} = 3 \cos(u)$$

Now we must rewrite dx in terms of du . But if $x = 3 \sin(u) - 2$, then $dx = 3 \cos(u) du$. We also need to change our bounds of integration. When $x = -2$, we have $0 = 3 \sin(u)$ so $u = 0$ (remember the bounds $-\pi/2 \leq u \leq \pi/2$). When $x = 1$, we have $3 = 3 \sin(u)$, so $u = \pi/2$. Therefore our integral becomes:

$$\begin{aligned} \int_{-2}^1 (5 - x^2 - 4x)^{3/2} dx &= \int_0^{\pi/2} (3 \cos(u))^3 (3 \cos(u) du) \\ &= 81 \int_0^{\pi/2} \cos^4(u) du \\ &= \frac{81}{4} \int_0^{\pi/2} (1 + \cos(2u))^2 du \\ &= \frac{81}{4} \int_0^{\pi/2} (1 + 2 \cos(2u) + \cos^2(2u)) du \\ &= \frac{81}{4} \int_0^{\pi/2} \left(1 + 2 \cos(2u) + \frac{1}{2}(1 + \cos(4u))\right) du \\ &= \frac{81}{4} \left[u + \sin(2u) + \frac{u}{2} + \frac{1}{8} \sin(4u) \right] \Big|_0^{\pi/2} \\ &= \frac{81}{4} \left[\frac{\pi}{2} + \frac{\pi}{4} \right] \\ &= \frac{243\pi}{16} \end{aligned}$$

(turn over)

2. (1 point) Find a general solution to

$$y'' - y = e^{-x}$$

using variation of parameters.

First we need to find y_c , the solution to the homogeneous equation $y'' - y = 0$. This differential equation has a characteristic equation of $r^2 - 1 = 0$, so $r = \pm 1$ and

$$y_c = c_1 e^x + c_2 e^{-x}$$

Now we set $y_p = u_1 e^x + u_2 e^{-x}$ and use variation of parameters to solve for u_1 and u_2 . Our two equations are:

$$\begin{aligned} 0 &= u_1' e^x + u_2' e^{-x} \\ e^{-x} &= 1 \cdot (u_1' (e^x)' + u_2' (e^{-x})') \end{aligned}$$

If we add these two equations together, we get $e^{-x} = 2u_1' e^x$, so $u_1' = e^{-2x}/2$. Integrating both sides yields $u_1 = -e^{-2x}/4$.

Next we plug the equation for u_1' back into our first equation, getting:

$$0 = \left(\frac{e^{-2x}}{2}\right) e^x + u_2' e^{-x}$$

so $u_2' = -1/2$ and $u_2 = -x/2$. Therefore

$$\begin{aligned} y_p &= u_1 e^x + u_2 e^{-x} \\ &= \left(\frac{-e^{-2x}}{4}\right) e^x - \frac{x e^{-x}}{2} \\ &= -\frac{e^{-x}}{4} - \frac{x e^{-x}}{2} \end{aligned}$$

and our general solution is

$$\begin{aligned} y &= y_p + y_c \\ &= -\frac{e^{-x}}{4} - \frac{x e^{-x}}{2} + c_1 e^x + c_2 e^{-x} \\ &= -\frac{x e^{-x}}{2} + c_1 e^x + c_2 e^{-x} \end{aligned}$$