

DYNAMICAL INVARIANTS FOR NONCOMMUTATIVE FLOWS

WILLIAM ARVESON

Department of Mathematics
University of California
Berkeley CA 94720, USA

18 November 1995

ABSTRACT. We show that semigroups of endomorphisms of $\mathcal{B}(H)$ can often be associated with a dynamical principle; that is, an infinitesimal structure that completely determines the dynamics. These dynamical invariants are similar to the second order differential equations of classical mechanics, in that one can locate a space of momentum operators, a “Riemannian metric”, and a potential. In the simplest cases these structures occur in $n \times n$ matrix algebras, and can be classified in terms of noncommutative geometric invariants. As a consequence, we obtain new information relating to the classification of E_0 -semigroups acting on type I_∞ factors.

CONTENTS

Introduction

1. Completely positive semigroups
2. Differential operators on matrix algebras
3. Semielliptic operators
4. Momentum
5. Modular Cohomology
6. Exactness
7. Classification of Semielliptic operators
8. Applications

References

1991 *Mathematics Subject Classification*. Primary 46L40; Secondary 81E05.

Key words and phrases. von Neumann algebras, automorphism groups, elliptic operators, Markov semigroups.

This research was supported by NSF grants DMS92-43893 and DMS95-00291

Introduction. The dynamical groups of quantum theory tend to be one-parameter groups of automorphisms of C^* -algebras generated by bounded functions of certain observables. Frequently, there is a distinguished state on the algebra that is invariant under the action of the automorphism group and which represents the vacuum. Using a standard GNS construction, one can arrange that the operator algebra acts on a Hilbert space, that the automorphism group is implemented by a one-parameter group of unitary operators on this representation space, and that there is a unit vector in the Hilbert space, fixed under the action of the unitary group, which gives rise to the vacuum state.

We have settled on the following general context for confronting dynamical systems of this type.

Definition A. A history is a triple (M_-, U, v) consisting of a one-parameter group of unitary operators $U = \{U_t : t \in \mathbb{R}\}$ acting on a separable Hilbert space H , a unit vector $v \in H$ satisfying

$$U_t v = v, \quad t \in \mathbb{R},$$

and a von Neumann algebra $M_- \subseteq \mathcal{B}(H)$ such that

$$\gamma_t(M_-) \subseteq M_-, \quad t \leq 0,$$

$\gamma = \{\gamma_t : t \in \mathbb{R}\}$ denoting the automorphism group of $\mathcal{B}(H)$ associated with U .

The group γ acts on $\mathcal{B}(H)$ by

$$(0.1) \quad \gamma_t(a) = U_t a U_t^*, \quad t \in \mathbb{R}.$$

There are two kinds of degeneracies that we have found it necessary to rule out. Suppose first that M_- is an abelian von Neumann algebra. Then so is the larger von Neumann algebra

$$M = \left(\bigcup_{t \geq 0} \gamma_t(M_-) \right)''$$

and M is invariant under the full dynamical group $\{\gamma_t : t \in \mathbb{R}\}$. After introducing appropriate coordinates we may realize M as the algebra of all bounded random variables on a standard probability space (Ω, P) , in such a way that the group action γ is represented by a one-parameter group of measure-preserving transformations of (Ω, P) . M_- becomes a subalgebra of $L^\infty(\Omega, P)$ that is invariant under negative time translations.

Now assume further that there is a von Neumann algebra $M_0 \subseteq M_-$ such that M_- is generated by the negative time translations of M_0 :

$$(0.2) \quad M_- = \left(\bigcup_{t \leq 0} \gamma_t(M_0) \right)''.$$

In this case we can find a single random variable

$$X_0 : \Omega \rightarrow \mathbb{R}$$

with the property that M_0 consists of all bounded measurable functions of X_0 (we may choose X_0 to be bounded if we wish). We can define a stationary stochastic process $\{X_t : t \in \mathbb{R}\}$ in the obvious way

$$X_t = \gamma_t(X_0), \quad t \in \mathbb{R}$$

and thus M_- appears as the “past” of the process $\{X_t : t \in \mathbb{R}\}$. Thus this case falls completely into the domain of stationary stochastic processes.

In general it may be impossible to find a “sharp time” subalgebra M_0 which satisfies (0.2). Nevertheless, that situation too is familiar in modern probability theory. For example, that would be the case when M_- is the “past” subalgebra associated with a stationary random distribution such as white noise.

For our purposes this probabilistic case is a degenerate one, and we rule it out by imposing the following requirement that forces us to the opposite extreme:

Definition B. A History (M_-, U, v) is called primary if M_- is a factor.

The second type of degeneracy represents a kind of *determinism*. Let (M_-, U, v) be a history. Then M_- is invariant under γ_t for every $t \leq 0$, and if there is a single $t_0 < 0$ for which $\gamma_{t_0}(M_-) = M_-$, then $\gamma_t(M_-) = M_-$ for every negative t and in fact M_- is invariant under the full dynamical group $\{\gamma_t : t \in \mathbb{R}\}$. One may interpret this as a deterministic situation in which the future is a function of the past (more precisely, every observable is a Borel function of an observable belonging to M_-). From our point of view, however, this simply represents a bad choice of the subalgebra M_- .

These considerations have led us to the following program for understanding the nature of noncommutative dynamical systems. One should seek to classify primary histories (M_-, U, v) which satisfy the condition

$$\gamma_t(M_-) \subsetneq M_-, \quad \text{for every } t < 0.$$

There are two significant equivalence relations for histories. The first is an obvious notion of conjugacy: (M_-, U, v) and $(\tilde{M}_-, \tilde{U}, \tilde{v})$ are said to be *conjugate* if there is a unitary operator $W : H \rightarrow \tilde{H}$ such that

- (i) $Wv = \tilde{v}$
- (ii) $WU_t = \tilde{U}_tW, \quad t \in \mathbb{R}$
- (iii) $WM_-W^{-1} = \tilde{M}_-$.

