Calculus 1B Quiz 3 Name:

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1. Compute:

$$\sum_{n=5}^{\infty} \left(\frac{-1}{\pi}\right)^n$$

This is a geometric series, but the indices are a little off.

$$\sum_{n=5}^{\infty} \left(\frac{-1}{\pi}\right)^n = \left(\frac{-1}{\pi}\right)^5 \sum_{n=5}^{\infty} \left(\frac{-1}{\pi}\right)^{n-5} = \left(\frac{-1}{\pi^5}\right) \sum_{n=0}^{\infty} \left(\frac{-1}{\pi}\right)^n =$$

$$= \frac{-1}{\pi^5}\left(\frac{1}{1+\frac{1}{\pi}}\right) = \frac{-1}{\pi^4(1+\pi)}$$

2. Set:

$$a_n = \frac{n-\sqrt{2}}{n^2-2}.$$

Does $\sum_{n=1}^{\infty} a_n$ converge?

(Hint: denominator factors!)

The denominator factors as $(n-\sqrt{2})(n+\sqrt{2})$ and thus $a_n = 1/(n+\sqrt{2})$. We will use integral test to determine $\sum_{n=1}^{\infty} a_n$ diverges. Set $f(x) = 1/(x+\sqrt{2})$, which is positive for positive x, clearly $f(n) = a_n$, and it's decreasing because the denominator gets bigger as x does and everything else is constant.

The improper integral

$$\int_1^{\infty} \frac{dx}{x+\sqrt{2}} = \lim_{t\to\infty} \left[ln(x+\sqrt{2})\right]\Big|_1^t = \infty$$

diverges, so by the integral test the series diverges as well.

(turn over)

3. Fibonacci Sequence. Define the sequence F_n as follows:

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_n + F_{n+1}$$

and set

$$a_n = \frac{F_{2n}}{2n}.$$

Does $\{a_n\}$ converge?

(Hint: Determine whether a_n is monotone or bounded. Do this by estimating F_{2n+2} in terms of F_{2n}.)

First thing to notice is that the F_n are always positive and increasing, because we are adding positive numbers in each step. The question is how fast they are growing. Looking at the definition of the F_n's, we see:

$$F_{2n+2} \;=\; F_{2n} + F_{2n+1} \;=\; F_{2n} + (F_{2n-1} + F_{2n}) \;=\; 2F_{2n} + F_{2n-1}.$$

Since the Fibonacci sequence is always positive, F_{2n-1} is positive and therefore $F_{2n+2} > 2F_{2n}$. This is enough to show that $\{a_n\}$ is monotone increasing:

$$a_{n+1} \;=\; \frac{F_{2(n+1)}}{2(n+1)} \;>\; \frac{2F_{2n}}{2(n+1)} \;>\; \frac{F_{2n}}{2n} \;=\; a_n$$

since $2n > n + 1$.

From class, we know that a monotone, bounded sequence is always convergent, and that a monotone, unbounded sequence is always divergent. So the question is whether $\{a_n\}$ is unbounded. Look again at the inequality $F_{2n+2} > 2F_{2n}$. It says by going up to F_{2n+2} we more than doubled F_{2n}. So $F_4 > 2F_2 = 2$, $F_6 > 2F_4 > 4F_2 = 4$, and in general $F_{2n} > 2^{n-1}$. Therefore

$$a_n = \frac{F_{2n}}{2n} > \frac{2^{n-1}}{2n}$$

which is unbounded. So $\{a_n\}$ is unbounded and thus divergent.