Calculus 1B Final Exam Review Sheet GSI: Paul Baginski

This problems on this review primarily concern the material from the final third of the course. For practice problems for the material from the first two-thirds of the course, see the two midterm review sheets.

Do not forget that quizzes for *both* sections are available on the webpage
http://math.berkeley.edu/∼baginski/Teaching.html
Each section gets rather different problems.

Integrate:

1. $\int \frac{\sin^2(\ln(x))}{x}\,dx$
2. $\int \frac{x^5}{x^3-x^2+x-1}\,dx$
3. $\int \sec^7(x)\tan^3(x)\,dx$
4. $\int \frac{x^5}{(x^2+1)^3}\,dx$
5. $\int xe^{\sqrt{x^2-4}}\,dx$
6. $\int xe^{\sqrt{x^2-\sqrt{12}\,x+3}}\,dx$
7. $\int \sqrt{4+\sqrt{16-x^2}}\,dx$
8. $\int \sin^2(3x)\cos(6x)\,dx$
9. $\int x^2\ln(x^2)\,dx$
10. $\int \frac{e^{2x}}{(e^x+1)(e^x+2)}\,dx$

1. We have a complicated function (sin) being applied to another complicated function (ln), so we should work to try to simplify the expression within the sin. So setting $u=\ln(x)$ gives $du=dx/x$,

so our integral becomes:

$$\begin{aligned}\int \frac{\sin^2(\ln(x))}{x}\,dx &= \int \sin^2(u)\,du\\ &= \int \frac{1}{2}(1-\cos(2u))du\\ &= \frac{u}{2}-\frac{\sin(2u)}{4}+C\\ &= \frac{\ln(x)}{2}-\frac{\sin(2\ln(x))}{4}+C\end{aligned}$$

2. This is a polynomial divided by a polynomial, so likely it will be partial fractions (unless some useful u-substitution is apparent). To apply partial fractions, we have to make the numerator have strictly small order than the denominator, so we have to long divide:

$$x^5 = (x^3-x^2+x-1)(x^2+x)+x$$

and our integral becomes:

$$\begin{aligned}\int \frac{x^5}{x^3-x^2+x-1}\,dx &= \int \frac{(x^3-x^2+x-1)(x^2+x)+x}{x^3-x^2+x-1}\,dx\\ &= \int (x^2+x)\,dx + \int \frac{x}{x^3-x^2+x-1}\,dx\\ &= \frac{x^3}{3}+\frac{x^2}{2}+\int \frac{x}{x^3-x^2+x-1}\,dx\end{aligned}$$

Now in order to proceed by partial fractions, we have to factor x^3-x^2+x-1. Since every polynomial of degree greater than 2 can be split into smaller degree polynomials, this degree 3 polynomial must factor into at least two factors. One of those factors is degree 1, so that means that x^3-x^2+x-1 has a root. A fair guess gives that $x=1$ is a root, and so x^3-x^2+x-1 factors as $(x-1)(x^2+1)$ and since x^2+1 has no real roots, this is the factorization we need for partial fractions. Therefore we need to rewrite

$$\begin{aligned}\frac{x}{x^3-x^2+x-1} &= \frac{A}{x-1}+\frac{Bx+C}{x^2+1}\\ &= \frac{Ax^2+A+Bx^2-Bx+Cx-C}{(x-1)(x^2+1)}\end{aligned}$$

So $A+B=0$ and $C-B=1$ and $A-C=0$. These three equations yield that $B=-\frac{1}{2}$ and $A=C=\frac{1}{2}$. Therefore

$$\begin{aligned}\int \frac{x}{x^3-x^2+x-1}\,dx &= \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{-\frac{x}{2}+\frac{1}{2}}{x^2+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2+1}\\ &= \frac{1}{2}\ln|x-1| - \frac{\ln(x^2+1)}{4} + \frac{1}{2}\arctan(x)+C\end{aligned}$$

so that means our original integral was:

$$\int \frac{x^5}{x^3-x^2+x-1}\,dx = \frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{2}\ln|x-1|-\frac{\ln(x^2+1)}{4}+\frac{1}{2}\arctan(x)+C$$

3. This one is a standard trig identity integral. Set $u = \sec(x)$, then $du = \sec(x)\tan(x)dx$ and therefore:

$$\begin{aligned}\int \sec^7(x)\tan^3(x)\,dx &= \int u^6 \tan^2(x)du \\ &= \int u^6(sec^2(x)-1)du \\ &= \int u^6(u^2-1)du \\ &= \frac{u^9}{9} - \frac{u^7}{7} + C \\ &= \frac{\sec^9(x)}{9} - \frac{\sec^7(x)}{7} + C\end{aligned}$$

