Calculus 1B Quiz 6 Name:

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1. (1 point) Write the Maclaurin series for:

$$2^x$$

and estimate the error when you approximate 2^π using the first three terms of the series.

$$2^x = e^{\ln(2^x)} = e^{x\ln(2)} = \sum_{n=0}^{\infty} \frac{(\ln(2))^n x^n}{n!}$$

The first three terms, evaluated at $x = \pi$, are:

$$1 + \ln(2)\pi + \frac{\ln(2)^2\pi^2}{2}$$

and this estimate of 2^π has an error of $R_2(\pi)$. We will use Taylor's Inequality on the interval $[-4, 4]$ (though any interval centered at 0 and containing π will work). The third derivative of 2^x is $\ln(2)^3 2^x$, which is an increasing function so it achieves its minimum at the left endpoint and its maximum at the right endpoint. The min is $\ln(2)^3/8$ and its max is $8\ln(2)^3$. The max is bigger in absolute value so $|f^(3)(x)| \leq 8\ln(2)^3$ on $[-4, 4]$. Therefore

$$|R_2(x)| \leq \frac{8\ln(2)^3}{3!}|x|^3 \quad \text{for } x \in [-4, 4]$$

$$|R_2(\pi)| \leq \frac{4\ln(2)^3\pi^3}{3}$$

2. (1 point) Write the Maclaurin series for

$$(1 + x)^\pi$$

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Binomial Series:

$$(1+x)^\pi = \sum_{n=0}^{\infty} \binom{\pi}{n} x^n$$

$$= 1 + \pi x + \frac{\pi(\pi-1)}{2!}x^2 + \frac{\pi(\pi-1)(\pi-2)}{3!}x^3 + \dots$$

(turn over)

3. (1 point) Approximate 2^π using the first two terms of the Maclaurin series in Problem 2. Estimate the error.

Set $f(x) = (1+x)^\pi$. Then $2^\pi = f(1)$. The first two terms of the Maclaurin series, evaluated at $x = 1$, are $1+\pi$. The error is $R_1(1)$, which we will bound using Taylor's Inequality. We will work on the interval $[-1,1]$, though any interval centered at 0 and containing 1 will theoretically work. In this case, though, the choice of interval as $[-1,1]$ makes the analysis *much* easier.

The second derivative of $f(x)$ is $f^{(2)}(x) = \pi(\pi-1)(1+x)^{\pi-2}$. This function is increasing on $[-1,1]$ since

$$(f^{(2)}(x))' = f^{(3)}(x) = \pi(\pi-1)(\pi-2)(1+x)^{\pi-3}$$

and on $[-1,1]$ we have $0 \le 1+x \le 2$. Raising a positive number to any power always yields a positive number, so the derivative of $f^{(2)}(x)$ is non-negative on $[-1,1]$ and so $f^{(2)}(x)$ is increasing on that interval. Its minimum therefore is $f^{(2)}(-1) = 0^{\pi-2} = 0$ and its maximum is

$$f^{(2)}(1) = 2^{\pi-2} < 2^2 = 4.$$

Therefore $|f^{(2)}(x)| \le 4$ on $[-1,1]$.

Taylor's Inequality then says

$$|R_1(x)| \le \frac{4}{2}|x|^2$$

for x in $[-1,1]$. Therefore $|R_1(1)| \le 2|1|^2 = 2$ and so $2^\pi = 1+\pi$ with an error of at most 2.