

Answers to Hutchings' 03 Midterm 2

Note that you will be expected to provide more detail in your solutions than what is presented below.

1. The only critical point of the function is at $(1, 1)$, which is outside the region $x^2 + y^2 \leq 1$. Thus, we just have to check on the boundary $x^2 + y^2 = 1$. Using the method of Lagrange multipliers, the equation $\nabla f = \lambda \nabla g$ gives that $x = y$. From the constraint equation $x^2 + y^2 = 1$, we get $x = y = \pm \frac{1}{\sqrt{2}}$. These two points give function values of:

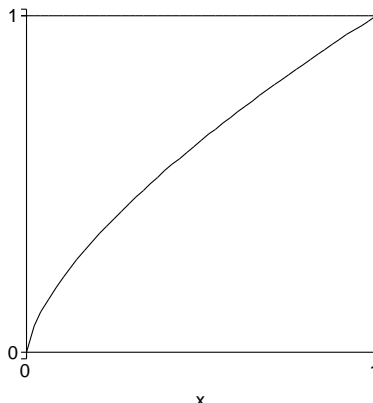
$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (\sqrt{2} + 1)^2$$

and

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)^2$$

Thus, the maximum of the function on the region $x^2 + y^2 \leq 1$ is $(\sqrt{2} + 1)^2$ and occurs at $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, and the minimum of the function on the region is $(\sqrt{2} - 1)^2$ and occurs at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

2. To evaluate this integral, we change the order of integration. We graph the region first:



From the picture above, the integral is:

$$\int_0^1 \int_0^{y^{3/2}} x \cos(y^4) dx dy$$

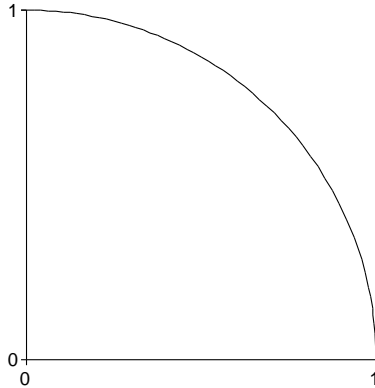
Evaluating this integral is simple, and the answer is $\frac{\sin(1)}{8}$.

3. To evaluate this integral, we change to polar coordinates. We first graph the region (shown below).

The integral becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^1 (r^2)^{2003} r dr d\theta$$

and evaluates to $\frac{\pi}{8016}$.



4. Using the given substitution, the equation $x^2 + xy + y^2 = 1$ becomes:

$$3u^2 + 3v^2 = 1$$

Thus, we'll be integrating over a circle of radius $\frac{1}{\sqrt{3}}$. We calculate the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{vmatrix} = -2\sqrt{3}$$

Using the change of variables formula, the integral is:

$$\int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \int_{-\sqrt{\frac{1}{3}-v^2}}^{\sqrt{\frac{1}{3}-v^2}} 2\sqrt{3} \, du \, dv$$

Since the area of a circle of radius $\frac{1}{\sqrt{3}}$ is $\frac{\pi}{3}$, the integral is $\frac{2\pi}{\sqrt{3}}$.

5. $f(x, y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3$ is a potential function for \mathbf{F} (i.e. $\nabla f = \mathbf{F}$). Thus, by the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = \frac{5}{3}$$

for every C .

6. We parameterize a curve around the circle: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Since $\langle \sin(\sin x), \sin(\sin y) \rangle$ is a conservative vector field, we can drop it from the integral. Now:

$$\begin{aligned} \int_C \langle \langle -y^3, x^3 \rangle \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -(\sin(t))^3, (\cos(t))^3 \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt \\ &= \int_0^{2\pi} \sin^4 t + \cos^4 t \, dt \\ &= \frac{3\pi}{2} \end{aligned}$$