Calculus 1B Quiz 6 Name:

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1. (1 point) Write a Maclaurin series for:

$$\left(1+x^2\right)^{-2}$$

Binomial Series:

$$\begin{aligned}
\left(1+x^2\right)^{-2} &= \sum_{n=0}^{\infty} \binom{-2}{n} (x^2)^n \\
&= 1 - 2x^2 + \frac{(-2)(-2-1)}{2!}x^4 + \frac{(-2)(-2-1)(-2-2)}{3!}x^6 + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)!}{n!} x^{2n} \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n}
\end{aligned}$$

2. (1 point) Let

$$f(x) = e^x - 3x^2$$

If we use the first two terms in the Maclaurin series to approximate $\sqrt{e} - \frac{3}{4}$, estimate the error in the approximation.

$f'(x) = e^x - 6x$, $f^{(2)}(x) = e^x - 6$, $f^{(n)}(x) = e^x$ for all $n \geq 3$. Therefore the Maclaurin series is $\sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 1, c_1 = 1, c_2 = (1-6)/2! = -5/2$ and $c_n = 1/n!$ for all $n \geq 3$.

We wish to approximate $\sqrt{e} - \frac{3}{4} = f(1/2)$ with the first two terms of the Maclaurin, evaluated at $x = 1/2$. In other words, our approximation is $1 + 1 \cdot \frac{1}{2} = \frac{3}{2}$. The error is $R_1(\frac{1}{2})$, which we will bound using Taylor's Inequality.

We will work on the interval $[-1, 1]$, though any interval centered at 0 and containing $1/2$ will work. On this interval, $f^{(2)} = e^x - 6$ is increasing, with its minimum being $e^{-1} - 6$ and its maximum being $e^1 - 6$. The larger of these two values in magnitude is $e^{-1} - 6$, so on $[-1, 1]$ we have that

$$|f^{(2)}(x)| \leq |e^{-1} - 6| = 6 - \frac{1}{e}$$

and so Taylor's Inequality says

$$|R_1(x)| \leq \frac{6 - \frac{1}{e}}{2!}|x|^2$$

and thus $R_1(1/2) \leq (3 - \frac{1}{2e})(\frac{1}{4})$.

(turn over)

3. (1 point) Suppose m is a positive integer. Find a series $\sum_{n=0}^{\infty} a_n$ that represents:

$$\left(\frac{m+1}{m}\right)^{m\pi}.$$

This series should have infinitely many nonzero terms, a_n, and ln (the natural logarithm) should not not appear in any of the terms.
(In class hint: rewrite $(m+1)/m$ as $1+1/m$.)

As Professor Hald would say,

$$\left(1+\frac{1}{m}\right)^{m\pi}.$$

"smells of" Binomial Series. Indeed, if we let $f(x) = (1+x)^{m\pi}$, then our number is just $f(1/m)$.
The Binomial Series for $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \binom{m\pi}{n} x^n$$

so

$$\left(\frac{m+1}{m}\right)^{m\pi} = \sum_{n=0}^{\infty} \binom{m\pi}{n} \frac{1}{2^n} = 1 + \frac{m\pi}{2} + \frac{m\pi(m\pi-1)}{2\cdot 2!} + \frac{m\pi(m\pi-1)(m\pi-2)}{4\cdot 3!} + \ldots$$