

For the following sequences, perform these tasks:

- (1) Write a general formula for $\{a_n\}$.
- (2) Determine if the sequence $\{a_n\}$ converges.
- (3) Determine if the series $\sum_{n=1}^{\infty} a_n$ converges.
- (4) Determine if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- (5) If $\sum_{n=1}^{\infty} a_n$ converges, find an upper and lower bound for the sum.

The sequences:

$$1. \frac{5}{2}, \frac{5}{4}, \frac{5}{6}, \frac{5}{8}, \dots$$

This sequence has the formula $a_n = \frac{5}{2n} = \frac{5}{2} \frac{1}{n}$. The sequence $\{a_n\}$ converges, and its limit is $\lim_{n \rightarrow \infty} a_n = 0$. The a_n are just the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ multiplied by the constant $5/2$. Since the harmonic series diverges, so does $\sum_{n=1}^{\infty} a_n$.

$$2. \frac{11}{5}, \frac{3}{2}, \frac{19}{15}, \frac{23}{20}, \frac{27}{25}, \frac{31}{30}, \dots$$

$\frac{3}{2}$ looks weird, so let's look if there's some pattern in the majority of the terms (and then $\frac{3}{2}$ might just be some anomaly). In the other terms, the numerator seems to be growing by 4 with each iteration, and the denominator seems to be growing by 5. So the numerator is going to have $4n$ appearing somewhere, and the denominator will have $5n$. Adjusting for the initial values at a_1 , we get

$$a_n = \frac{4n + 7}{5n}$$

and sure enough, $\frac{3}{2}$ was just $\frac{15}{10}$ written in lowest terms, so it wasn't out of place after all.

The a_n converge, with $\lim_{n \rightarrow \infty} a_n = \frac{4}{5}$. Since $\lim a_n \neq 0$, the Divergence Test tells us that $\sum_{n=1}^{\infty} a_n$ diverges.

$$3. \frac{1}{\ln(2)}, \frac{1}{\ln(3)} - \frac{1}{\ln(2)}, \frac{1}{\ln(4)} - \frac{1}{\ln(3)}, \frac{1}{\ln(5)} - \frac{1}{\ln(4)}, \dots$$

Here, for $n \geq 2$, we get a nice pattern that

$$a_n = \frac{1}{\ln(n+1)} - \frac{1}{\ln(n)}$$

and the only problem is that a_1 doesn't have this form (because we'd be dividing by $\ln(1) = 0$ and that's undefined). So, we just make an exception for a_1 and say $a_1 = 1/\ln(2)$ and then a_n has the form above for $n \geq 2$.

The $1/\ln(n)$ goes to 0 as n goes to ∞ , so $\lim_{n \rightarrow \infty} a_n = 0 - 0 = 0$ and the a_n converge. Now let's see if the series $\sum_{n=1}^{\infty} a_n$ converges.

Here the important thing to notice is that if you add a_n to a_{n+1} , a part of a_n cancels out with a part of a_{n+1} , namely the $1/\ln(n)$. This means that we're dealing with a telescoping sum, so we can get an explicit formula for the partial sums s_n .

$$s_n = a_1 + a_2 + \dots + a_n = \left[\frac{1}{\ln(2)} \right] + \left[\frac{1}{\ln(3)} - \frac{1}{\ln(2)} \right] + \dots + \left[\frac{1}{\ln(n+1)} - \frac{1}{\ln(n)} \right] = \frac{1}{\ln(n+1)}$$

since in this case everything else cancels. So

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

and we have not only found out that the series converges, but found out the exact number this infinite sum adds up to: 0.

$$4. \frac{1}{11}, \frac{7}{101}, \frac{25}{1001}, \frac{79}{10001}, \frac{241}{100001}, \dots$$

This is a bit more complicated. The denominator is the easier to figure out: it's just $10^n + 1$. Now we have to look how fast the numerator is growing. Each numerator looks to be approximately 3 times the previous numerator: $1 * 3$ is (relatively) close to 7; $7 * 3$ is close to 25; $25 * 3$ is close to 79. This indicates that the numerator is growing similarly to 3^n . And once we've made that observation, it's only a step further to notice that the numerator is $3^{n-1} + 4$. So

$$a_n = \frac{3^{n-1} + 4}{10^n + 1}$$

and we can take the limit to see if a_n converges:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{10}\right)^{n-1} + \frac{4}{10^{n-1}}}{10 + \frac{1}{10^{n-1}}} = 0$$

and therefore the a_n converge to 0.

