Do not forget that quizzes for *both* sections are available on the webpage
http://math.berkeley.edu/∼baginski/Teaching.html
Each section gets rather different problems.

For the following functions, perform these tasks:

(1) Write the Maclaurin series.
(2) Determine the radius of convergence for the Maclaurin series.

The functions:

1. $f(x) = 8$
2. $f(x) = \frac{6}{3x+4}$
3. $f(x) = \sin(\pi x^2)$
4. $f(x) = \frac{x}{8-x^3}$
5. $f(x) = \frac{\arctan(x^2)}{x}$
6. $f(x) = \sqrt{3-x}$

1. $8 = \sum_{n=0}^{\infty} c_n x^n$, where $c_0 = 8$ and $c_n = 0$ for every $n > 0$. Since this in actuality a finite sum, it must converge for all x.

2. Use the Maclaurin series for $1/(1-x)$ (geometric series).

$$\begin{aligned} \frac{6}{3x+4} &= \frac{3}{2}\frac{1}{1-(-\frac{3x}{4})} \\ &= \frac{3}{2}\sum_{n=0}^{\infty}(-\frac{3x}{4})^n \\ &= \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^n}{4^n} \end{aligned}$$

The right hand side is a geometric series with $r = -3x/4$, so it will converge exactly when $1 > |r| > |3x/4|$. This implies $-3/4 < x < 3/4$ so the interval of convergence is $(-4/3, 4/3)$.

3. Here we use the Maclaurin series for $\sin(z)$ (we're just using a different variable name) and

then substitute $z = \pi x^2$.

$$\begin{aligned}
\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x^2)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{4n+2}}{(2n+1)!}
\end{aligned}$$

Since the Maclaurin series $\sin(z)$ converges for all z, it will in particular converge for all z of the form πx^2. So the Maclaurin series for $\sin(\pi x^2)$ converges for all x and interval of convergence is $(-\infty, \infty)$.

4. We will find a geometric series inside this expression to be able to get the Maclaurin series.

$$\begin{aligned}
\frac{x}{8 - x^3} &= \frac{x}{8} \frac{1}{(1 - \frac{x^3}{8})} \\
&= \frac{x}{8} \sum_{n=0}^{\infty} \left(\frac{x^3}{8} \right)^n \\
&= \frac{x}{8} \sum_{n=0}^{\infty} \frac{x^{3n}}{8^n} \\
&= \sum_{n=0}^{\infty} \frac{x^{3n+1}}{8^{n+1}}
\end{aligned}$$

The geometric series that we used was $\sum_{n=0}^{\infty} \left(\frac{x^3}{8} \right)^n$, which converges exactly when $|x^3/8| < 1$, i.e. $|x| < 2$. So the interval of convergence for the geometric series is $(-2, 2)$ and multiplying this series on the outside by $x/8$ does not change the interval of convergence.

5. This will just be $1/x$ times the Maclaurin series for $\arctan(x^2)$. Therefore:

$$\begin{aligned}
\frac{\arctan(x^2)}{x} &= \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(x^{4n+2}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(x^{4n+-1}}{2n+1}
\end{aligned}$$

The interval of convergence for $\arctan(x)$ is $[-1, 1]$. Therefore the Maclaurin series of $\arctan(x^2)$ will converge only if $x^2 \in [-1, 1]$, which happens only when $x \in [-1, 1]$. Multiplying our Maclaurin series on the outside by $1/x$ will not change the interval of convergence, so it is $[-1, 1]$.

6. The raising a sum to an exponent of $1/2$ should cue that we use Binomial Series, but for

this we need to obtain the proper form.

$$\begin{aligned}
\sqrt{3-x} &= \sqrt{3}\left(1+\left(-\frac{x}{3}\right)\right)^{1/2} \\
&= \sqrt{3}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(-\frac{x}{3}\right)^{n} \\
&= \sqrt{3}\left(1+\frac{1}{1!}\frac{1}{2}\frac{-x}{3}+\frac{1}{2!}\frac{1}{2}\frac{(-1)}{2}\frac{(-x)^2}{3^2}+\frac{1}{3!}\frac{1}{2}\frac{(-1)}{2}\frac{(-3)}{2}\frac{(-x)^3}{3^3}+\dots\right) \\
&= \sqrt{3}\left(1-\frac{1}{1!}\frac{1}{2}\frac{x}{3}-\frac{1}{2!}\frac{(1\cdot 3)}{2^2}\frac{x^2}{3^2}-\frac{1}{3!}\frac{(1\cdot 3\cdot 5)}{2^3}\frac{(-x)^3}{3^3}+\dots\right) \\
&= \sqrt{3}-\sqrt{3}\sum_{n=1}^{\infty}\frac{(1\cdot 3\cdot 5\cdots(2n-1))}{n!}\frac{x^n}{6^n}
\end{aligned}$$