4. This integral, like #2 also suggests using partial fractions. But this time, we do not have to long divide, since the numerator has order 5, which is strictly less than 6, the order of the denominator. Since we have repeated factors in the denominator, we must rewrite

$$\begin{aligned}\frac{x^5}{(x^2+1)^3} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} \\ &= \frac{Ax^5+2Ax^3+Ax+Bx^4+2Bx^2+B+Cx^3+Cx+Dx^2+D+Ex+F}{(x^2+1)^3}\end{aligned}$$

The resulting equations are: $1 = A$; $0 = B$; $0 = 2A+C$; $0 = 2B+D$; $0 = A+C+E$; $0 = B+D+F$. These simplify to $A = E = 1$, $B = D = F = 0$, and $C = -2$. This gives:

$$\begin{aligned}\int \frac{x^5}{(x^2+1)^3}\,dx &= \int \left(\frac{x}{x^2+1} + \frac{-2x}{(x^2+1)^2} + \frac{x}{(x^2+1)^3}\right)dx \\ &= \frac{1}{2}\ln(x^2+1) + \frac{1}{x^2+1} - \frac{1}{4}\frac{1}{(x^2+1)^2} + C\end{aligned}$$

5. For this integral, we have e being raised to a very nasty exponent, so our first goal should be trying to make this exponent have a nicer form, perhaps at the cost of making the rest of the integral uglier. One could try substituting $u = x^2 - 4$, but then you get an $e^{\sqrt{u}}$ inside the integral, and that is still ugly. So we have to deal with the square root the only other way we know how:

trig substitution. Set $x = 2\sec(u)$. Then $dx = 2\sec(u)\tan(u)du$ and so our integral becomes:

$$\begin{aligned}
\int xe^{\sqrt{x^2-4}}\,dx &= \int 2\sec(u)e^{\sqrt{4\sec^2(u)-4}}(2\sec(u)\tan(u)du) \\
&= 4\int e^{2\tan(u)}\tan(u)\sec^2(u)du \\
(\text{set } v = 2\tan(u)) &= \int ve^v dv \\
(\text{by parts, } f = v,\ g' = e^v dv) &= ve^v - \int e^v dv \\
&= ve^v - e^v + C \\
&= 2\tan(u)e^{2\tan(u)} - e^{2\tan(u)} + C \\
&= \sqrt{x^2-4}\,e^{\sqrt{x^2-4}} - e^{\sqrt{x^2-4}} + C
\end{aligned}$$

6. For this integral, we also have an ugly exponent. If we were to do a trig substitution, we would need to complete the square. In doing so, we'd discover that $x^2 - \sqrt{12}\,x + 3$ is a perfect square: $(x-\sqrt{3})^2$. So actually, we have:

$$\begin{aligned}
\int xe^{\sqrt{x^2-\sqrt{12}\,x+3}}\,dx &= \int xe^{x-\sqrt{3}}\,dx \\
&= \frac{1}{e^{\sqrt{3}}}\int xe^x\,dx \\
(\text{same parts as previous problem}) &= \frac{1}{e^{\sqrt{3}}}(xe^x - e^x)
\end{aligned}$$

7. With all the square roots, the only thing that stands a chance is a trig substitution. Set $x = 4\sin(u)$ and $dx = 4\cos(u)du$. Then

$$\begin{aligned}
\int\sqrt{4+\sqrt{16-x^2}}\,dx &= \int\sqrt{4+\sqrt{16-16\sin^2(u)}}\,4\cos(u)du \\
&= 8\int\sqrt{1+\cos(u)}\cos(u)du \\
&= 8\sqrt{2}\int\sqrt{\frac{1}{2}(1+\cos(u))}\cos(u)du \\
&= 8\sqrt{2}\int\sqrt{\cos^2\left(\frac{u}{2}\right)}\cos(u)du \\
&= 8\sqrt{2}\int\cos\left(\frac{u}{2}\right)\cos(u)du \\
&= 8\sqrt{2}\int\cos\left(\frac{u}{2}\right)\left(1-2\sin^2\left(\frac{u}{2}\right)\right)du \\
&= 16\sqrt{2}\sin\left(\frac{u}{2}\right) - 32\sqrt{2}\sin^3\left(\frac{u}{2}\right) + C
\end{aligned}$$

The integral is now solved in terms of u, but we still have to replace things back in terms of x to be finished, so just replace $u = \arcsin(\frac{x}{4})$ in the final line of our solution.