In taking this limit, we actually found what the dominant terms in a_n were: the 3^{n-1} in the numerator and the 10^{n-1} in the denominator. That means that as n gets big, a_n acts similarly to $(3/10)^{n-1}$. This suggests that we can use Comparison Test or Limit Comparison Test. Since we want to also get bounds, I'll use Comparison Test, which requires a bit of cleverness:

$$a_n \leq \frac{3^{n-1} + 4}{10^n} < \frac{3^{n+1}}{10^n} = b_n$$

since $3^{n-1} + 4 < 3^{n+1}$ when $n \geq 1$. Then b_n is just a geometric series with $r = 3/10$ and $a = 9/10$ (remember a is just the first term of the geometric series, and n in this case started at 1). So since $\sum_{n=1}^{\infty} b_n$ is convergent, the by Comparison Test, $\sum_{n=1}^{\infty} a_n$ is convergent too, and it is bounded above by $\sum_{n=1}^{\infty} b_n = \frac{9/10}{1-3/10} = 9/7$. For a lower bound, just note that a_n is always positive, so the partial sums are increasing. That means that $a_1 = 5/10 = 1/2$ is a lower bound for $\sum_{n=1}^{\infty} a_n$.

$$5. \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{18}, -\frac{1}{16}, -\frac{\sqrt{2}}{50}, 0, \frac{\sqrt{2}}{98}, \dots$$

This one is quite a hard one to figure out the formula for a_n . There's a few weird things going on: the sign changes for a while and then switches back; there are these 0's occurring as terms; and there are occasional $\sqrt{2}$ appearing in the numerator. What to make of it all?

The first tip-off is actually the first term: $\sqrt{2}/2$ should be a very familiar number, as it's one of the values spit out by sin and cos. It is $\sin(\pi/4)$ and $\cos(\pi/4)$. So maybe some other values of sin or cos occur in the sequence too--this could explain the 0's popping up since $0 = \sin(0) = \cos(\pi/2)$. So now the first two terms don't seem too mysterious and the key to choosing between sin and cos lies in the third term. There is a $\sqrt{2}$ in the numerator, but then that's actually going to give us

a $\sqrt{2}/2$ if we pull a 2 out of the denominator. The negative sign then means that either we have $\cos(3\pi/4)$ or $\sin(-\pi/4)$. It actually doesn't matter which one we pick, but the cos has a cleaner description: we're getting a $\cos(n\pi/4)$ floating around. This was the hard part and from there it's quick to see that

$$a_n = \frac{\cos(\frac{n\pi}{4})}{n^2}.$$

(Determining the formula for such a difficult sequence is pretty time-consuming, so probably for such a sequence you'd just get the formula outright on the exam.)

Now we determine whether a_n converges. We have that $-1 \leq \cos(n\pi/4) \leq 1$, so $-1/n^2 \leq a_n \leq 1/n^2$. By the Squeeze Theorem, since $\lim_{n \rightarrow \infty} -1/n^2 = 0 = \lim_{n \rightarrow \infty} 1/n^2$, we also get $\lim_{n \rightarrow \infty} a_n = 0$.

Now let us determine convergence. Since the sign does not switch every other term, we cannot use Alternating Series Test. So maybe we are lucky and the series absolutely converges, which would then imply that $\sum^\infty a_n$ converges. So:

$$|a_n| = \frac{|\cos(\frac{n\pi}{4})|}{n^2} \leq \frac{1}{n^2}$$

Since $\frac{1}{n^2}$ is a p -series with $p > 1$, it converges and so by Comparison Test, $\sum^\infty |a_n|$ converges. This exactly means that $\sum^\infty a_n$ converges absolutely and so $\sum^\infty a_n$ converges.