The Binomial Series $(1+x)^k$ has an interval of convergence $(-1,1)$. We used the series $(1+(-x/3))^{1/2}$, so we need $-x/3\in(-1,1)$, signifying that $x\in(-3,3)$ is the interval of convergence for our series.

For the following functions $f(x)$, perform these tasks:

(1) Write the Taylor series at $x=2$.
(2) Determine the 6th derivative of the function at $x=2$.
(3) Use the second Taylor polynomial $T_2(x)$ to estimate $f(2.5)$.
(4) Bound the error on this estimate.

The functions:

1. $f(x)=x^3+4x+1$
2. $f(x)=\sin(\pi x)$
3. $f(x)=\frac{-1}{x-3}$
4. $f(x)=(x-2)^2e^{5x}$

1. To write the Taylor series for this function, we actually have to compute the derivatives and evaluate them at $x=2$.

$$\begin{aligned}
f(x) &= x^3+4x+1, \quad f(2)=8+8+1=17 \\
f'(x) &= 3x^2+4, \quad f'(2)=12+4=16 \\
f''(x) &= 6x, \quad f''(2)=12 \\
f'''(x) &= 6, \quad f'''(2)=6 \\
f^{(n)}(x) &= 0, \text{ for } n\geq 4
\end{aligned}$$

so our expansion is:

$$\begin{aligned}
f(x) &= 17+16(x-2)+\frac{12}{2!}(x-2)^2+\frac{6}{3!}(x-2)^3 \\
&= 17+16(x-2)+6(x-2)^2+(x-2)^3
\end{aligned}$$

Since $f(x)$ is a third degree polynomial, its 6th derivative is constantly 0. The second Taylor polynomial at $x = 2$ is:

$$\begin{aligned} T_2(2.5) &= 17 + 16(2.5 - 2) + 6(2.5 - 2)^2 \\ &= 17 + 8 + \frac{3}{2} = \frac{53}{2} \end{aligned}$$

Since we are dealing with a finite sum, we don't need to bound the error–we can explicitly compute it.

$$|f(2.5) - T_2(2.5)| = |(2.5 - 2)^3| = 1/8$$

2. $\sin(\pi x)$ is another case where we have to explicitly compute the derivatives and evaluate them at $x = 2$ (the Maclaurin series for sin doesn't give us any information about the Taylor series at $x = 2$). So computing:

$$\begin{aligned} f(x) &= \sin(\pi x),\ \ f(2) = \sin(2\pi) = 0 \\ f'(x) &= \pi\cos(\pi x),\ \ f'(2) = \pi\cos(2\pi) = \pi \\ f''(x) &= -\pi^2\sin(\pi x),\ \ f''(2) = 0 \\ f'''(x) &= -\pi^3\cos(\pi x),\ \ f'''(2) = -\pi^3 \\ f^{(4)}(x) &= \pi^4\sin(\pi x),\ \ f^{(4)}(2) = 0 \\ f^{(5)}(x) &= \pi^5\cos(\pi x),\ \ f^{(5)}(2) = \pi^5 \end{aligned}$$

This is enough for us to get a general pattern: $f^{(2n)}(2) = 0$ and $f^{(2n+1)}(2) = (-1)^n\pi^{2n+1}$. So that means our Taylor series is

$$\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n+1}(x-2)^{2n+1}}{(2n+1)!}$$

We've explicitly had to compute the derivatives, so we calculated that $f^{(6)}(2) = 0$.