8. For this integral, we need to get all the arguments for the trig functions to be the same. I.e., we need trig functions of $3x$ or of $6x$, but not both. Let's switch down to $6x$. Then

$$\begin{aligned}\int \sin^2(3x)\cos(6x)\,dx &= \frac{1}{2}\int (1+\cos(6x))\cos(6x)\,dx \\ &= -\frac{1}{12}\sin(6x) + \frac{1}{2}\int \cos^2(6x)dx \\ &= -\frac{1}{12}\sin(6x) + \frac{1}{4}\int 1+\cos(12x)dx \\ &= -\frac{1}{12}\sin(6x) + \frac{x}{4} + \frac{x}{2}\sin(12x)dx\end{aligned}$$

9. Recall that $\ln(x^2) = 2\ln(x)$, so we have a very natural candidate integral for integration by parts. Set $f = \ln(x)$ and $g' = x^2dx$. Then

$$\begin{aligned}\int x^2\ln(x^2)\,dx &= 2\frac{x^3\ln(x)}{3} - 2\int \frac{x^3}{3}\frac{1}{x}dx \\ &= 2\frac{x^3\ln(x)}{3} - \frac{2}{9}x^3 + C\end{aligned}$$

10. This last integral has the shape of a typical partial fractions integral, except instead of x we have e^x. This is especially apparent when we note that $e^{2x} = e^xe^x$ in the numerator. So let's first substitute $u = e^x$ (so $du = e^xdx$). Then

$$\int \frac{e^{2x}}{(e^x+1)(e^x+2)}\,dx = \int \frac{u}{(u+1)(u+2)}\,du$$

and now we do partial fractions.

$$\begin{aligned}\frac{u}{(u+1)(u+2)} &= \frac{A}{u+1} + \frac{B}{u+2} \\ &= \frac{Au+2A+Bu+B}{(u+1)(u+2)}\end{aligned}$$

and our equations are $A + B = 1$ and $2A + B = 0$. So $A = -1$ and $B = 2$. Hence

$$\begin{aligned}\int \frac{e^{2x}}{(e^x+1)(e^x+2)}\,dx &= \int \frac{u}{(u+1)(u+2)}\,du \\ &= \int \frac{-1}{u+1}\,du + \int \frac{2}{u+2}\,du \\ &= -\ln(u+1) + 2\ln(u+2) + C \\ &= 2\ln(e^x+2) - \ln(e^x+1) + C\end{aligned}$$

For the following functions $f(x)$, perform these tasks:

(1) Write the Maclaurin series.
(2) Determine the 6th derivative of the function at $x = 0$.
(3) Determine the interval of convergence for the Maclaurin series.

The functions:

1. $f(x) = \ln(x^4 + 1)$
2. $\sqrt{4 - x}$
3. $\dfrac{\sin(\sqrt{x})}{\sqrt{x}}$
4. $e^{-x} - \cos(x)$

1. If we start taking larger and larger derivatives of $\ln(x^4 + 1)$, then they very quickly become complicated because of product rules and quotient rules. So there has to be a slicker way to solve it. In fact, there are at least two.

Solution 1:. $\ln(x^4 + 1)$ is complicated, but maybe we could compute the Maclaurin series of something simpler and then substitute. For example, if we could compute the Maclaurin series of $\ln(x + 1)$, then we'd just have to replace x with x^4 in the series to get the Maclaurin series of $\ln(x^4 + 1)$. It turns out that computing the derivatives of $\ln(x + 1)$ is much easier:

$$
\begin{aligned}
f(x) &= \ln(x + 1) & f(0) &= 0 \\
f'(x) &= \frac{1}{x + 1} & f'(0) &= 1 \\
f''(x) &= \frac{-1}{(x + 1)^2} & f''(0) &= -1 \\
f^{(3)}(x) &= \frac{2}{(x + 1)^3} & f^{(3)}(0) &= 2 \\
f^{(4)}(x) &= \frac{-2 \cdot 3}{(x + 1)^4} & f^{(4)}(0) &= -2 \cdot 3 \\
f^{(n)}(x) &= \frac{(-1)^{n-1}(n - 1)!}{(x + 1)^n} & f^{(n)}(0) &= (-1)^{n-1}(n - 1)!
\end{aligned}
$$

So this gives us the Maclaurin series we want:

$$
\begin{aligned}
\ln(x + 1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n - 1)!}{n!} x^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n
\end{aligned}
$$

$$\ln(x^4+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x^4)^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{4n}$$

Recall that Maclaurin series have the general form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

So $f^{(6)}(0)$ appears in the coefficient of x^6 in the Maclaurin series. Examining the Maclaurin series for $\ln(x^4+1)$, we need to find an n such that $x^{4n} = x^6$. There is no such n, meaning that the coefficient of x^6 in the Maclaurin series is 0. Therefore $0 = f^{(6)}(0)/6!$ so $f^{(6)}(0) = 0$.

Lastly, to determine the interval of convergence, we must use Ratio test to get the radius of convergence and then test the endpoints. Ratio test is:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|\frac{(-1)^n}{n+1} x^{4(n+1)}|}{|\frac{(-1)^{n-1}}{n} x^{4n}|} = \lim_{n\to\infty} \frac{n}{n+1}|x|^4 = |x|^4$$

So, in order to converge, we need $|x|^4 < 1$, so $|x| < 1$. Now we must test the endpoints: $x = \pm 1$. For $x = 1$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which is an alternating harmonic and converges by alternating series test. For $x = -1$, we also get the alternating harmonic, so the interval of convergence is $[-1, 1]$.