Now we need upper and lower bounds on $\sum^\infty a_n$. We have $-|a_n| \leq a_n \leq |a_n|$ by the definition of absolute value, so

$$-\frac{1}{n^2} \leq -|a_n| \leq a_n \leq |a_n| \leq \frac{1}{n^2}$$

and taking partial sums and limits, we get

$$-\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

If you happened to remember from the book (or class) that $\sum^\infty 1/n^2 = \pi^2/6$, then this gives you explicit numerical bounds, but having these bounds in terms of well-known convergent series is fine too.

For the following integrals, perform these tasks:

- (1) Determine if the integral is improper.
- (2) If improper, determine if the integral converges.
- (3) If improper and convergent, evaluate or find bounds.

The integrals:

1. $\int \sqrt{x} \ln(x) dx$

There are no definite bounds, so this is not an improper integral. We solve by parts. The $\ln(x)$ is annoying, but if we differentiate, we get $1/x$, which is less annoying. So set $f = \ln(x)$ and $g' = \sqrt{x} dx$. Then $f' = dx/x$ and $g = \frac{2}{3}x^{3/2}$, so

$$\int \sqrt{x} \ln(x) dx = \frac{2}{3} \ln(x) x^{\frac{3}{2}} - \frac{2}{3} \int x^{\frac{1}{2}} dx = \frac{2}{3} \ln(x) x^{\frac{3}{2}} - \frac{4}{9} x^{\frac{3}{2}}$$

2. $\int_2^3 \frac{x dx}{4x^2 - 25}$

The bounds of integration are finite, but the function has a point of discontinuity at $x = 5/2$, which is in the interval $[2, 3]$, so this is an improper integral. This means we have to evaluate

$$\lim_{t \rightarrow \frac{5}{2}^-} \int_2^t \frac{x dx}{4x^2 - 25}, \text{ and}$$

$$\lim_{t \rightarrow \frac{5}{2}^+} \int_t^3 \frac{x dx}{4x^2 - 25}.$$

If either one of these improper integrals diverges, then the full integral diverges. Otherwise, everything converges and the full integral is equal to the sum of these two values.

Set $u = 4x^2 - 25$, then $du = 8x$, so after integration, the first improper integral becomes

$$\lim_{t \rightarrow \frac{5}{2}^-} [\ln(|4x^2 - 25|)] \Big|_2^t = \ln(0) - \ln(9) = -\infty$$

so the first improper integral diverges. Therefore we do not even have to evaluate the second integral, because the full integral from 2 to 3 must diverge.

3. $\int_{-\infty}^{\infty} x e^{-x^2} dx$

This function is being integrated from $-\infty$ to ∞ , so definitely improper integral. Additionally, $x e^{-x^2}$ is always defined so it has no discontinuities. Therefore we need to break up the integral into two improper integrals, and for no special reason, we'll pick 7 as the splitting point. So we have to check the following two integrals:

$$\int_{-\infty}^7 x e^{-x^2} dx, \text{ and}$$

$$\int_7^{\infty} x e^{-x^2} dx$$

Set $u = -x^2$, then $du = -2x dx$ so our first integral becomes:

$$\lim_{t \rightarrow -\infty} -\frac{1}{2} \int_{-t^2}^{-49} e^u du = \lim_{t \rightarrow -\infty} -\frac{1}{2} [e^u]_{-t^2}^{-49} = -e^{-49}$$

and so this improper integral converges to $-e^{-49}$. The second one has a similar computation, and gives the value e^{-49} . So the full integral from $-\infty$ to ∞ converges, and its value is 0.

4. $\int x \cos(x) \sin(x) dx$.