The second Taylor polynomial $T_2(x)$ is all the terms of the Taylor series where the exponent on $x - 2$ is at most 2. So $T_2(x) = \pi(x - 2)$ and $T_2(2.5) = \pi/2$. To estimate the error, we have two options. The first is to notice that when we evaluate the Taylor series at $x = 2.5$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n+1}(0.5)^{2n+1}}{(2n+1)!}$, which is an alternating series. So we can use the error estimate on alternate series, which is just the absolute value of the next term, namely $|(-1)^1\pi^3(0.5)^3/3!| = 9\pi^3/16$.

Our second option is to use Taylor's inequality. The 3rd derivative of $f(x)$ is $-\pi^3\cos(\pi x)$, which never exceeds $M = \pi^3$ in absolute value for all x. Therefore, we can bound the remainder $|R_2(2.5)|$ by:

$$\begin{aligned} |R_2(x)| &\leq \frac{M}{3!}|x-a|^3, \text{ so} \\ |R_2(2.5)| &\leq \frac{\pi^3}{6}(0.5)^3 = 9\pi^3/16 \end{aligned}$$

In this case, the bounds we get are the same.

3. The function $f(x) = \frac{-1}{x-3}$ looks very similar to the sort of functions that would yield a geometric series as their Maclaurin series. That perhaps suggests that the Taylor series would also look like a geometric series $\sum ar^n$. But since it's a Taylor series about $x = 2$, we need something of the form $\sum c_n(x-2)^n$, so that implies that our r should have $(x - 2)$ as a factor. So we need to

rearrange $f(x)$ to the form $1/(1-r)$, where $(x-2)$ is a factor of r. If we rearrange our function, we get:

$$\begin{aligned}\frac{-1}{x-3} &= \frac{1}{3-x} \\ &= \frac{1}{1-(x-2)} \\ &= \sum_{n=0}^{\infty}(x-2)^n\end{aligned}$$

and this is our Taylor series at $x=2$. We know from the form of the Taylor series that the coefficient of $(x-2)^6$ is $f^{(6)}(2)/6!$. In our case, this coefficient is 1, so that means $f^{(6)}(2)=6!$.

Next, we need to evaluate $T_2(x) = 1+(x-2)+(x-2)^2$ at $x=2.5$. This comes out to: $T_2(2.5) = 1.75$. Since we are dealing with geometric series, we can compute the error explicitly (rather than just bounding it):

$$\begin{aligned}|f(2.5)-T_2(2.5)| &= |\sum_{n=0}^{\infty}(2.5-2)^n - 1 - (2.5-2) - (2.5-2)^2| \\ &= \sum_{n=3}^{\infty}(2.5-2)^n \\ &= \sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n \\ &= \left(\frac{1}{2}\right)^3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \\ &= \frac{1}{8}\frac{1}{1-\frac{1}{2}} \\ &= \frac{1}{4}\end{aligned}$$

4. Consider $f(x)=(x-2)^2e^{5x}$. We want a power series of the form $\sum c_n(x-2)^n$, so the $(x-2)^2$ that appears in $f(x)$ poses no problem. We'll just compute the Taylor series for e^{5x} at $x=2$ and then multiply through by $(x-2)^2$:

$$\begin{aligned}g(x) &= e^{5x},\ g(2)=e^{10} \\ g'(x) &= 5e^{5x},\ g'(2)=5e^{10} \\ g''(x) &= 5^2e^{5x},\ g''(2)=5^2e^{10} \\ g'''(x) &= 5^3e^{5x},\ g'''(2)=5^3e^{10}\end{aligned}$$

From this pattern we get that the Taylor series for e^{5x} at $x=2$ is:

$$e^{5x} = \sum_{n=0}^{\infty}\frac{5^ne^{10}}{n!}(x-2)^n$$

So that means the Taylor series for $f(x)$ at $x = 2$ is:

$$\begin{aligned} f(x) &= (x-2)^2 e^{5x} \\ &= (x-2)^2 \sum_{n=0}^{\infty} \frac{5^n e^{10}}{n!}(x-2)^n \\ &= \sum_{n=0}^{\infty} \frac{5^n e^{10}}{n!}(x-2)^{n+2} \\ &= \sum_{n=2}^{\infty} \frac{5^{n-2} e^{10}}{(n-2)!}(x-2)^n \end{aligned}$$

(the first two terms of the Taylor series turned out to have coefficients of 0).

