

1. Does

$$\int_1^{\infty} \frac{dx}{x + x^{2007}}$$

converge?

For $x \geq 1$ we have that $x + x^{2007} \geq x^{2007} \geq 1$; therefore

$$\frac{1}{x + x^{2007}} \leq \frac{1}{x^{2007}}.$$

Since $2007 > 1$, $\int_1^{\infty} dx/x^{2007}$ converges, so by Comparison Test our desired integral also converges.

2. Does:

$$\int_1^{\infty} \frac{dx}{(1 + x^2)(\arctan(x))^2}$$

converge? If so, evaluate.

First, the denominator is never equal to 0 on the interval $[1, \infty)$, so there are no discontinuities.

Set $u = \arctan(x)$, then $du = dx/(1 + x^2)$ and

$$\int \frac{dx}{(1 + x^2)(\arctan(x))^2} = \int \frac{du}{u^2} = \frac{-1}{u} = \frac{-1}{\arctan(x)}.$$

Therefore:

$$\int_1^{\infty} \frac{dx}{(1 + x^2)(\arctan(x))^2} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{(1 + x^2)(\arctan(x))^2} = \lim_{t \rightarrow \infty} \left[\frac{-1}{\arctan(x)} \right] \Big|_1^t,$$

and since $\arctan(t)$ converges to $\pi/2$ as $t \rightarrow \infty$ and $\arctan(1) = \pi/4$, we get that our improper integral converges to $2/\pi$.

(turn over)

3. Suppose $P(x)$ is a polynomial whose coefficients are all non-negative. Let $n \geq 2$ be an integer.

Show that

$$\int_1^{\infty} \frac{dx}{P(x) + x^n}$$

always converges.

(in class hint: for $x \geq 1$, $P(x) \geq 0$.)

The in-class hint is true because if $P(x)$ has all non-negative coefficients, then $P(x)$ will map a non-negative number to a non-negative number (each term in the polynomial is just ax^k for some $a \geq 0$ and $k \geq 0$).

Since $x \geq 1$, we have $x^n \leq P(x) + x^n$ so

$$\frac{1}{P(x) + x^n} \leq \frac{1}{x^n}.$$

But $n \geq 2$ so we know that $\int_1^{\infty} dx/x^n$ must converge. By Comparison Theorem, our original integral must also converge.