Solution 2: We can also use the technique we used to find the Maclaurin series of $\arctan(x)$: find the Maclaurin series of the derivative of $\arctan(x)$ and then integrate it. This technique is promising because the derivative of $\ln(x^4+1)$ is $4x^3/(1+x^4)$, which has the form of a geometric series. Let's work it out:

$$\frac{4x^3}{1+x^4} = 4x^3 \frac{1}{1-(-x^4)}$$
$$= 4x^3 \left(\sum_{n=0}^{\infty} (-x^4)^n\right)$$
$$= 4x^3 \left(\sum_{n=0}^{\infty} (-1)^n x^{4n}\right)$$
$$= 4\sum_{n=0}^{\infty} (-1)^n x^{4n+3}$$

and now we integrate to get

$$\begin{aligned}
\ln(x^4+1) &= \int \left(4\sum_{n=0}^{\infty}(-1)^n x^{4n+3}\right)dx \\
&= 4\sum_{n=0}^{\infty}\int (-1)^n x^{4n+3}dx \\
&= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+4}x^{4n+4} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{4n+4} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{4n}
\end{aligned}$$

The nice thing about this method is that we get the radius of convergence automatically. The radius is the same as the radius for $4x3/(1+x^4)$, which is the same as for $1/(1+x^4)$. Since this is a geometric series, it converges when $|-x^4|<1$, so when $|x|<1$. We then compute the endpoints $x=\pm 1$ as in Solution 1, and finding $f^{(6)}(0)$ is the same as in Solution 1 as well.

2. The only way we know of dealing with a square root is with Binomial Formula, so we must get it into that form.

$$\begin{aligned}
\sqrt{4-x} &= 2\left(1+\left(\frac{-x}{4}\right)\right)^{\frac{1}{2}} \\
&= 2\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{-x}{4}\right)^n \\
&= \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{x^n}{2^{2n-1}} \\
&= \sum_{n=0}^{\infty}\left(\frac{(-1)^{n-1}}{2^n}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\right)(-1)^n\frac{x^n}{2^{2n-1}} \\
&= -\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{x^n}{2^{3n-1}}
\end{aligned}$$

where these last steps were done by computing the binomial coefficients

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}}{2^n}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}$$

The coefficient of x^6 in the Maclaurin expansion is generally $f^{(6)}(0)/6!$, and specifically here, it is:

$$\frac{f^{(6)}(0)}{6!} = -\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{6!}\frac{1}{2^{17}}$$

So

$$f^{(6)}(0) = -\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^{17}}$$

The radius of convergence on a Binomial series $(1+x)^k$ is $|x|<1$, so for $(1+(-x/4))^{1/2}$ we need $|-x/4|<1$ so $|x|<4$. So our Maclaurin series converges on $(-4,4)$ and now we must test the endpoints ± 4. For Binomial series, this is generally rather difficult (as explained in the book), so on an exam you'd only be expected to determine the radius of convergence. As an example of the argument involved, when we take $x = -4$, we get

$$-\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{(-4)^n}{2^{3n-1}} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{1}{2^{n-1}}$$

This is an alternating series, so we want to use alternating series test. First, we check whether the terms are decreasing:

$$\frac{1\cdot 3\cdot 5\cdots(2(n+1)-3)}{(n+1)!}\frac{1}{2^{(n+1)-1}} \overset{?}{<} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{1}{2^{n-1}}$$

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(n+1)!}\frac{1}{2^{n}} \overset{?}{<} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{1}{2^{n-1}}$$

$$\frac{2n-1}{(n+1)2} \overset{\checkmark}{<} 1$$

so they are decreasing. Next we have to check that the limit is zero. For this we first note that:

$$\begin{aligned}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\frac{1}{2^{n-1}} &< \frac{2\cdot 4\cdot 6\cdots(2n-2)}{n!}\frac{1}{2^{n-1}}\\ &= \frac{2^{n-1}(n-1)!}{n!}\frac{1}{2^{n-1}}\\ &= \frac{1}{n}\end{aligned}$$

So since $\lim_{n\to\infty} 1/n = 0$, the terms in our series go to zero. Therefore, by alternating series test, the Maclaurin series converges at $x = -4$. The argument for $x = 4$ is even more complicated, so I'm omitting it.

3. We know the Maclaurin series for $\sin(x)$, so let's try replacing x with $\sqrt{x}$ in it and see what happens.

$$\begin{aligned}\sin(\sqrt{x}) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\sqrt{x})^{2n+1}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sqrt{x}\,x^n\end{aligned}$$

This is a power series, but not yet a Maclaurin series since we have that $\sqrt{x}$ floating around. But when we divide it out, we get a legitimate Maclaurin series, so

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,x^n$$

The coefficient of x^6 in a Maclaurin series is $f^{(6)}(0)/6!$. The coefficient of x^6 in this Maclaurin series is $(-1)^6/13!$. Therefore $f^{(6)}(0) = 6!/13!$.