The second equivalence relation is *cocycle conjugacy*, and will be taken up elsewhere.

There are several ways of finding histories among other objects that arise naturally in quantum physics. For example, suppose that we start quite simply with a pair (U, v) consisting of a one-parameter unitary group U acting on a Hilbert space H and a unit vector $v \in H$ such that $U_tv = v$ for every $t \in \mathbb{R}$. Assume further that K is another Hilbert space and we are given a representation W_0 of the canonical commutation relations over K . Thus

$$f \in K \mapsto W_0(f) \in \mathcal{B}(H)$$

is a strongly continuous mapping from K to the unitary operators on H for which

$$W_0(f)W_0(g) = e^{i\omega(f,g)}W_0(f+g), \quad f, g \in K,$$

$\omega(f, g)$ denoting the imaginary part of the inner product $\langle f, g \rangle$ in K . Then for every $t \in \mathbb{R}$ we can define a representation $W_t : K \rightarrow \mathcal{B}(H)$ of the CCRs by $W_t(f)$

$U_t W_0(f) U_t^*$. One may think of the family $\{W_t : t \in \mathbb{R}\}$ as a noncommutative quantum process. In this case we have a natural candidate for M_- :

$$M_- = \left(\bigcup_{t \leq 0} W_t(K) \right)'',$$

and thus we have a history (M_-, U, v) .

In this example there is a ‘‘sharp time’’ von Neumann algebra associated with time zero

$$M_0 = W_0(K)''$$

with the property that M_- is generated by M_0 and its negative time images under the group of automorphisms associated with U . In general, one cannot expect to have sharp time algebras as this example does. Nevertheless, in very general circumstances (e.g., in any case where one has a quantum system obeying the Haag-Kastler axioms [16, pp. 106–107]) one can find significant examples of histories (M_-, U, v) .

This paper deals with the classification of histories (M_-, U, v) of the simplest type, namely

$$(0.3.1) \quad M_- \text{ is a factor of type } I, \text{ and}$$

$$(0.3.2) \quad \gamma_t(M_-) \subsetneq M_-, \text{ for every } t < 0.$$

This leads to the problem of classifying *pairs* of E_0 -semigroups in the following way.

Let M_+ be the commutant of M_- . Then we obtain a pair of E_0 -semigroups α^+ , α^- acting respectively on M_+ and M_- by

$$\begin{aligned} \alpha_t^+(a) &= U_t a U_t^*, & t \geq 0, a \in M_+ \\ \alpha_t^-(b) &= U_t^* b U_t, & t \geq 0, b \in M_-. \end{aligned}$$

Moreover, the vector state of $\mathcal{B}(H)$ defined by v , $\omega(a) = \langle av, v \rangle$, can be restricted to these two subalgebras to give normal states ρ^+ , ρ^- which are invariant under the actions of the respective E_0 -semigroups. Thus the classification problem for type I histories reduces to the problem of classifying *pairs* of E_0 -semigroups which have distinguished normal invariant states. The theory of E_0 -semigroups was initiated by Powers in [24], and has undergone considerable development during the past several years, see [1], [3], [4], [5], [25], [26], [27] and references cited in these papers.

Notice too that while $\omega(a) = \langle av, v \rangle$ is a pure state of $\mathcal{B}(H)$ its restrictions to the subfactors M_+ and M_- are *not* necessarily pure states on these algebras. Thus we are led to the problem of classifying pairs (α, ρ) where $\alpha = \{\alpha_t : t \geq 0\}$ is an E_0 -semigroup acting on a type I_∞ factor (which we may take as $\mathcal{B}(H_\alpha)$ for some separable Hilbert space H_α) and ρ is a normal state on $\mathcal{B}(H_\alpha)$ satisfying

$$\rho(\alpha_t(a)) = \rho(a), \quad a \in \mathcal{B}(H_\alpha), t \geq 0.$$

Two such pairs (α, ρ) and $(\tilde{\alpha}, \tilde{\rho})$ are said to be *conjugate* if there exists a $*$ -isomorphism $\theta : \mathcal{B}(H_\alpha) \rightarrow \mathcal{B}(H_{\tilde{\alpha}})$ such that

$$\tilde{\alpha}_t \circ \theta = \theta \circ \alpha_t, \quad t \geq 0,$$

Our main results below apply to the case where where ρ is *not* a pure state and is weakly continuous. Equivalently, if we realize ρ in the form

$$\rho(a) = \text{trace}(\Omega a)$$

where Ω is a positive trace-class operator on H_α , then we are assuming that

$$(0.4) \quad 2 \leq \text{rank } \Omega < \infty.$$

In the extreme case where Ω is of rank 1, ρ is a pure state of $\mathcal{B}(H_\alpha)$ and in this case all of our invariants become trivial. However, we point out that Powers has recently worked out a *standard form* for E_0 -semigroups [26] and any E_0 -semigroup in standard form admits an invariant vector state. Thus the case where Ω is rank one should not be considered mysterious. The other extreme case (Ω is of infinite rank) appears to be amenable to the techniques developed below, but there are technical difficulties associated with the infinite rank case that require special attention. We have elected to postpone discussion of the infinite rank case to a subsequent paper.

