Calculus 1B Quiz 5 Name:

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1. (1 point) Write the Maclaurin series for:

$$xe^{-2x^2}$$

The Maclaurin series for e^z is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

so now let's replace z with $-2x^2$. Then we get

$$\begin{aligned} xe^{-2x^2} &= x\left(1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \frac{(-2x^2)^4}{4!} + \dots\right) \\ &= x\left(1 - 2x^2 + \frac{4x^4}{2!} - \frac{8x^6}{3!} + \frac{16x^8}{4!} + \dots\right) \\ &= x - 2x^3 + \frac{4x^5}{2!} - \frac{8x^7}{3!} + \frac{16x^9}{4!} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n+1}}{n!} \end{aligned}$$

and this is the Maclaurin series for xe^{-2x^2}.

(turn over)

2. (2 points) Approximate

$$\int_0^2 \cos(\sqrt{x})dx$$

so that your error is less than $\frac{1}{100}$.

The Maclaurin series for $\cos(z)$ is

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

so let's substitute $z = \sqrt{x}$. This gives:

$$\begin{aligned}
\cos(\sqrt{x}) &= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \frac{(\sqrt{x})^8}{8!} - \dots \\
&= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} - \frac{x^4}{8!} - \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}
\end{aligned}$$

and since this is a power series centered at $a = 0$, this power series must be the Maclaurin series for $\cos(\sqrt{x})$. Now we integrate term by term:

$$\begin{aligned}
\int_0^2 \cos(\sqrt{x})dx &= \left[\int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \right] \Bigg|_0^2 \\
&= \left[\sum_{n=0}^{\infty} \int \frac{(-1)^n x^n}{(2n)!} \right] \Bigg|_0^2 \\
&= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)(2n)!} \right] \Bigg|_0^2 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{(n+1)(2n)!} \\
&= 2 - 1 + \frac{8}{3 \cdot (4!)} - \frac{16}{4 \cdot (6!)} + \frac{32}{5 \cdot (8!)} - \dots \\
&= 2 - 1 + \frac{1}{9} - \frac{1}{180} + \frac{1}{6300} - \dots
\end{aligned}$$

This is an alternating series, so the error on the nth partial sum is $|a_{n+1}|$. We want the error to be smaller than $1/100$ and the fourth term has absolute value $1/180 < 1/100$, so the sum of the first three terms gives us the value of the integral with the desired precision. So

$$\int_0^2 \cos(\sqrt{x})dx \approx 2 - 1 + \frac{1}{9} = \frac{10}{9}$$

and this value is accurate up to two decimal places.