To determine the radius of convergence, we use the Ratio Test.

$$\lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}}{(2(n+1)+1)!}x^{n+1}\right|}{\left|\frac{(-1)^n}{(2n+1)!}\sqrt{x}\,x^n\right|} = \lim_{n\to\infty}\frac{|x|}{(2n+3)(2n+2)} = 0$$

so the series converges everywhere.

4. The Maclaurin series for e^{-x} is

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

and the Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Therefore, to find the Maclaurin series for $e^{-x} - \cos(x)$, we simply must subtract these two series. This kills the powers of x which are multiples of 4.

$$\begin{aligned}
e^{-x} - \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
&= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \\
&= -x + \frac{2x^2}{2!} - \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{2x^6}{6!} - \frac{x^7}{7!} - \dots \\
&= \sum_{n=0}^{\infty} c_n x^n
\end{aligned}$$

where

$$c_n = \begin{cases} \frac{-1}{n!} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ is multiple of 4} \\ \frac{2}{n!} & \text{if } n \text{ is even, not multiple of 4} \end{cases}$$

For a general Maclaurin series, the coefficient of x^6 is $f^{(6)}(0)/6!$, so here that is $c_6 = 2/6!$. Therefore $f^{(6)}(0) = 2$. As for the interval of convergence, we know e^{-x} converges everywhere since e^x does, and $-\cos(x)$ also converges everywhere. So for every x, we are adding two convergent series, so the sum is also convergent. Therefore the Maclaurin series for $e^{-x} - \cos(x)$ converges everywhere.

Find the general solution to the following differential equations, or if initial values are given, solve the initial value problem.

1. $y'' - 2y' - 3y = e^{3x+2}$
2. $y'' - 9y' - 10 = (x^2 + 3)e^{-x}$
3. $y'' + 9y = \sin^4(3x)$
4. $y' = x^2 y + \frac{y}{x} + 5$
5. $y'' + xy = 0$

1. We have constants in front of the y'', y' and y, so a method like undetermined coefficients

or variation of parameters should work. At first glance, you may think undetermined coefficients is inapplicable because we don't have e^{ax} for a constant a, but $e^{3x+2} = e^2 e^{3x}$, a constant times an suitable function. So let's use undetermined coefficients.

First we find the general solution y_c to $y'' - 2y' - 3y = 0$. This has characteristic equation $r^2 - 2r - 3 = 0$, which has roots $r = 3, -1$. So

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Now we suppose $y_p = Ae^{3x}$. But this is no good since it lies in y_c, so we multiply by x: $y_p = Axe^{3x}$. Therefore $y_p' = Ae^{3x} + 3xAe^{3x}$ and $y_p'' = 9Axe^{3x} + 6Ae^{3x}$. Therefore

$$\begin{aligned} e^2 e^{3x} &= y_p'' - 2y_p' - 3y_p \\ &= (9Ax + 6A)e^{3x} - (2A + 6Ax)e^{3x} - 3Axe^{3x} \end{aligned}$$

Divide out by e^{3x}, and we get two equations: $0 = 9A - 6A - 3A$ and $e^2 = 6A - 2A$. So $A = e^2/4$ and $y_p = xe^{3x+2}/4$. Therefore our general solution is:

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} + \frac{xe^{3x+2}}{4}$$

2. This equation also has a natural form for undetermined coefficients. First we solve the homogeneous equation $y'' - 9y' - 10y = 0$. This has characteristic equation $r^2 - 9r - 10 = 0$ which has roots $r = 10, -1$. So

$$y_c = c_1 e^{10x} + c_2 e^{-x}$$

Now we guess that our particular solution is $y_p = (Ax^2 + Bx + C)e^{-x}$. This has Ce^{-x} in common with y_c, so we must multiply by x. Therefore $y_p = (Ax^3 + Bx^2 + Cx)e^{-x}$ is our guess. Then

$$\begin{aligned} y_p' &= (3Ax^2 + 2Bx + C)e^{-x} - (Ax^3 + Bx^2 + Cx)e^{-x} \\ y_p'' &= (6Ax + 2B)e^{-x} - (3Ax^2 + 2Bx + C)e^{-x} - (3Ax^2 + 2Bx + C)e^{-x} + (Ax^3 + Bx^2 + Cx)e^{-x} \end{aligned}$$

So plugging back in, we get:

$$\begin{aligned} (x^2 + 3)e^{-x} &= y_p'' - 9y_p' - 10y_p \\ \\ &= (-36Ax^2 + (6A - 22B)x + 2B - 21C)e^{-x} \end{aligned}$$

So $1 = -36A$ and $0 = 6A - 22B$ and $3 = 2B - 21C$. The solution is: $A = -1/36$, $B = -1/132$, and $C = -196/(66 \cdot 21)$. Therefore

$$y = y_c + y_p = c_1 e^{10x} + c_2 e^{-x} - (\frac{1}{36}x^3 + \frac{1}{132}x^2 + \frac{196}{66 \cdot 21}x)e^{-x}$$

3. For this differential equation, we have constant coefficients on all the y, y' and y'', so either undetermined coefficients or variation of parameters is needed. But $\sin^4$ of something is not a function that works with undetermined coefficients, so we are forced to use variation of parameters.