In broad terms, our results show that there is an infinitesimal “dynamical principle” that governs the behavior of such a pair (α, ρ) . This invariant is analogous to the Hamiltonian structure (on the cotangent bundle of a Riemannian manifold) that governs the behavior of a constrained classical mechanical system with a finite number of degrees of freedom. Indeed, the work carried out in the following sections amounts to a classification of Riemannian type structures in finite dimensional matrix algebras.

The geometric nature of these invariants shows that the “dynamical principles” that govern the behavior of E_0 -semigroups are closely akin to second order differential equations. As in classical mechanics, we are able to identify two fundamental aspects of the dynamics: a metric term corresponding to momentum and kinetic energy, and a potential term corresponding to the driving force. Moreover, one is free to specify the momentum space, the metric, and the potential operator arbitrarily. Regardless of how this is done one obtains an E_0 -semigroup pair (α, ρ) (see remark 8.8 for more detail). In particular, this gives an entirely new way of constructing E_0 -semigroups.

On the other hand, not all pairs (α, ρ) (satisfying the finiteness condition (0.4)) arise in this way. But we are able to give a useful characterization of those that do. Broadly speaking, a pair (α, ρ) arises in this way iff it satisfies two conditions. First, α must obey a certain *minimality* condition (which can always be arranged by replacing α with a compression to a suitable hereditary subalgebra of $\mathcal{B}(H_\alpha)$). More significantly, (α, ρ) must also be *exact* in an appropriate sense. Exactness is equivalent to several important properties, and it is closely analogous to the hypothesis of classical mechanics in which one supposes that the force (a vector field on configuration space) should be the gradient of a potential.

1. Markov semigroups. The purpose of this section is to make some observations about the relationship that exists between E_0 -semigroups and semigroups of completely positive maps.

Let $\phi = \{\phi_t : t \geq 0\}$ be a semigroup of normal completely positive linear maps of $\mathcal{B}(H)$ such that $\phi_t(\mathbf{1}) = \mathbf{1}$, and which is continuous in the sense that for every $a \in \mathcal{B}(H)$, $\xi, \eta \in H$, $\langle \phi_t(a)\xi, \eta \rangle$ is a continuous function of t . We will refer to such a semigroup simply as a *completely positive semigroup*.

Definition 1.1. A Markov semigroup is a pair (ϕ, ρ) consisting of a completely positive semigroup ϕ acting on $\mathcal{B}(H)$ and a faithful normal state ρ of $\mathcal{B}(H)$ which is invariant under ϕ in that $\rho \circ \phi_t = \rho$ for every t .

Remark. We emphasize that the normal state should be *faithful*: $\rho(a^*a) = 0 \implies a = 0$, for every $a \in \mathcal{B}(H)$.

Two Markov semigroups (ϕ, ρ) , (ϕ', ρ') are said to be *conjugate* if there is a $*$ -isomorphism $\theta : \mathcal{B}(H) \rightarrow \mathcal{B}(H')$ such that

$$(1.2.1) \quad \theta \circ \phi_t = \phi'_t \circ \theta, \quad t \geq 0,$$

$$(1.2.2) \quad \rho' \circ \theta = \rho.$$

In this paper, we will be primarily concerned with Markov semigroups acting on finite dimensional spaces; that is, Markov semigroups which act on $n \times n$ matrix algebras, $2 \leq n < \infty$.

The connection between Markov semigroups and E_0 -semigroups is based on the following observations. By an E_0 -semigroup we mean a semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of normal $*$ -endomorphisms of the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space H , such that $\alpha_t(\mathbf{1}) = \mathbf{1}$ and which is continuous in the sense described above.

Lemma 1.3. Let α be an E_0 -semigroup acting on $\mathcal{B}(H)$ and let ω be a normal state of $\mathcal{B}(H)$ which is invariant under the action of α_t for every $t \geq 0$. Let p_0 be the support projection of ω . Then $\alpha_t(p_0) \geq p_0$ for every $t \geq 0$.

Remark. The support projection of ω is the smallest projection p with the property that $\omega(\mathbf{1} - p) = 0$. One has $\omega(a) = \omega(p_0 a p_0)$ for every $a \in \mathcal{B}(H)$, and $\omega(a^*a) = 0$ iff $a p_0 = 0$.

proof of 1.3. Let $t \geq 0$, and consider the projection $\alpha_t(p_0)$. Because of the invariance of ω we have

$$\omega(\mathbf{1} - \alpha_t(p_0)) = \omega \circ \alpha_t(\mathbf{1} - p_0) = \omega(\mathbf{1} - p_0) = 0,$$

and hence $\alpha_t(p_0) \geq p_0$ \square

It follows immediately that the family of projections $p_t = \alpha_t(p_0)$ is increasing on the interval $0 \leq t < \infty$. Let us write M for the von Neumann algebra $\mathcal{B}(H)$. Then we can define a family $\phi = \{\phi_t : t \geq 0\}$ of completely positive maps on the hereditary subalgebra $M_0 = p_0 M p_0$ as follows:

$$\phi_t(a) = p_0 \alpha_t(a) p_0, \quad a \in M_0.$$

We have $\phi_t(p_0) = p_0$, and the semigroup property follows from the observation that if $s, t \geq 0$ then

$$\phi_s(\phi_t(a)) = p_0 \alpha_s(p_0 \alpha_t(a) p_0) p_0 = p_0 p_s \alpha_{s+t}(a) p_s p_0 = \phi_{s+t}(a),$$

because $p_0 p_s = p_s p_0 = p_0$. Thus, if we let ρ be the restriction of ω to M_0 and then identify M_0 with $\mathcal{B}(p_0 H)$, we find that we have a Markov semigroup (ϕ, ρ) . If ω is weakly continuous (equivalently, p_0 is finite dimensional) then we may consider that (ϕ, ρ) acts on an $n \times n$ matrix algebra.

