Calculus 1B Quiz 8 Name:

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1. (1 point) A pizza that comes out of the oven has a temperature of 370° Fahrenheit. Three minutes later you can eat it because it is 80° Fahrenheit. If the room temperature is 70° Fahrenheit, find the cooling constant for pizza (in degrees per minute).

This is a Newton's Law of Cooling problem. If $T(t)$ is the temperature of the pizza at time T, then $T(t)$ satisfies the differential equation:

$$\frac{dT}{dt} = k(T - 70)$$

This is a separable equation, so we solve to obtain:

$$\begin{aligned} \int \frac{dT}{T-70} &= \int k dt \\ \ln|T-70| &= kt + C \\ T - 70 &= Ae^{kt} \end{aligned}$$

for some constant A. To determine A, we plug in $T(0) = 370$ to find:

$$370 - 70 = Ae^0$$

so $A = 300$. Now we use $T(3) = 80$ to find k, because:

$$80 - 70 = 300e^{3k}$$

and therefore $e^{3k} = 1/30$ and so

$$k = \frac{\ln(\frac{1}{30})}{3} = -\frac{\ln(30)}{3}$$

(turn over)

2. (2 points) Blue whales feed on plankton. With an infinite supply of plankton, each year there would be a new whale for every 4 whales. However there is only a finite supply of plankton and this supply can feed at most 50 whales each year. There are currently 20 whales.

Write a differential equation that models the whale population under the restrictions of the plankton supply. Then solve the differential equation to find a formula for the number of whales after t years.

Since this is population growth with environmental restrictions, we need to use the logistic equation, namely

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

where $P(t)$ is the whale population at time t.

In ideal conditions, the environmental restrictions are negligible and $\frac{dP}{dt} \approx kP$. With infinite supply of plankton, every 4 whales produce a new whale, so $\frac{dP}{dt} \approx P/4$ and therefore $k = 1/4$.

The plankton supply can only support 50 whales maximum each year, so as long as $P(t) < 50$ the whales should be increasing in number, but if $P(t) > 50$ there should be a die-off of whales. In other words, $\frac{dP}{dt}$ should be negative exactly when $P(t) > 50$. From the form of the logistic equation, this can only happen when $K = 50$. Therefore our differential equation is:

$$\frac{dP}{dt} = \frac{1}{4}P\left(1 - \frac{P}{50}\right)$$

with initial value $P(0) = 20$.

The solution to the logistic equation is:

$$\begin{aligned} P(t) &= \frac{K}{1 + \frac{K - P(0)}{P(0)} e^{-kt}} \\ &= \frac{50}{1 + \frac{50-20}{20} e^{-\frac{t}{4}}} \\ &= \frac{50}{1 + \frac{3}{2} e^{-\frac{t}{4}}} \end{aligned}$$