This integral is not improper since it does not have definite bounds. There are two flavors of function on the inside of the integral sign: there's a polynomial x and then a combination of trigonometrics $\cos(x) \sin(x)$. So this suggests to use integration by parts, to eventually get integrals of only one flavor, polynomial or trigonometric. How to choose the parts? The interesting thing about the trigonometric part is that it's a function $\sin(x)$ times its derivative $\cos(x)$ so we can integrate it easily. The remainder is x , which will disappear when we differentiate, so this sounds pretty good, because we'll be left with an integral that's all trigonometric. In detail: $f = x$, $f' = dx$, $g = \cos(x) \sin(x) dx$, $g = \sin^2(x)$, so we get:

$$\int x \cos(x) \sin(x) dx = x \sin^2(x) - \int \sin^2(x) dx.$$

Now we have a purely trigonometric integral to deal with, and it's one that we've done before for homework. Set $f = \sin(x)$ and $g' = \sin(x)dx$. Then we get

$$\int \sin^2(x)dx = -\sin(x)\cos(x) + \int \cos^2(x)dx$$

and then we use $\cos^2(x) = 1 - \sin^2(x)$ to obtain after rearrangement:

$$\int \sin^2(x)dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{x}{2}$$

so our original integral becomes:

$$\int x \cos(x) \sin(x)dx = x \sin^2(x) + \frac{1}{2}\sin(x)\cos(x) - \frac{x}{2}.$$

and we're done.

For the following sequences, determine if the corresponding series converges.

$$1. a_n = \frac{\cos(\pi n)}{n^2 + 9n + 8}$$

The $\cos(\pi n)$ is just $(-1)^n$ in disguise, so this is an alternating series with $b_n = 1/(n^2 + 9n + 8)$. As $n \rightarrow \infty$, the denominator of b_n gets bigger and the numerator stays constant, so $\lim_{n \rightarrow \infty} b_n = 0$. You can also see that the b_n are decreasing by taking the derivative of $1/(x^2 + 9x + 8)$. But in this particular case, it's easier to just observe that since everything is positive:

$$(n+1)^2 + 9(n+1) + 8 > n^2 + 9n + 8$$

so $b_n > b_{n+1}$. Therefore Alternating Series Test applies and the series converges.

Incidentally, this series also absolutely converges, so that's an alternate way to go about the problem. We need to show $\sum_{n=1}^{\infty} b_n$ converges, but we can just do that with Comparison Test since $b_n \leq 1/n^2$, which is a p -series with $p > 1$ so it converges.

$$2. a_n = \frac{3n^4 + \sqrt{1+4n^2}}{6n^5\sqrt{n} - n^3 + n^2 + 10}$$

The dominant term in the numerator is n^4 , while the dominant term in the denominator is $n^5\sqrt{n}$, which tells us that the a_n act like $b_n = \frac{n^4}{n^5\sqrt{n}} = 1/(n^{3/2})$ as n gets big. So let's do a Limit Comparison Test between a_n and b_n . The limit you get is $\frac{1}{2} > 0$, so a_n and b_n both converge or both diverge. But b_n is a p -series with $p > 1$ so it converges and therefore $\sum_{n=1}^{\infty} a_n$ also converges.

$$3. a_n = \frac{(4n)!}{(2n+3)!}$$

The factorials indicate that Ratio Test would be useful.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{(4n+4)!(2n+3)!}{(2n+5)!(4n)!} = \\ \lim_{n \rightarrow \infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!(2n+3)!}{(2n+5)(2n+4)(2n+3)!(4n)!} &= \lim_{n \rightarrow \infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(2n+5)(2n+4)} \end{aligned}$$

So since the limit is ∞ , the Ratio Test says that $\sum_{n=1}^{\infty} a_n$ diverges.

$$4. a_n = \frac{2^n}{n^2 + n^n}$$

The dominant term in the numerator is 2^n while the dominant term in the denominator is n^n . So as n gets big, the a_n start acting like $b_n = (2/n)^n$. A Limit Comparison Test between a_n and b_n grounds our intuition:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n}{n^2 + n^n} \frac{n^n}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{n^2}{n^n} + 1}$$

and $\lim_{n \rightarrow \infty} n^2/n^n = \lim_{n \rightarrow \infty} 1/n^{n-2} = 0$. So

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 > 0$$

and so a_n and b_n both will converge or both will diverge.