Notice that if ω happens to be a pure (vector) state, then p_0H is one-dimensional and ϕ is the trivial semigroup. Thus all information has been lost in the passage from (α, ω) to (ϕ, ρ) . On the other hand, in all other cases the Markov semigroup will contain significant information about (α, ω) .

Indeed, it is often the case that (ϕ, ρ) completely determines (α, ω) . In order to discuss this issue briefly, let $H_0 = p_0H$, and consider the subspace $H_+ \subseteq H$ defined by

$$H_+ = [\alpha_{t_1}(a_1)\alpha_{t_2}(a_2) \dots \alpha_{t_n}(a_n)\xi : a_i \in M_0, \xi \in H_0, n = 1, 2, \dots].$$

There is a corresponding von Neumann subalgebra M_+ of $\mathcal{B}(H)$, namely the weakly closed algebra generated by the family of operators $\{\alpha_t(a) : a \in M_0, t \geq 0\}$. It can be shown that M_+ is a hereditary subalgebra of $\mathcal{B}(H)$, and in fact

$$M_+ = p_+\mathcal{B}(H)p_+,$$

p_+ denoting the projection of H onto H_+ . The original pair (α, ω) is called *minimal* if $H_+ = H$, or equivalently $M_+ = \mathcal{B}(H)$.

If (α, ω) is minimal, then it is determined up to conjugacy by its associated Markov semigroup (ϕ, ρ) (this follows from work of Bhat described in the following paragraph). But even when (α, ρ) is not minimal, one may compress α to the invariant subalgebra $M_+ = p_+\mathcal{B}(H)p_+$ to obtain a minimal pair (α', ω') . Thus one may always assume that minimality is satisfied provided one is willing to compress α to an invariant corner of $\mathcal{B}(H)$.

Moreover, a recent dilation theorem of B. V. R. Bhat [6], building on and clarifying earlier partial results ([17], [18], [19], [11], [14], [21]), implies that *every* Markov semigroup (α, ρ) arises in this way from a minimal pair (α, ω) , where α is an E_0 -semigroup and ρ is a normal α -invariant state. We will discuss Bhat's theorem, minimality criteria, and properties of the “ n -point functions”

$$\omega(\alpha_{t_1}(a_1)\alpha_{t_2}(a_2) \dots \alpha_{t_n}(a_n))$$

defined for $a_1, \dots, a_n \in M_0, t_i \geq 0, n = 1, 2, \dots$ in a subsequent paper.

2. Differential operators on matrix algebras.

There is a natural notion of the *order* of a differential operator which is easily adapted to the context of linear operators on arbitrary commutative algebras. Indeed, if A is a complex commutative algebra and $L : A \rightarrow A$ is a linear operator, one may define a sequence of multilinear mappings $\Delta^n L : A^{n+1} \rightarrow A$ as follows: $\Delta^0 L = L$ and

$$\Delta^{n+1} L(f_1, \dots, f_n, f_{n+1}; a) = \Delta^n L(f_1, \dots, f_n; f_{n+1}a) - f_{n+1}\Delta^n L(f_1, \dots, f_n; a).$$

One finds that $\Delta^n L(f_1, \dots, f_n; a)$ is a symmetric function of its first n variables f_1, \dots, f_n . L is said to be a differential operator of order n if $\Delta^{n+1} L = 0$ and $\Delta^n L \neq 0$.

The purpose of this section is to discuss the extent to which the notion of order is appropriate for operators on noncommutative algebras, and to introduce a suitable definition of *symbol* of a differential operator. For definiteness, we will consider the case where A is the algebra of all $n \times n$ matrices over \mathbb{C} , since it is this case that is relevant for our purposes. However, the reader will note that we make no essential use of structures specific to $M_n(\mathbb{C})$. Throughout, $\mathcal{L}(A)$ will denote the algebra of all linear operators $L : A \rightarrow A$.

Definition. L is said to be a first order differential operator if for every $a, x, y \in A$ we have

$$(2.1) \quad L(xay) - xL(ay) - L(xa)y + xL(a)y = 0.$$

Notice that the trilinear form appearing on the left side of (2.1) is a noncommutative version of the trilinear form $\Delta^2 L(x, y; a)$ discussed above. Moreover, since A has a unit (2.1) is equivalent to the somewhat simpler condition

$$L(xy) - xL(y) - L(x)y + xL(\mathbf{1})y = 0$$

for all $x, y \in A$ (this will be discussed more fully later in the section).

A simple computation shows that any operator of the form

$$(2.2) \quad L(x) = D(x) + ax$$

where a is a fixed element of A and D is a derivation must be a first order differential operator. The fact that the multiplier a appears on the left in (2.2) is inessential, since any operator of the form $L(x) = D(x) + ax + xb$ can be put into the form (2.2) by replacing D with the derivation $D'(x) = D(x) + xb - bx$. Noting that the first order differential operators L satisfying $L(\mathbf{1}) = 0$ are derivations, it follows that the first order differential operators are *exactly* those of the form (2.2).

In contrast to the commutative case, operators having the form $L(x) = ax$ or $L'(x) = xa$ should not be regarded as “order zero” differential operators, since the difference of two such operators $L(x) - L'(x) = ax - xa$ is in this case a nontrivial derivation which must be assigned order 1. Thus *there is no meaningful definition of “order zero” for operators on noncommutative algebras.*

Similarly, there is no viable concept of “order n ” for $n > 2$. For in the case where $A = M_n(\mathbb{C})$ is a matrix algebra, every linear operator L on A is a finite sum of double multipliers $x \mapsto axb$, a, b being fixed elements of A , and thus by the preceding paragraph L has the form

$$L = A_1 B_1 + A_2 B_2 + \cdots + A_r B_r,$$

where A_k and B_k are first order differential operators. We conclude that *every* operator on A is of “order” at most 2.