We know that the coefficient of $(x-2)^6$ is $f^{(6)}(2)/6!$. Here that coefficient is $5^4 e^{10}/4!$. So that means $f^{(6)}(2) = 6 \cdot 5^5 e^{10}$.

The second Taylor polynomial $T_2(x)$ is all the terms of the Taylor series up to the $(x-2)^2$ term. In this case that's only: $T_2(x) = e^{10}(x-2)^2$. So $T_2(2.5) = e^{10}/4$.

To bound the error, we need to use Taylor's Inequality and for that we need to explicitly compute the third derivative of $f(x)$.

$$\begin{aligned} f'''(x) &= 5e^{5x}(6 + 30(x-2) + 25(x-2)^2) \\ &= 5e^{5x}(25x^2 - 70x + 46) \end{aligned}$$

and now we must bound it on some interval $[2-d, 2+d]$ that contains 2.5. This function is kinda ugly, so let's keep our interval as small as possible: $[1.5, 2.5]$. The factor of $5e^{5x}$ is a strictly increasing function, so on the interval $5e^{7.5} \leq 5e^{5x} \leq 5e^{12.5}$. Now we must deal with the polynomial $25x^2 - 70x + 46$. Could it be increasing on this interval as well? The derivative is $50x - 70$, which is positive for all x in $[1.5, 2.5]$ (but becomes negative just a little outside of this interval, so it's good we picked one so small!). So since the polynomial is increasing, for all x in $[1.5, 2.5]$ we have

$$\begin{aligned} 25(1.5)^2 - 70(1.5) + 46 \leq & \; 25x^2 - 70x + 46 & \leq 25(2.5)^2 - 70(2.5) + 46 \\ -2.75 \leq & \; 25x^2 - 70x + 46 & \leq 27.25 \end{aligned}$$

Therefore on our interval, $|5e^{5x}| \leq 5e^{12.5}$ and $|25x^2 - 70x + 46| \leq 27.25$, so combined we get that $|f(x)| \leq 136.25e^{12.5}$. Set this number to be M. Then

$$\begin{aligned} |R_2(2.5)| &\leq \frac{M}{3!}(2.5-2)^3 \\ &= 136.25e^{12.5}(0.5)^3/6 \\ &\approx 2.839e^{12.5} \end{aligned}$$

Find the general solution to the following differential equations, or if initial values are given, solve the initial value problem.

1. $xy' + y = xe^{2x}$
2. $y' + xy^2 = -x, \; y(0) = \pi$

3. $y'' - 3y' - 4y = 5e^{-x}$

4. $y'' + y = 2\sin(x) + 4x\cos(x)$

5. $3y'' + 2y' - 5y = 0,\ y(0) = 1,\ y'(0) = 1$

6. $y'' - \frac{y'}{x} + xy = 0$

1. This is a first order linear differential equation with $P(x) = 1/x$ and $Q(x) = e^{2x}$. We first calculate $I(x) = e^{\int P(x)dx}$

$$\begin{aligned} I(x) &= e^{\int P(x)dx} \\ &= e^{\int \frac{dx}{x}} \\ &= e^{ln(x)} \\ &= x \end{aligned}$$

Therefore the general solution is:

$$\begin{aligned} y &= \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right] \\ &= \frac{1}{x}\left[\int xe^{2x}dx + C\right] \\ &= \frac{1}{x}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}dx + C\right] \\ &= \frac{1}{x}\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C\right] \end{aligned}$$

We performed an integration by parts in the middle, with $u = x$ and $dv = e^{2x}dx$.

2. We can separate the variables in this equation to obtain:

$$\begin{aligned} y' + xy^2 &= -x \\ y' &= -x(1 + y^2) \\ \frac{dy}{1 + y^2} &= -xdx \\ \int \frac{dy}{1 + y^2} &= -\int xdx \\ \arctan y &= -\frac{x^2}{2} + C \\ y &= \tan\left(-\frac{x^2}{2} + C\right) \end{aligned}$$

And now we use the initial value.

$$\pi = y(0) = \tan(C)$$

so $C = \arctan(\pi)$ and $y = \tan\left(-\frac{x^2}{2} + \arctan(\pi)\right)$.

