Calculus 1B Quiz 8 Name:

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1. (1 point) The radioactive isotope polonium 208 has a half-life of 3 years. If we begin with 30 grams, find a formula for the mass that remains after $t$ years.

Radioactive elements have exponential decay, so if we set $P(t)$ to be the number of grams of polonium at time $t$, then

$$\frac{dP}{dt} = kP$$

for some constant $t$. We also know that $P(0) = 30$ and that in three years, there will be half as many grams, so $P(3) = 15$. Solving the differential equation for $P$, we get

$$P(t) = Ae^{kt}$$

for some constant $A$. But $P(0) = 30 = Ae^0 = A$, so $A = 30$. Therefore

$$P(3) = 15 = 30e^{3k}$$

so

$$k = \frac{\ln(\frac{1}{2})}{3} = \frac{-\ln(2)}{3}$$

(turn over)

2. (2 points) 100,000 people live in Berkeley. Today, 10 zombies rose from the grave and began attacking people. Any person who is bitten by a zombie also becomes a zombie. During the initial panic and confusion, each zombie will convert two people into new zombies each day. However, as the number of people decreases, it becomes more difficult for the zombies to find them and convert them. If all the people have turned into zombies, then there will be no new zombies.

Write down a differential equation that models the number of zombies. Then solve this equation to obtain an explicit function for the number of zombies after $t$ days.

We need the logistic equation, since we are considering a population growing in a restricted environment. Let $z(t)$ be the number of zombies after $t$ days.

Initially, while the environmental restrictions were still negligible, the zombies were doubling in number every day. So initially, we have

$$\frac{dz}{dt} \approx 2z$$

However, as the zombie population grows, the environmental restrictions start to play a factor. This is the role of $(1 - \frac{z}{K})$ in the logistic equation. So our differential equation is:

$$\frac{dz}{dt} = 2z(1 - \frac{z}{K})$$

where $K$ is the carrying capacity. As long as $z(t)$ stays smaller that $K$ the zombies will be increasing in population. We know that when $z = 100,000$ all the people have been turned into zombies and the rate of growth should be 0. Plugging into the differential equation, we get

$$0 = 2(100,000)\left(1 - \frac{100,000}{K}\right)$$

which is true exactly when $K = 100,000$. So our differential equation is:

$$\frac{dz}{dt} = 2z\left(1 - \frac{z}{100,000}\right)$$

with initial value $z(0) = 10$. The solution to this logistic equation is:

$$z(t) = \frac{100,000}{1 + Ae^{-2t}}, \text{ where } A = \frac{100,000 - 10}{10} = 9999$$

(CORRECTION: A sharp student noticed a mistake in the calculation of $K$, the carrying capacity. When all 100,000 people have been turned into zombies, then the rate of growth should be 0. However when all people have turned into zombies, the number of zombies will be 100,010 because the original 10 zombies that rose from the grave did not come from the 100,000 living Berkeley residents. Therefore $K = 100,010$ since the population of zombies cannot exceed this number.)