Despite the somewhat negative tone of these remarks, we will find Definition (2.1) to be quite useful. In order to discuss this issue, we recall the differential algebra $\Omega^*(A)$ associated with A . This was introduced in [2], and has become a basic constituent of Connes’ noncommutative differential calculus [9]. Actually, we only require the two modules $\Omega^1(A)$ and $\Omega^2(A)$, which can be defined for unital $*$ -algebras as follows. Consider the tensor product $A \otimes A$ as an involutive bimodule over A , with

$$\begin{aligned} a(x \otimes y)b &= ax \otimes yb, \\ (x \otimes y)^* &= y^* \otimes x^*. \end{aligned}$$

The map $d : A \rightarrow A \otimes A$ defined by $dx = \mathbf{1} \otimes x - x \otimes \mathbf{1}$ is a derivation for which $(dx)^* = -d(x^*)$, and it is a universal derivation of A in the sense that if E is any bimodule and $D : A \rightarrow E$ is a derivation, then there is a unique homomorphism of

bimodules $\theta : \Omega^1(A) \rightarrow E$ such that $\theta \circ d = D$. Every element of $\Omega^1(A)$ is a finite sum of the form

$$\omega = \sum_{k=1}^r a_k dx_k.$$

Finally, $\Omega^1(A)$ is the kernel of the multiplication map $\mu : A \otimes A \rightarrow A$ defined by $\mu(a \otimes b) = ab$, and thus we have an exact sequence of involutive bimodules

$$(2.3) \quad 0 \rightarrow \Omega^1(A) \rightarrow A \otimes A \xrightarrow{\mu} A \rightarrow 0.$$

For brevity, we will use the notation Ω^1, Ω^2 rather than $\Omega^1(A), \Omega^2(A)$ whenever it does not lead to confusion. Ω^2 is defined by

$$\Omega^2 = \Omega^1 \otimes_A \Omega^1,$$

and every element of Ω^2 is a finite sum of the form

$$\omega = \sum_{k=1}^r a_k dx_k dy_k.$$

Alternately, Ω^2 can be viewed as the submodule of the bimodule $A^{\otimes 3} = A \otimes A \otimes A$ that is linearly spanned by elements of the form

$$\begin{aligned} adxdy &= a(\mathbf{1} \otimes x - x \otimes \mathbf{1})(\mathbf{1} \otimes y - y \otimes \mathbf{1}) \\ &= a \otimes x \otimes y - ax \otimes \mathbf{1} \otimes y + ax \otimes y \otimes \mathbf{1} - a \otimes xy \otimes \mathbf{1}. \end{aligned}$$

There are two multiplication maps defined on $A^{\otimes 3}$ which together give rise to a homomorphism of bimodules $\mu : A^{\otimes 3} \rightarrow A^{\otimes 2} \oplus A^{\otimes 2}$:

$$\mu(a \otimes b \otimes c) = (ab \otimes c, a \otimes bc),$$

and we have an exact sequence of modules

$$(2.4) \quad 0 \rightarrow \Omega^2 \rightarrow A^{\otimes 3} \xrightarrow{\mu} A^{\otimes 2} \oplus A^{\otimes 2} \rightarrow 0.$$

The involutions defined in $A^{\otimes 3}$ and $A^{\otimes 2} \oplus A^{\otimes 2}$ by

$$\begin{aligned} (a \otimes b \otimes c)^* &= c^* \otimes b^* \otimes a^* \\ (\xi, \eta)^* &= (\eta^*, \xi^*) \end{aligned}$$

make (2.4) into an exact sequence of involutive bimodules.

We will make frequent use of the following observation.

Lemma 2.5. *Let $L \in \mathcal{L}(A)$. There is a unique homomorphism of bimodules $\theta_L \in \text{hom}(\Omega^2, A)$ such that*

proof. The uniqueness of θ_L is clear from the fact that Ω^2 is spanned by elements of the form $a dx dy$, and

$$\theta_L(a dx dy) = a(L(xy) - xL(y) - L(x)y + xL(\mathbf{1})y).$$

For existence, we can define $\theta \in \text{hom}(A^{\otimes 3}, A)$ by $\theta(a \otimes b \otimes c) = -aL(x)b$, and one finds that the restriction of θ to elements of the form $dx dy$ is as required \square

A straightforward computation shows that, more generally,

$$\theta_L(a dx b dy c) = aL(xby)c - axL(by)c - aL(xb)yc + axL(b)yc.$$

Note too that L is a first order differential operator if, and only if, $\theta_L = 0$.