3. $y'' - 3y' - 4y = 5e^{-x}$. This is a second-order linear differential equation, so we must find the general solution y_c to the homogeneous equation $y'' - 3y' - 4y = 0$ and a particular solution y_p to our original differential equation.

We can solve $y'' - 3y' - 4y = 0$ by considering the auxiliary equation $r^2 - 3r - 4$, which has the roots $r = -1, 4$. So

$$y_c = c_1 e^{-x} + c_2 e^{4x}$$

Our first guess for y_p is Ae^{-x}, however that appears in y_c. Therefore we multiply our first guess by x to get $y_p = Axe^{-x}$. This is not in y_c, so we may proceed with determining A.

$$\begin{aligned} y_p' &= Ae^{-x} - Axe^{-x} \\ y_p'' &= Axe^{-x} - 2Ae^{-x} \end{aligned}$$

$$\begin{aligned} 5e^{-x} &= [Axe^{-x} - 2Ae^{-x}] - 3[Ae^{-x} - Axe^{-x}] - 4[Axe^{-x}] \\ 5e^{-x} &= -5Ae^{-x} \end{aligned}$$

Therefore $A = -1$, $y_p = -xe^{-x}$ and our general solution is:

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{4x} - xe^{-x}$$

4. $y'' + y = 2\sin(x) + 4x\cos(x)$. This is also a second-order linear differential equation, so to obtain y_c we need to solve the auxiliary equation $r^2 + 1 = 0$, which has roots $r = \pm i$. Therefore

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

Now, normally we would find a particular solution y_{p_1} to $y'' + y = 2\sin(x)$ and a particular solution y_{p_2} to $y'' + y = 4x\cos(x)$ and then add them. But if we were clever, we'd notice that our first guess for y_{p_2} would encompass our first guess for y_{p_1}. So really, we can save ourselves some time by finding a particular solution y_p to our full differential equation $y'' + y = 2\sin(x) + 4x\cos(x)$.

The first guess for y_p is:

$$y_p = (Ax + B)\cos(x) + (Cx + D)\sin(x)$$

However, we notice that the $B\cos(x)$ and $D\sin(x)$ terms appear in y_c, so we must multiply our first guess of y_p by x:

$$y_p = (Ax^2 + Bx)\cos(x) + (Cx^2 + Dx)\sin(x)$$

This guess does not have any terms in y_c, so we are okay. Now we solve for A, B, C, D.

$$y_p' = (Cx^2 + (D + 2A)x + B)\cos(x) + (-Ax^2 + (2C - B)x + D)\sin(x)$$

$$y_p'' = (-Ax^2 + (4C - B)x + 2A + 2D)\cos(x) + (-Cx^2 - (4A + D)x - 2B + 2C)\sin(x)$$

$$\begin{aligned} 2\sin(x) + 4x\cos(x) &= y_p'' + y_p \\ &= (4Cx + 2A + 2D)\cos(x) + (-4Ax - 2B + 2C)\sin(x) \end{aligned}$$

So equating coefficients, we get four equations:

$$\begin{aligned} 4 &= 4C \\ 0 &= 2A + 2D \\ 0 &= -4A \\ 2 &= -2B + 2C \end{aligned}$$

and these equations have the solution: $A = B = D = 0$ and $C = 1$. Therefore $y_p = x^2 \sin(x)$ and our general solution is:

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x)$$

5. $3y'' + 2y' - 5y = 0$, $y(0) = 1$, $y'(0) = 1$. This is a homogeneous second-order linear differential equation, so to solve it we need to solve the auxiliary equation: $3r^2 + 2r - 5 = 0$. This has roots $r = 1, -5/3$. So the general solution is

$$y = c_1 e^x + c_2 e^{-\frac{5}{3}x}$$

Then $1 = y(0) = c_1 + c_2$ and

$$\begin{aligned} y' &= c_1 e^x - \frac{5}{3} c_2 e^{-\frac{5}{3}x} \\ 1 = y'(0) &= c_1 - \frac{5}{3} c_2 \end{aligned}$$

These two equations have the solution $c_1 = 1$ and $c_2 = 0$, so our solution is $y = e^x$.

