Calculus 1B Quiz 9 Name:

GSI: Paul Baginski Student ID:

1. (3 points) Find the general solution to:

$$y'' - 4y' + 4 = xe^{2x}$$

NOTE: I made a typo with this quiz. The left hand side *should* have been $y'' - 4y' + 4y$. The solution for that is below and the solution for the miswritten problem is on the back. Take a close look at the intended solution–it illustrates a very important point about multiplying by x.

Intended solution:

First we must find the general solution y_c to the homogeneous equation $y'' - 4y' + 4 = 0$. This equation has an auxiliary equation of $0 = r^2 - 4r + 4 = (r-2)^2$. Therefore the general solution is

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Now we must find a particular solution y_p to our differential equation. Our first guess is $y_p = (Ax+B)e^{2x}$, however this is a solution to the complementary equation so it yields 0 instead of xe^2x. Therefore we need to multiply by x to get $y_p = Ax^2e^{2x} + Bxe^{2x}$. While Ax^2e^{2x} is not a solution to the complementary equation, Bxe^{2x} is, so this y_p does not work either. This means we have to multiply by x *again* to try $y_p = Ax^3e^{2x} + Bx^2e^{2x}$. No terms in this function are solutions to the complementary equation, so this y_p should work.

Computing:

$$\begin{aligned} y_p' &= (3Ax^2 + 2Bx)e^{2x} + (2Ax^3 + 2Bx^2)e^{2x} \\ y_p'' &= (4Ax^3 + (12A + 4B)x^2 + (6A + 8B)x + 2B)e^{2x} \end{aligned}$$

and plugging this into the equation yields:

$$\begin{aligned} xe^{2x} &= y'' - 4y' + 4 \\ &= [(4Ax^3 + (12A+4B)x^2 + (6A+8B)x + 2B)e^{2x}] \\ &\quad -4[(3Ax^2 + 2Bx)e^{2x} + (2Ax^3 + 2Bx^2)e^{2x}] \\ &\quad +4[Ax^3e^{2x} + Bx^2e^{2x}] \\ &= 6Axe^{2x} + 2Be^{2x} \end{aligned}$$

Divide out by e^{2x} and equate coefficients of like powers of x to obtain: $1 = 6A$ and $0 = 2B$. Therefore $A = 1/6$ and $B = 0$ and our particular solution is

$$y_p = \frac{1}{6}x^3e^{2x}$$

so the general solution to our differential equation is:

$$y = y_p + y_c = \frac{1}{6}x^3e^{2x} + c_1e^{2x} + c_2xe^{2x}$$

NOTE: We absolutely needed to multiply by x^2 instead of just x. Otherwise we would have never found our particular solution.

Unintended Solution

We wish to solve the differential equation $y'' - 4y' = xe^{2x} - 4$. First we must find the general solution y_c to the homogeneous equation $y'' - 4y' = 0$. This has an auxiliary equation of $r^2 - 4r = 0$, which has roots $r = 0, 4$. Therefore

$$y_c = c_1 e^{0x} + c_2 e^{4x} = c_1 + c_2 e^{4x}$$

The constant function 1 and e^{4x} are linearly independent functions, so this general solution does make sense.

Now we find y_1 a solution to $y'' - 4y' = -4$. Our initial guess is $y_1 = A$, but this is in y_c so it cannot work. Therefore we multiply our first guess by x to get $y_1 = Ax$, which is not in y_c. Then $y_1' = A$ and $y_1'' = 0$ so we get the equation $0 - 4A = -4$. So $A = 1$ and $y_1 = x$.

Lastly, we find a solution y_2 to $y'' - 4y' = xe^{2x}$. Our initial guess is $y_2 = (Ax + B)e^{2x}$. This is fine since e^{2x} and xe^{2x} are not in y_c. Then

$$\begin{aligned} y_2' &= (A + 2B)e^{2x} + 2Axe^{2x} \\ y_2'' &= (4A + 4B)e^{2x} + 4Axe^{2x} \end{aligned}$$

so

$$\begin{aligned} xe^{2x} &= y_2'' - 4y_2' \\ &= (4A + 4B)e^{2x} + 4Axe^{2x} - 4(A + 2B)e^{2x} - 8Axe^{2x} \\ &= -4Axe^{2x} - 4Be^{2x} \end{aligned}$$

Dividing out by e^{2x} and equating coefficients of like powers of x, we get: $1 = -4A$ and $0 = -4B$ so $A = -1/4$, $B = 0$ and $y_2 = -xe^{2x}/4$.

Therefore our general solution is:

$$y = y_1 + y_2 + y_c = x - \frac{xe^{2x}}{4} + c_1 + c_2 e^{4x}$$