Now it follows from the definition of Ω^1 that for any derivation D of A there is a unique $\theta_D \in \text{hom}(\Omega^1, A)$ such that

$$\theta_D(dx) = D(x), \quad x \in A.$$

Moreover, the map $D \mapsto \theta_D$ is a linear isomorphism of the space of derivations of A onto $\text{hom}(\Omega^1, A)$. There is a somewhat similar characterization of $\text{hom}(\Omega^2, A)$, valid for any unital algebra A for which the Hochschild cohomology space $H^2(A, A)$ is trivial:

Lemma 2.6. *Let $\theta \in \text{hom}(\Omega^2, A)$. There is a linear operator $L \in \mathcal{L}(A)$ satisfying $L(\mathbf{1}) = 0$ and $\theta = \theta_L$. L is unique up to a perturbation of the form $L' = L + D$ where D is a derivation of A .*

proof. Fix θ , and consider the bilinear map $T : A \times A \rightarrow A$ defined by $T(x, y) = \theta(dx dy)$. The Hochschild coboundary of T

$$\begin{aligned} bT(x, y, z) &= xT(y, z) - T(xy, z) + T(x, yz) - T(x, y)z \\ &= \theta(x dy dz - d(xy) dz + dx d(yz) - dx dy z) \end{aligned}$$

vanishes because

$$\begin{aligned} d(xy) dz &= x dy dz + dx y dz, & \text{and} \\ dx d(yz) &= dx y dz + dx dy z. \end{aligned}$$

Since $H^2(A, A) = 0$ when A is a matrix algebra, there exists a linear operator $L \in \mathcal{L}(A)$ for which

$$\theta(dx dy) = bL(x, y) = xL(y) - L(xy) + L(x)y.$$

Setting $L_0(x) = L(x) - xL(\mathbf{1})$ we find that $L_0(\mathbf{1}) = 0$ and $\theta = \theta_{-L_0}$.

If L_1 and L_2 are two linear operators satisfying $L_1(\mathbf{1}) = L_2(\mathbf{1}) = 0$ and $\theta = \theta_{L_1} = \theta_{L_2}$, then $\theta_{L_1 - L_2} = 0$. Since $L_1 - L_2$ is a first order differential operator that annihilates $\mathbf{1}$, it must be a derivation \square

These remarks about the differential nature of linear operators on A are summarized in the following exact sequence of complex vector spaces, in which $\mathcal{C}(A)$

$\{L \in \mathcal{L}(A) : L(\mathbf{1}) = 0\}$, ι is the identification of $\text{hom}(\Omega^1, A)$ with derivations by way of $\iota\theta(x) = \theta(dx)$ $x \in A$, and $\theta : L \mapsto \theta_L$ is the mapping of Lemma 2.5,

$$0 \rightarrow \text{hom}(\Omega^1, A) \xrightarrow[\iota]{} \mathcal{L}_0(A) \xrightarrow[\theta]{} \text{hom}(\Omega^2, A) \rightarrow 0.$$

Throughout the remainder of this paper we will be concerned with *pairs* (A, ρ) consisting of a finite dimensional C^* -algebra A (usually $M_n(\mathbb{C})$) together with a distinguished *faithful* state ρ , that is a linear functional satisfying

$$\begin{aligned} \rho(\mathbf{1}) &= 1, & \text{and} \\ \rho(a^*a) &> 0, & \text{for every nonzero } a \in A. \end{aligned}$$

The associated space $\mathcal{D}(A, \rho)$ of differential operators is defined by the three requirements:

$$(2.7.1) \quad \text{Normalization:} \quad L(\mathbf{1}) = 0$$

$$(2.7.2) \quad \text{Divergence zero:} \quad \rho \circ L = 0$$

$$(2.7.3) \quad \text{Symmetry:} \quad L(x^*) = L(x)^*, \quad x \in A.$$

Conditions (2.7.1) and (2.7.2) are dual to each other, in a sense that will be exploited in the following sections. In general, an operator L satisfying (2.7.3) will be called *symmetric*. $\mathcal{D}(A, \rho)$ is a real vector space of linear operators on A . For every $L \in \mathcal{D}(A, \rho)$ we define the *symbol* of L by $\sigma_L = \rho \circ \theta_L$. More explicitly, σ_L is the linear functional defined on Ω^2 by

$$\sigma_L(a \, dx \, dy) = \rho(aL(xy) - axL(y) - aL(x)y).$$

We collect the following elementary properties of the symbol for later use.

Proposition 2.9. *For every $L \in \mathcal{D}(A, \rho)$, σ_L has the following properties:*

- (i) $\sigma_L(\xi^*) = \overline{\sigma_L(\xi)}, \quad \xi \in \Omega^2$
- (ii) $\sigma_L = 0$ iff L is a derivation.

proof. (i) follows from the formula $\theta_L(\xi^*) = \theta_L(\xi)^*$, which in turn reduces to the property $L(x^*) = L(x)^*$ after evaluating both sides at elements of the form $\xi = dx \, dy$.

For (ii), we note that the bilinear form $a, b \in A \mapsto \rho(ab)$ is nondegenerate because ρ is a faithful state. Thus $\theta_L(dx \, dy) = 0$ for all $x, y \in A$ iff $\rho(a\theta_L(dx \, dy)) = 0$ for every $a, x, y \in A$, hence (ii). \square

Finally, we note that for any finite set D_0, D_1, \dots, D_r of symmetric derivations of A satisfying $\rho \circ D_k = 0$ for $0 \leq k \leq r$, the operator

$$L(x) = D_0(x) + \sum_{k=1}^r D_k^2(x)$$

belongs to $\mathcal{D}(A, \rho)$, and has symbol

$$(2.10) \quad \sigma_L(a \, dx \, dy) = 2 \sum_{k=1}^r \rho(aD_k(x)D_k(y)).$$

More generally, one has

$$(2.11) \quad \sigma_L(a \, dx \, b \, dy \, c) = 2 \sum_{k=1}^r \rho(aD_k(x)bD_k(y)c).$$

It is operators of this type that will be central to our analysis of Markov semigroups

3. Semielliptic operators. Throughout this section, A will denote a finite dimensional matrix algebra and (ϕ, ρ) will denote a Markov semigroup acting on A . In this case ϕ is uniformly continuous

$$\lim_{t \rightarrow 0} \|\phi_t - \text{id}\| = 0.$$

The infinitesimal generator L exists in all senses, and obeys the three properties (2.7). Hence L belongs to $\mathcal{D}(A, \rho)$. There are two characterizations of the generators of completely positive semigroups that are significant for our purposes. The first is due to Lindblad [20] and independently to Gorini *et al* [15] (also see [10, Theorem 4.2]). The second characterization is due to Evans and Lewis [14], based on work of Evans [12]. These two results can be paraphrased in our context as follows.