The characteristic equation is $r^2 + 9 = 0$, so $r = \pm 3i$ and

$$y_c = c_1 \cos(3x) + c_2 \sin(3x)$$

so $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$. Now we want $y_p = u_1y_1 + u_2y_2$ and we need to solve our equations $u_1'y_1 + u_2'y_2 = 0$ and $a(u_1'y_1' + u_2'y_2') = G(x)$. These become:

$$
\begin{aligned}
u_1' \cos(3x) + u_2' \sin(3x) &= 0 \\
-3u_1' \sin(3x) + 3u_2' \cos(3x) &= \sin^4(3x)
\end{aligned}
$$

The first equation gives us $u_2' = -u_1' \cos(3x)/\sin(3x)$. So that gives:

$$
\begin{aligned}
\sin^4(3x) &= -3u_1' \sin(3x) + 3\left(\frac{-u_1' \cos(3x)}{\sin(3x)}\right)\cos(3x) \\
&= -3u_1' \sin(3x) - 3u_1' \frac{\cos^2(3x)}{\sin(3x)} \\
&= -3u_1' \sin(3x) - 3u_1' \frac{1 - \sin^2(3x)}{\sin(3x)} \\
&= -3u_1' \frac{1}{\sin(3x)}
\end{aligned}
$$

So

$$
\begin{aligned}
u_1' &= -\frac{1}{3}\sin^5(3x) \\
u_1 &= -\frac{1}{3}\int \sin^5(3x)\, dx \\
(\text{ using trig identities}) &= \frac{1}{9}\left(\cos(3x) - \frac{2}{3}\cos^3(3x) + \frac{1}{5}\cos^5(3x)\right)
\end{aligned}
$$

Now we solve for u_2'. Since $u_1' = -\frac{1}{3}\sin^5(3x)$and $u_2' = -u_1' \cos(3x)/\sin(3x)$, we get

$$
\begin{aligned}
u_2' &= \frac{1}{3}\sin^4(3x)\cos(3x) \\
u_2 &= \frac{1}{3}\int \sin^4(3x)\cos(3x)dx \\
&= \frac{\sin^5(3x)}{45}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_p &= u_1 \cos(3x) + u_2 \sin(3x) \\
&= \frac{1}{9}\left(\cos^2(3x) - \frac{2}{3}\cos^4(3x) + \frac{1}{5}\cos^6(3x)\right) + \frac{\sin^6(3x)}{45}
\end{aligned}
$$

and

$$
y = y_c + y_p \;=\; c_1\cos(3x) + c_2\sin(3x) + \frac{1}{9}\left(\cos^2(3x) - \frac{2}{3}\cos^4(3x) + \frac{1}{5}\cos^6(3x)\right) + \frac{\sin^6(3x)}{45}
$$

4. This is a general linear differential equation, so we must use the formula for it. Rearranging gives $y' - (x^2 + \frac{1}{x})y = 5$, so $P(x) = -x^2 - \frac{1}{x}$ and $Q(x) = 5$. Then

$$\begin{aligned} I(x) &= e^{\int P(x)dx} \\ &= e^{-\frac{x^3}{3} - \ln(x)} \\ &= \frac{1}{xe^{\frac{x^3}{3}}} \end{aligned}$$

So our general solution is:

$$\begin{aligned} y &= \frac{y(0)}{I(x)} + \frac{1}{I(x)} \int I(x)Q(x)dx \\ &= y(0)xe^{\frac{x^3}{3}} + xe^{\frac{x^3}{3}} \int \frac{5}{xe^{\frac{x^3}{3}}}\, dx \end{aligned}$$

5. Since we don't have a constant coefficient in front of y, we must use the series solutions method. So we assume $y = \sum_{n=0}^{\infty} c_n x^n$. So $y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$. Therefore our differential equation becomes

$$\begin{aligned} 0 &= y'' + xy \\ &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_2 + \sum_{n=1}^{\infty} (c_{n+2}(n+2)(n+1) + c_{n-1})x^n \end{aligned}$$

So we obtain the equations:

$$\begin{aligned} 0 &= 2c_2 && \text{from } x^0 \\ 0 &= c_{n+2}(n+2)(n+1) + c_{n-1} && \text{from } x^n \text{ for } n \geq 1 \end{aligned}$$

