

1. (2 points) Integrate

$$4 \int \frac{\sqrt{-x^2 - 6x - 5}}{x^2 + 6x + 9} dx$$

To make the square root more manageable, we need to complete the square.

$$-x^2 - 6x - 5 = -x^2 - 6x - 9 + 9 - 5 = 4 - (x + 3)^2$$

We can also factor the denominator as $(x + 3)^2$.

Since $4 - (x + 3)^2$ is in the square root, we should substitute $2 \sin(u)$. However, it needs to be substituted for the thing that's getting squared, i.e. $x + 3 = 2 \sin(u)$. Then we can simplify:

$$\sqrt{-x^2 - 6x - 5} = \sqrt{4 - (x + 3)^2} = \sqrt{4 - 4 \sin^2(u)} = 2 \cos(u)$$

Before we can evaluate the integral in terms of u , we need to express dx in terms of u . Since $x = 2 \sin(u) - 3$, we get $dx = 2 \cos(u) du$. Therefore our integral becomes:

$$\begin{aligned} 4 \int \frac{\sqrt{-x^2 - 6x - 5}}{x^2 + 6x + 9} dx &= 4 \int \frac{2 \cos(u)}{4 \sin^2(u)} (2 \cos(u) du) \\ &= 4 \int \cot^2(u) du \\ &= 4 \int (\csc^2(u) - 1) du \\ &= -4 \cot(u) - 4u + C \end{aligned}$$

Now we must put this back in terms of x . Since $\sin(u) = (x + 3)/2$, we get $u = \arcsin\left(\frac{x+3}{2}\right)$. Also,

$$\cos(u) = \sqrt{1 - \sin^2(u)} = \sqrt{1 - \frac{(x + 3)^2}{4}} = \frac{1}{2} \sqrt{4 - (x + 3)^2}$$

Therefore,

$$\begin{aligned} 4 \int \frac{\sqrt{-x^2 - 6x - 5}}{x^2 + 6x + 9} dx &= -4 \cot(u) - 4u + C \\ &= -4 \frac{\left(\frac{1}{2} \sqrt{4 - (x + 3)^2}\right)^2}{\left(\frac{x+3}{2}\right)^2} - 4 \arcsin\left(\frac{x + 3}{2}\right) + C \\ &= -4 \frac{(4 - (x + 3)^2)}{(x + 3)^2} - 4 \arcsin\left(\frac{x + 3}{2}\right) + C \\ &= \frac{-16}{(x + 3)^2} + 4 - 4 \arcsin\left(\frac{x + 3}{2}\right) + C \\ &= \frac{-16}{(x + 3)^2} - 4 \arcsin\left(\frac{x + 3}{2}\right) + C \end{aligned}$$

(turn over)

2. (1 point) Find a general solution to

$$y'' - y = e^x$$

using variation of parameters.

First we need to find y_c , the solution to the homogeneous equation $y'' - y = 0$. This differential equation has a characteristic equation of $r^2 - 1 = 0$, so $r = \pm 1$ and

$$y_c = c_1 e^x + c_2 e^{-x}$$

Now we set $y_p = u_1 e^x + u_2 e^{-x}$ and use variation of parameters to solve for u_1 and u_2 . Our two equations are:

$$\begin{aligned} 0 &= u_1' e^x + u_2' e^{-x} \\ e^x &= 1 \cdot (u_1'(e^x)' + u_2'(e^{-x})') \end{aligned}$$

If we add these two equations together, we get $e^x = 2u_1' e^x$, so $u_1' = 1/2$. Integrating both sides yields $u_1 = x/2$.

Next we plug the equation for u_1' back into our first equation, getting:

$$0 = \left(\frac{1}{2}\right) e^x + u_2' e^{-x}$$

so $u_2' = -e^{2x}/2$ and $u_2 = -e^{2x}/4$. Therefore

$$\begin{aligned} y_p &= u_1 e^x + u_2 e^{-x} \\ &= \frac{x e^x}{2} - \left(\frac{e^{2x}}{4}\right) e^{-x} \\ &= \frac{x e^x}{2} - \frac{e^x}{4} \end{aligned}$$

and our general solution is

$$\begin{aligned} y &= y_p + y_c \\ &= \frac{x e^x}{2} - \frac{e^x}{4} + c_1 e^x + c_2 e^{-x} \\ &= \frac{x e^x}{2} + c_1 e^x + c_2 e^{-x} \end{aligned}$$