Theorem 3.1. *An operator $L \in \mathcal{D}(A, \rho)$ generates a Markov semigroup iff there is a completely positive linear map $P : A \rightarrow A$ and an operator $a \in A$ such that*

$$L(x) = P(x) + ax + xa^*, \quad x \in A.$$

Theorem 3.2. *An operator $L \in \mathcal{D}(A, \rho)$ generates a Markov semigroup iff for every $n \geq 1$ and every set of elements $a_1, b_1, \dots, a_n, b_n \in A$, satisfying $b_1 a_1 + b_2 a_2 + \dots + b_n a_n = 0$ we have*

$$\sum_{i,j=1}^n a_j^* L(b_j^* b_i) a_i \geq 0.$$

An operator L satisfying the condition of (3.2) is called *conditionally completely positive* [13]. While 3.1 tells us exactly which operators are generators of Markov semigroups, the cited decomposition of L into a sum of more familiar operators is unfortunately not unique.

The purpose of this section is to use (3.2) to give a new characterization of generators of Markov semigroups in terms of their symbols $\sigma_L : \Omega^2(A) \rightarrow \mathbb{C}$. Recall that the involution in Ω^1 is defined by

$$(a dx)^* = -dx^* a^* = -d(x^* a^*) + x^* da^*,$$

while that of Ω^2 is defined by

$$(a dx dy)^* = dy^* dx^* a^*.$$

Thus for any two 1-forms $\omega_1, \omega_2 \in \Omega^1$ we can form various products in Ω^2 : $\omega_1 \omega_2$, $\omega_1^* \omega_2$, etc.

Theorem 3.3. *An operator $L \in \mathcal{D}(A, \rho)$ generates a Markov semigroup on (A, ρ) iff $\sigma_L(\omega^* \omega) \leq 0$ for every $\omega \in \Omega^1$.*

proof. Let L be an arbitrary operator in $\mathcal{L}(A)$, and let $\theta_L : \Omega^2 \rightarrow A$ be the module homomorphism defined in Lemma 2.5. Choose any sequence of elements $a_1, b_1, \dots, a_n, b_n \in A$ satisfying $\sum_k b_k a_k = 0$ and define an element $\omega \in A \otimes A$ by

$$(3.5) \quad \omega = \sum_{k=1}^n b_k \otimes a_k.$$

Notice that ω belongs to the kernel of the multiplication map $\mu : A \otimes A \rightarrow A$ and hence $\omega \in \Omega^1$. Conversely, every element $\omega \in \Omega^1$ can be decomposed into a sum of the form (3.5) which belongs to the kernel of the multiplication map. Now for such a 1-form ω we have

$$\omega^* \omega = \sum_{i,j=1}^n a_j^* \otimes b_j^* b_i \otimes a_i.$$

Let $\theta \in \text{hom}(\Omega^2, A)$ be defined by $\theta(a \otimes x \otimes b) = -aL(x)b$. The proof of Lemma 2.5 shows that θ_L is obtained by restricting θ to Ω^2 , hence

$$\theta_L(\omega^* \omega) = - \sum_{i,j=1}^n a_j^* L(b_j^* b_i) a_i.$$

This observation, together with Theorem 3.2, shows that an operator $L \in \mathcal{D}(A, \rho)$ generates a Markov semigroup if, and only if, $\theta_L(\omega^* \omega) \leq 0$ for every $\omega \in \Omega^1$. It follows that for every generator L and every $\omega \in \Omega^1$ we have $\sigma_L(\omega^* \omega) = \rho(\theta_L(\omega^* \omega)) \leq 0$.

Conversely, if $\sigma_L(\omega^* \omega) \leq 0$ for every $\omega \in \Omega^1$ then for every $b \in A$ we have

$$\rho(b^* \theta_L(\omega^* \omega) b) = \rho(\theta_L((\omega b)^* \omega b)) = \sigma_L((\omega b)^* \omega b) \leq 0.$$

Since ρ is a faithful state and the operator $T = \theta_L(\omega^* \omega)$ obeys $\rho(b^* T b) \leq 0$ for every b , it follows that $T \leq 0$. Hence $\theta_L(\omega^* \omega) \leq 0$ for every $\omega \in \Omega^1$ and we may conclude from the preceding paragraphs that L generates a Markov semigroup \square

In view of Theorem 3.3, we make the following

Definition 3.4. *A semielliptic operator is an operator $L \in \mathcal{D}(A, \rho)$ satisfying*

$$\sigma_L(\omega^* \omega) \leq 0, \quad \text{for every } \omega \in \Omega^1.$$

The classification problem for Markov semigroups is now reduced to the problem of classifying semielliptic operators up to the natural notion of conjugacy: $L \in \mathcal{D}(A, \rho)$ and $L' \in \mathcal{D}(A', \rho')$ are said to be *conjugate* if there is a $*$ -isomorphism $\theta : A \rightarrow A'$ satisfying

$$(3.5) \quad \begin{aligned} \rho' \circ \theta &= \rho \\ L' \circ \theta &= \theta \circ L. \end{aligned}$$

4. Momentum. Let (A, ρ) be a matrix algebra endowed with a faithful state ρ . In this section we show how semielliptic operators are constructed from more basic structures.