The first equation yields $c_2 = 0$ and we always have $c_0 = y(0)$ and $c_1 = y'(0)$. The second equation gives us our recursion: for $n \geq 1$ we have $c_{n+2} = -c_{n-1}/(n+2)(n+1)$. Reindexing, this is $c_n = -c_{n-3}/n(n-1)$ for $n \geq 3$. This equation says indices drop by 3 each time, so this gives us three cases. If n is a multiple of 3 (i.e. $n = 3k$) then after multiple applications for the recursive formula, we get down to c_0. If $n = 3k+1$, then after repeated application we get down to c_1. And lastly if $n = 3k+2$, then we get down to c_2. For example:

$$\begin{aligned} c_{3k+1} &= -\frac{c_{3(k-1)+1}}{(3k+1)(3k)} \\ &= -\frac{1}{(3k+1)(3k)} \frac{-c_{3(k-2)+1}}{(3(k-1)+1)(3(k-1))} \\ &\vdots \\ &= \frac{-1}{(3k+1)(3k)} \frac{-1}{(3(k-1)+1)(3(k-1))} \frac{-1}{(3(k-2)+1)(3(k-2))} \cdots \frac{-c_{3(k-k)+1}}{(3(k-(k-1))+1)(3(k-(k-1))} \\ &= \frac{(-1)^k}{(3k+1)(3k)(3k-2)(3k-3)\cdots 4 \cdot 3}\, c_1 \end{aligned}$$

Using this method we get the above formula for c_{3k+1}, and we can expand recursively to also find that

$$\begin{aligned}
c_{3k} &= \frac{(-1)^k}{(3k)(3k-1)(3k-3)(3k-4)\cdots 3\cdot 2}\, c_0 \\
c_{3k+2} &= \frac{(-1)^k}{(3k+2)(3k+1)(3k-1)(3k-2)\cdots 5\cdot 4}\, c_2 \\
&= 0
\end{aligned}$$

Find the arc lengths.

1. $y = \ln\left(\frac{e^x+1}{e^x-1}\right),\ \ln(2) \leq x \leq \ln(3)$

Let's first rewrite $y = \ln(e^x+1) - \ln(e^x-1)$ to make it easier to compute the derivative. The formula for arc length gives us:

$$\begin{aligned}
L &= \int_{\ln(2)}^{\ln(3)} \sqrt{1+(y')^2}\, dx \\
&= \int_{\ln(2)}^{\ln(3)} \sqrt{1+\left(\frac{e^x}{e^x+1} - \frac{e^x}{e^x-1}\right)^2}\, dx \\
&= \int_{\ln(2)}^{\ln(3)} \sqrt{1+\left(\frac{-2e^x}{e^{2x}-1}\right)^2}\, dx \\
&= \int_{\ln(2)}^{\ln(3)} \sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}}\, dx \\
&= \int_{\ln(2)}^{\ln(3)} \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx \\
&= \int_{\ln(2)}^{\ln(3)} \frac{e^{2x}+1}{e^{2x}-1}\, dx \\
&= \frac{1}{2}\int_{\ln(2)}^{\ln(3)} \frac{2e^{2x}}{e^{2x}-1}\, dx + \int_{\ln(2)}^{\ln(3)} \frac{1}{e^{2x}-1}\, dx \\
&= \frac{1}{2}\ln(e^{2x}-1)\Big|_{\ln(2)}^{\ln(3)} + \int_{\ln(2)}^{\ln(3)} \frac{1}{(e^x+1)(e^x-1)}\, dx \\
&= \frac{1}{2}(\ln(8)-\ln(3)) + \int_{\ln(2)}^{\ln(3)} \frac{e^x}{e^x(e^x+1)(e^x-1)}\, dx
\end{aligned}$$

To deal with that remaining integral, we will set $u = e^x + 1$, then $du = e^x dx$. We won't change the bounds of integration, since we will revert back to x when we are done integrating.

$$\begin{aligned}
L &= \frac{1}{2}(\ln(8) - \ln(3)) + \int_{\ln(2)}^{\ln(3)} \frac{e^x}{e^x(e^x+1)(e^x-1)}\, dx \\
&= \frac{1}{2}(\ln(8) - \ln(3)) + \int_{-}^{-} \frac{1}{(u-1)u(u-2)}\, du \\
&= \frac{1}{2}(\ln(8) - \ln(3)) + \int_{-}^{-} \left[\frac{\frac{1}{2}}{u} + \frac{-1}{u-1} + \frac{\frac{1}{2}}{u-2}\right] du \\
&= \frac{1}{2}(\ln(8) - \ln(3)) + \left[\frac{1}{2}\ln(u) - \ln(u-1) + \frac{1}{2}\ln(u-2)\right]\Bigg|_{-}^{-} \\
&= \frac{1}{2}(\ln(8) - \ln(3)) + \left[\frac{1}{2}\ln(e^x+1) - \ln(e^x) + \frac{1}{2}\ln(e^x-1)\right]\Bigg|_{\ln(2)}^{\ln(3)} \\
&= \frac{1}{2}(\ln(8) - \ln(3)) + \frac{1}{2}\ln(4) - \frac{3}{2}\ln(3) + \frac{3}{2}\ln(2) \\
&= \frac{1}{2}\ln\left(\frac{256}{81}\right) \\
&= \ln\left(\frac{16}{9}\right)
\end{aligned}$$

