

1. (3 points) Find the general solution to:

$$y'' + 4y = 8x + \cos(2x)$$

First we find the general solution y_c to the complementary equation:

$$y'' + 4y = 0$$

The auxiliary equation for this is $r^2 + 4 = 0$, which has the solutions $r = \pm 2i$. Therefore the general solution is:

$$y_c = c_1 \cos(2x) + c_2 \sin(2x)$$

Now we must find a particular solution y_p . We do this by finding particular solutions y_1 and y_2 to the respective equations

$$\begin{aligned} y'' + 4y &= 8x \\ y'' + 4y &= \cos(2x) \end{aligned}$$

and then taking $y_p = y_1 + y_2$.

Let us consider a solution $y_1 = Ax + B$ to $y'' + 4y = 8x$. Then our equation becomes:

$$8x = y'' + 4y = (0) + 4(Ax + B)$$

so equating coefficients gives us $4A = 8$ and $4B = 0$. So $A = 2$ and $B = 0$ and $y_1 = 2x$ is a solution.

Now we must consider the equation $y'' + 4y = \cos(2x)$. Our first guess is $y_2 = A \cos(2x) + B \sin(2x)$, however we notice that these terms are both solutions to the complementary equation (so they yield 0 when placed into the left hand side). Therefore we multiply by x to try $y_2 = Ax \cos(2x) + Bx \sin(2x)$. Neither of these terms is a solution to the complementary equation. So,

$$\begin{aligned} y_2' &= A \cos(2x) - 2Ax \sin(2x) + B \sin(2x) + 2Bx \cos(2x) \\ y_2'' &= -2A \sin(2x) - 2A \sin(2x) - 4Ax \cos(2x) + 2B \cos(2x) + 2B \cos(2x) - 4Bx \sin(2x) \\ &= -4A \sin(2x) - 4Ax \cos(2x) + 4B \cos(2x) - 4Bx \sin(2x) \end{aligned}$$

and plugging in we get:

$$\begin{aligned} \cos(2x) &= y'' + 4y \\ &= (-4A \sin(2x) - 4Ax \cos(2x) + 4B \cos(2x) - 4Bx \sin(2x)) + 4(Ax \cos(2x) + Bx \sin(2x)) \\ &= -4A \sin(2x) + 4B \cos(2x) \end{aligned}$$

Since $\cos(2x)$ and $\sin(2x)$ are linearly independent, their coefficients on both sides must be equal. This means $1 = 4B$ and $0 = -4A$. Therefore $A = 0$ and $B = 1/4$ and $y_2 = \sin(2x)/4$.

Therefore the general solution is:

$$y = y_c + y_1 + y_2 = c_1 \cos(2x) + c_2 \sin(2x) + 2x + \sin(2x)/4$$