6. $y'' - \frac{y'}{x} + xy = 0$. We have a second-order differential equation with x's multiplying y and its derivatives. The only way we know how to solve this is with series solutions. So:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ y' &= \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \end{aligned}$$

Let's multiply the equation through by x (to make it cleaner) and then plug the y's back into the equation:

$$\begin{aligned} 0 &= xy'' - y' + x^2 y \\ &= x \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + x^2 \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+1} - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

For any x^n, the coefficient of x^n on the left-hand side equals the sum of the coefficients of x^n on the right-hand side. So we get the equations

$$\begin{array}{llcl}
(x^0) & 0 & = & 0 - c_1 + 0 \\
(x^1) & 0 & = & 2c_2 - 2c_2 + 0 \\
(x^2) & 0 & = & 3 \cdot 2c_3 - 3c_3 + c_0 \\
(x^3) & 0 & = & 4 \cdot 3c_4 - 4c_4 + c_1 \\
 & & \vdots & \\
(x^n) & 0 & = & (n+1)nc_{n+1} - (n+1)c_{n+1} + c_{n-1} \\
 & & = & (n+1)(n-1)c_{n+1} + c_{n-2}
\end{array}$$

where the last line is the equation for all $n \geq 2$.

The first equation gives us that $c_1 = 0$, and then the equation for x^2 gives us that $c_4 = 0$ also. Proceeding recursively, we see that $c_{3k+1} = 0$ for all k.

Next we see that $c_3 = -c_0/4 \cdot 2$ and $c_6 = -c_3/7 \cdot 5 = c_0/(7 \cdot 5 \cdot 4 \cdot 2)$. Continuing, $c_9 = -c_0/(10 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 2)$ and $c_{12} = c_0/(13 \cdot 11 \cdot 10 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 2)$. The denominator is almost a factorial, except all the multiples of three are missing. This can be written as:

$$c_{3k} = \frac{(-1)^k 3^k k!}{(3k+1)!} c_0$$

Lastly, we have to look at indices of the form $3k+2$. We get: $c_5 = -c_2/6 \cdot 4$ and $c_8 = -c_5/9 \cdot 7 = c_0/(9 \cdot 7 \cdot 6 \cdot 4)$ and $c_{11} = -c_0/(12 \cdot 10 \cdot 9 \cdot 7 \cdot 6 \cdot 4)$. This pattern can be written as:

$$c_{3k+2} = \frac{(-1)^k 3(3k+2)(3(k-1)+2) \cdots 2}{(3k+3)!} c_2$$

Therefore our solution is:

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n n!}{(3n+1)!} c_0 x^{3n} + \sum_{n=0}^{\infty} \frac{(-1)^n 3(3n+2)(3(n-1)+2) \cdots 2}{(3n+3)!} c_2 x^{3n+2}$$

Use differential equations to model the number of bead stores in Berkeley. There are currently 7 bead stores in Berkeley.

1. If everybody in Berkeley loved beads, the number of bead stores would double every 4 years. Write and solve a differential equation that models the number of bead stores.

Under ideal conditions our "population" of bead stores should grow exponentially. So our differential equation is

$$\frac{dP}{dt} = kP$$

which has a solution $P = P_0 e^{kt} = 7e^{kt}$. Now we must determine k. Our population will double every four years, so at time $t = 4$ we should have 14 bead stores: $P(4) = 14 = 7e^{4k}$, which gives us $k = \ln(2)/4$.

2. But seriously, even *Berkeley* can't sustain more than 20 bead stores. Write and solve a differential equation that models the number of bead stores that takes into account this restraint.

Our differential equation should be a logistic equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

Here, the factor $(1 - \frac{P}{K})$ is attached to temper the growth that would occur in ideal conditions. This ideal growth corresponds to the kP, so our k is actually the k from the previous half of the problem. Our K is 20 because the environment cannot support a larger population. So our differential equation is:

$$\frac{dP}{dt} = \frac{\ln(2)}{4}P(1 - \frac{P}{20})$$

The solution is:

$$\begin{aligned} P &= \frac{K}{1 + Ae^{-kt}} \\ &= \frac{20}{1 + Ae^{-\ln(2)t/4}} \\ &= \frac{20}{1 + A2^{-t/4}} \end{aligned}$$

and $A = (K - P_0)/P_0 = 13/7$.