Calculus 1B Quiz 3 Name:

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Set:

$$a_n = \frac{\arctan(n)}{1+n^2}$$

(1) Show that $\sum_{n=1}^{\infty} a_n$ converges.

(2) Give an upper bound for $\sum_{n=1}^{\infty} a_n$.

(Fact: $\arctan(1) = \pi/4$).

There are at least two ways to solve this problem: 1) Integral Test and 2) Comparison Test.

Solution 1: Integral Test. If you used integral test, I looked for three things: 1) showing $f(x)$ is decreasing so that you can validly apply integral test; 2) determining convergence; 3) finding a bound using Remainder Estimate for the Integral Test.

Take $f(x) = \frac{\arctan(x)}{1+x^2}$. The derivative of $f(x)$ is:

$$\frac{\frac{1}{1+x^2}(1+x^2) - 2x\arctan(x)}{(1+x^2)^2} = \frac{1-2x\arctan(x)}{(1+x^2)^2}$$

Since $\arctan(x)$ is increasing and $\arctan(1) = \pi/4$, the numerator is negative for all $x \geq 1$ and therefore the $f(x)$ is decreasing on $[1, \infty)$.

Now we determine convergence.

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \left[\frac{1}{2}(\arctan(x))^2\right]\Big|_1^t = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$$

Since the improper integral converges, so does the series.

Lastly, we use the Remainder Estimate for Integral Test to determine an upper bound for $S = \sum_{n=1}^{\infty} a_n$. By the Remainder Estimate, we know

$$S - s_1 = R_1 \leq \int_1^{\infty} \frac{\arctan(x)}{1+x^2}dx = \frac{3\pi^2}{32}$$

So

$$S \leq \frac{3\pi^2}{32} + s_1 = \frac{3\pi^2}{32} + \frac{\arctan(1)}{1+1^2} = \frac{3\pi^2}{32} + \frac{\pi}{8} = \frac{3\pi^2+4\pi}{32}$$

is an upper bound.

Solution 2: Convergence test. Even though this method was in Chapter 11.4, since I had gone over it in section when we reviewed CT for improper integrals, solving the problem with CT for series was totally acceptable. As long as CT was all you used, I looked for 1) determination of convergence and 2) upper bound. If, however, at any point you used integral test, then you were subjected to the grading criteria in solution 1 as well.

Since $\arctan(n)$ does not go above $\pi/2$, we have the following inequalities:

$$a_n = \frac{\arctan(n)}{1+n^2} \leq \frac{\pi}{2}\frac{1}{1+n^2} \leq \frac{\pi}{2}\frac{1}{n^2}.$$

For any constant c, the series $c\sum_{n=1}^{\infty} 1/n^2$ converges (p-series with $p = 2$), so with $c = \pi/2$ the Comparison Test indicates that the series $\sum_{n=1}^{\infty} a_n$ converges as well.

The Comparison Test also tells us that $\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ is an upper bound for our infinite sum. It is mentioned in Section 11.3 and was also mentioned in class that the exact sum of the important series $\sum_{n=1}^{\infty} 1/n^2$ is $\pi^2/6$. You were not expected to remember that, but if you did, it gave you a quick upper bound of $\pi^3/12$ to our series. This is only slightly higher than the upper bound given by the integral test.