2. The perimeter of the figure inscribed by the two functions: $f(x) = x^2 - 9$ and $g(x) = 9 - x^2$.

These two functions meet at the x-axis at $x = \pm 3$. So the perimeter we seek is $L_1 + L_2$, where L_1 is the arc length of $x^2 - 9$ from $x = -3$ to $x = 3$, and L_2 is the arc length of $9 - x^2$ from $x = -3$ to $x = 3$. We can save ourselves some trouble by observing symmetry to the problem: $L_1 = L_2$ because the arcs are mirror images of each other. Therefore:

$$\begin{aligned}
P &= 2L_1 \\
&= 2\int_{-3}^{3} \sqrt{1 + ((x^2+9)')^2}\, dx \\
&= 2\int_{-3}^{3} \sqrt{1 + 4x^2}\, dx \\
&= 2\int_{-}^{-} \sqrt{1 + 4\left(\frac{\tan(u)}{2}\right)^2}\, \frac{1}{2}\sec^2(u) du \\
&= \int_{-}^{-} \sec(u)\sec^2(u) du \\
&= \left[\frac{1}{2}(\sec(u)\tan(u) + \ln(|\sec(u) + \tan(u)|)\right]\Bigg|_{-}^{-} \\
&= \left[\frac{1}{2}(\sqrt{1+4x^2}\,(2x) + \ln(|\sqrt{1+4x^2} + 2x|)\right]\Bigg|_{-3}^{3} \\
&= 6\sqrt{37} + \ln(\sqrt{37} + 6) - \ln(\sqrt{37} - 6)
\end{aligned}$$

Find the surface area of the figure resulting from rotating $y = x^2$ around the x-axis, allowing $1 \leq x \leq 3$.

The formula for surface area gives:

$$\begin{aligned} S &= \int_1^3 2\pi(x^2)\sqrt{1 + ((x^2)')^2}\ dx \\ &= 2\pi \int_1^3 x^2\sqrt{1+4x^2}\ dx \\ &= 2\pi \int_-^- \left(\frac{\tan(u)}{2}\right)^2 \sqrt{1+\tan^2(u)}\ \frac{1}{2}\sec^2(u) du \\ &= \frac{\pi}{2}\int_-^- \tan^2(u)\sec^3(u)\ du \end{aligned}$$

To figure out $\int \tan^2(u)\sec^3(u)\ du$, we integrate by parts: set $f = \sec(u), g' = \tan^2(u)\sec^2(u)du$. Then

$$\begin{aligned} \int \tan^2(u)\sec^3(u)\ du &= \frac{\sec(u)\tan^3(u)}{3} - \frac{1}{3}\int \sec(u)\tan^4(u)\ du \\ &= \frac{\sec(u)\tan^3(u)}{3} - \frac{1}{3}\int \sec(u)(\sec^2(u)-1)\tan^2(u)\ du \\ &= \frac{\sec(u)\tan^3(u)}{3} - \frac{1}{3}\int \sec(u)(\sec^2(u)-1)\tan^2(u)\ du \\ &= \frac{\sec(u)\tan^3(u)}{3} - \frac{1}{3}\int \sec^3(u)\tan^2(u)\ du + \frac{1}{3}\int \sec(u)\tan^2(u)\ du \end{aligned}$$

So

$$\begin{aligned} \int \tan^2(u)\sec^3(u)\ du &= \frac{3}{4}\left[\frac{\sec(u)\tan^3(u)}{3} + \frac{1}{3}\int \sec(u)\tan^2(u)\ du\right] \\ &= \frac{3}{4}\left[\frac{\sec(u)\tan^3(u)}{3} + \frac{1}{3}\left[-\ln(\sec(u)+\tan(u)) + \frac{1}{2}(\sec(u)\tan(u) + \ln(\sec(u)+\tan(u)))\right]\right] \\ &= \frac{\sec(u)\tan^3(u)}{4} + \frac{1}{8}\left[\sec(u)\tan(u) - \ln(\sec(u)+\tan(u))\right] \end{aligned}$$

And we have:

$$\begin{aligned} S &= \frac{\pi}{2}\left[\frac{\sec(u)\tan^3(u)}{4} + \frac{1}{8}\left[\sec(u)\tan(u) - \ln(\sec(u)+\tan(u))\right]\right]\Bigg|_-^- \\ &= \frac{\pi}{2}\left[\frac{\sqrt{1+4x^2}\ 8x^3}{4} + \frac{1}{8}\left[2x\sqrt{1+4x^2} - \ln(\sqrt{1+4x^2}+2x)\right]\right]\Bigg|_1^3 \\ &= \frac{217\pi\sqrt{37}}{8} - \frac{\pi}{16}\ln(\sqrt{37}+6) - \frac{11\pi\sqrt{5}}{8} - \frac{\pi}{16}\ln(\sqrt{5}+2) \end{aligned}$$