

1. Does:

$$\int_4^5 \frac{x dx}{\sqrt{x^2 - 16}}$$

converge? If so, evaluate.

The polynomial in the denominator has roots at 4 and -4 , so since 4 appears in the limits of integration, this is an improper integral.

Take $u = x^2 - 16$, then $du = 2x dx$. So

$$\int \frac{x dx}{\sqrt{x^2 - 16}} = \frac{1}{2} \int u^{-\frac{1}{2}} du = u^{\frac{1}{2}} = \frac{1}{4} \sqrt{x^2 - 16}$$

Now, with the actual bounds:

$$\int_4^5 \frac{x dx}{\sqrt{x^2 - 16}} = \lim_{t \rightarrow 4^+} \int_t^5 \frac{x dx}{\sqrt{x^2 - 16}} = \lim_{t \rightarrow 4^+} \left[\sqrt{x^2 - 16} \right]_t^5 = \lim_{t \rightarrow 4^+} 3 - \sqrt{t^2 - 16} = 3$$

2. Does:

$$\int_0^{\infty} x^{1+\sin^2(x)} dx$$

converge? If so, evaluate.

Diverges. For any x we have $1 \leq 1 + \sin^2(x) \leq 2$ so $x^1 \leq x^{1+\sin^2(x)}$ for every $x \geq 0$. But we know that $\int_0^{\infty} x dx$ diverges, so by Comparison Test

$$\int_0^{\infty} x^{1+\sin^2(x)} dx$$

diverges. This is not an integral you actually want to compute, so Comparison Test is the key to showing convergence.

Note: you can even just consider

$$\int_1^{\infty} x^{1+\sin^2(x)} dx.$$

Then $x^1 \leq x^{1+\sin^2(x)}$ again, but since $x \geq 1$ we have $x^1 \geq 1$. So now do Comparison Test with $\int_1^{\infty} dx$, which diverges, so $\int_1^{\infty} x^{1+\sin^2(x)}$ diverges and therefore the full improper integral diverges.

(turn over)

3. Does

$$\int_1^{\infty} \frac{e^{\sin(x)} \arctan(x)}{x^2} dx$$

converge?

This is an integral you REALLY don't want to compute, so Comparison Test has to be the key.

The denominator is x^2 , and we know $\int_1^{\infty} dx/x^2$ converges, so hopefully the numerator doesn't do anything too crazy.

Recall that $\arctan(x)$ is bounded above by $\pi/2$. Also remember that e^x is a monotone increasing function, meaning that if $a < b$ then $e^a < e^b$. Since $\sin(x) \leq 1$, $e^{\sin(x)} \leq e^1 = e$. So the numerator is always less than $\frac{e\pi}{2}$. Therefore

$$\frac{e^{\sin(x)} \arctan(x)}{x^2} \leq \frac{e\pi}{2} \frac{1}{x^2}.$$

Since multiplying by a non-zero constant does not affect convergence,

$$\int_1^{\infty} \frac{e\pi}{2} \frac{dx}{x^2}$$

converges and by Comparison Test the desired integral also converges.

(In principle, to apply Comparison Test as I just did, then you have to show that $\frac{e^{\sin(x)} \arctan(x)}{x^2}$ is always non-negative on $[1, \infty]$. The denominator is clearly never negative, $\arctan(x)$ is only negative when x is negative, and e^{anything} is always positive. Therefore the whole integrand is always positive on $[1, \infty)$.)