Calculus 1B Quiz 7 Name:

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1. (1 point) Find the solution to the differential equation:

$$y' = \frac{y}{x^2}, \qquad y(1) = 5$$

This is separable, so let's separate the variables.

$$\begin{aligned} y' &= \frac{y}{x^2} \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x^2} \\ \int \frac{1}{y}\frac{dy}{dx}dx &= \int \frac{1}{x^2}dx \\ \ln|y| &= \frac{-1}{x} + C \\ |y| &= e^{\frac{-1}{x}+C} \\ y &= Ae^{\frac{-1}{x}} \end{aligned}$$

where A is a constant. Using the initial value $y(1) = 5$, we can solve for A:

$$5 = Ae^{\frac{-1}{1}}$$

So $A = 5e$ and $y = 5e^{1-\frac{1}{x}}$ is the solution to the initial value problem.

(turn over)

2. (2 points) Find a solution y to the differential equation

$$y' = \frac{x+y}{x-y}$$

when x is positive. You do not need to express y explicitly.

(Hint: try substituting $v = y/x$).

Notice that if we multiply x and y by the same nonzero constant a, then the right hand side of the differential equation is unaffected (because $(ax + ay)/(ax - ay) = (x + y)(x - y)$). So that means the differential equation only depends on the ratio y/x, which is why the substitution $v = y/x$ might be fruitful.

If $v = y/x$, then $y = vx$. So then

$$\frac{x+y}{x-y} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

On the other hand, we have:

$$y' = (vx)' = vx' + v'x = v + v'x$$

So combined we get

$$\begin{aligned}
v + v'x &= \frac{1+v}{1-v} \\
v'x &= \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v} \\
\frac{1-v}{1+v^2}v' &= \frac{1}{x} \\
\int \frac{1}{1+v^2}\frac{dv}{dx}dx - \int \frac{v}{1+v^2}\frac{dv}{dx}dx &= \int \frac{dx}{x} \\
\arctan(v) - \frac{1}{2}\ln(1+v^2) &= \ln(x) + C
\end{aligned}$$

and now we substitute $v = y/x$ to get an equation in terms of only x and y:

$$\begin{aligned}
\arctan(y/x) - \frac{1}{2}\ln(1+y^2/x^2) &= \ln(x) + C \\
\arctan(y/x) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) &= \ln(x) + C \\
\arctan(y/x) - \frac{1}{2}\left(\ln(x^2+y^2) - 2\ln(x)\right) &= \ln(x) + C \\
\arctan(y/x) - \frac{1}{2}\ln(x^2+y^2) &= C
\end{aligned}$$