The question now becomes whether $\sum_{n=3}^{\infty} b_n$ converges. Intuitively, if $n > 2$, then $\frac{2}{n} < 1$. If we then raise that number to the n th power, that will shrink it down to 0 very quickly. So $\sum_{n=3}^{\infty} b_n$ should probably converge. We can formalize this intuition with a Comparison Test: let $c_n = (2/n)^2 = 4/n^2$. For every $n \geq 3$ and every positive number $r < 1$, we have $r^n < r^2$. In particular, this inequality holds for $r = 2/n$ for any $n \geq 3$, so we have just argued that $b_n < c_n$ for all $n \geq 3$. But c_n is a constant times a p -series with $p > 1$, so $\sum_{n=3}^{\infty} c_n$ converges and therefore $\sum_{n=3}^{\infty} b_n$ converges by Comparison Test. Then Limit Comparison Test already told us that this implies $\sum_{n=3}^{\infty} a_n$ converges.

$$5. a_n = \frac{n}{(\ln(n^n))^2} \text{ for } n \geq 2.$$

The first step is to recall that \ln of a number to an exponent is equal to the exponent times the \ln of the number. So $\ln(n^n) = n \ln(n)$. That means that a_n simplify to:

$$a_n = \frac{1}{n(\ln(n))^2}.$$

Note that we've got an $\ln(n)$ and its derivative $1/n$, so an Integral Test might be in order. Set $f(x) = 1/[x(\ln(x))^2]$. Then f is continuous, positive since $\ln(x)$ is positive for $x > 1$ and decreasing since we are dividing by the product of three increasing functions, x , $\ln(x)$ and $\ln(x)$. So Integral Test can validly be performed. We will substitute $u = \ln(x)$ during this integration:

$$\begin{aligned} \int_2^{\infty} f(x) dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{dx}{x(\ln(x))^2} = \lim_{t \rightarrow \infty} \int_{\ln(2)}^{\ln(t)} u^{-2} du = \\ \lim_{t \rightarrow \infty} \left[-\frac{1}{\ln(x)} \right]_2^t &= \frac{1}{\ln(2)} \end{aligned}$$

This improper integral converges, so the series converges as well.

For the following series, determine whether the series converges, and if it does, give an upper bound U and a lower bound L for the sum, such that $U - L < \frac{3}{4}$.

1.

$$\sum_{n=1}^{\infty} \left(\frac{\arctan(n)}{\pi} \right)^n$$

For this series, we need to recall properties about $\arctan(n)$. First, it's bounded above by $\pi/2$. Secondly, it's positive on $[1, \infty)$ and increasing, so $\arctan(n) \geq \arctan(1) = \pi/4$ for all n . So this means that $(1/4)^n \leq a_n \leq (1/2)^n$ for all n . Since $(1/2)^n$ is a geometric series with $r = 1/2 < 1$, it converges and we get

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \left(\frac{1}{2} \right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

So $U = 1$ is an upper bound. But $a_n \geq (1/4)^n$ for all n so we also get

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \left(\frac{1}{4} \right)^n = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

and so our series has $L = \frac{1}{3}$ as a lower bound. But $U - L = \frac{2}{3} < \frac{3}{4}$ and so we're done.

2.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

This series is precisely the last series in the previous section, so we will not repeat certain computations. We used Integral Test to determine convergence, so we should use Remainder Form for the Integral Test to get our bounds. Suppose this series converges to a number S . Then the Remainder Formula says that for every $n \geq 2$, we have

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq s_n + \int_n^{\infty} f(x)dx$$

So these give us lower and upper bounds on S for every $n \geq 2$; set L_n to be the expression on the left-hand side, and U_n to be the expression on the right-hand side. Then we have

$$U_n - L_n = \left[s_n + \int_n^{\infty} f(x)dx \right] - \left[s_n + \int_{n+1}^{\infty} f(x)dx \right] = \int_n^{n+1} f(x)dx$$

Now we may use our previous computations about $\int f(x)dx$ to see that for this problem

$$U_n - L_n = \frac{1}{\ln(n)} - \frac{1}{\ln(n+1)}$$

and using a calculator, we would have to find some n such that $U_n - L_n < \frac{3}{4}$. It turns out that the first value, $n = 2$ already works since $U_2 - L_2$ is approximately 0.5. Therefore we can set $U = U_2$ and $L = L_2$ to solve the problem.