Let p be an element of A satisfying $p^* = -p$. If we write $D_p(a) = [p, a] = pa - ap$, then D_p is a symmetric derivation of A ; replacing p with $p - \rho(p)\mathbf{1}$ if necessary, we can assume that p is normalized so that $\rho(p) = 0$. With this convention for normalization, the operator p is uniquely determined by the derivation D_p . Letting τ be the tracial state of A , we can define the density matrix h of ρ by $\rho(a) = \tau(ha)$, $a \in A$. h is a self adjoint matrix with strictly positive spectrum. Notice, finally, that $\rho(D_p a) = 0$ iff a commutes with p , and in that case we have $D_p a \in \mathcal{D}(A, \rho)$.

Definition 4.1. A momentum space is a pair (P, \langle, \rangle) consisting of a real linear space P of skew-adjoint operators $p \in A$ satisfying $\rho(p) = 0$ and $\rho \circ D_p = 0$, together with a real inner product $\langle, \rangle : P \times P \rightarrow \mathbb{R}$.

We emphasize that the inner product on P can be specified arbitrarily, and in particular if A is realized concretely as $\mathcal{B}(H)$ for a finite dimensional Hilbert space H then there need be no relation between the inner products on P and H . Given a momentum space (P, \langle, \rangle) , we can construct an operator $\Delta \in \mathcal{D}(A, \rho)$ as follows. Choose an orthonormal basis p_1, p_2, \dots, p_r for P and let $D_k(a) = [p_k, a]$, $1 \leq k \leq r$. Δ is defined by

$$(4.2) \quad \Delta = \sum_{k=1}^r D_k^2.$$

Proposition 4.3. The operator Δ of (4.2) does not depend on the orthonormal basis chosen for P , and its symbol obeys

$$(4.3.1) \quad \sigma_\Delta(dx dy) = -2\rho(\Delta(x)y),$$

for all $x, y \in A$.

proof. Since D_k is a derivation in $\mathcal{D}(A, \rho)$ we have

$$\rho(D_k^2(x)y) = \rho(D_k(D_k(x)y)) - \rho(D_k(x)D_k(y)) = -\rho(D_k(x)D_k(y))$$

for each k , so that by (2.10)

$$\rho(\Delta(x)y) = -\sum_{k=1}^r \rho(D_k(x)D_k(y)) = -\frac{1}{2}\sigma_\Delta(dx dy).$$

This establishes (4.3.1).

Let p'_1, \dots, p'_r be another orthonormal basis for P and set $\Delta' = \sum_k D_k'^2$, where $D'_k = [p'_k, \cdot]$. Since ρ is a faithful state it suffices to show that $\rho(\Delta(x)y) = \rho(\Delta'(x)y)$ for all $x, y \in A$; and by (4.3.1) this will follow from the assertion

$$\sum_{k=1}^r \rho(D'_k(x)D'_k(y)) = \sum_{k=1}^r \rho(D_k(x)D_k(y)),$$

for all $x, y \in A$. To prove the latter we make the substitution $p'_k = \sum_i \langle p'_k, p_i \rangle p_i$ in the expression $D'_k(x)D'_k(y) = [p'_k, x][p'_k, y]$ to obtain

$$D'_k(x)D'_k(y) = \sum_{i,j=1}^r \langle p'_k, p_i \rangle \langle p'_k, p_j \rangle D_i(x)D_j(y).$$

When the right side is summed on k , we may use the orthonormality of $\{p_k\}$ and $\{p'_k\}$ in the form $\sum_k \langle p'_k, p_i \rangle \langle p'_k, p_j \rangle = \delta_{ij}$ to obtain

$$\sum_{k=1}^r D'_k(x)D'_k(y) = \sum_{k=1}^r D_k(x)D_k(y),$$

and the claim follows \square

The operator Δ of equation (4.2) is called the *Laplacian* of the momentum space (P, \langle, \rangle) . Finally, let v be any skew-adjoint element of A for which $D_v(a) = [v, a]$ obeys $\rho \circ D_v = 0$, and set

$$(4.3) \quad L(x) = \Delta(x) + [v, x], \quad x \in A.$$

Proposition 4.4. *The operator L of (4.3) is a semielliptic operator in $\mathcal{D}(A, \rho)$.*

proof. It is obvious that L satisfies the criteria (2.7) for membership in $\mathcal{D}(A, \rho)$.

Since D_v is a derivation we have $\sigma_L = \sigma_\Delta$, and thus it suffices to show that Δ is semielliptic. For that, choose

$$\omega = \sum_{k=1}^n a_k dx_k$$

in Ω^1 . Then

$$\omega^* \omega = \sum_{i,j=1}^n (dx_i)^* a_i^* a_j dx_j = - \sum_{i,j=1}^n dx_i^* a_i^* a_j dx_j.$$

Thus by (2.11) we have

$$\sigma_L(\omega^* \omega) = - \sum_{k=1}^r \sum_{i,j=1}^n \rho(D_k(x_i^*) a_i^* a_j D_k(x_j)) = - \sum_{k=1}^r \rho(z_k^* z_k) \leq 0,$$

where $z_k = \sum_j a_j D_k(x_j)$, using the fact that $D_k(x^*) = D_k(x)^*$ \square

The following observation shows that an operator of the form (4.3) determines both of its summands uniquely. $L^2(A, \rho)$ denotes the finite dimensional Hilbert space obtained by endowing A with the inner product $\langle a, b \rangle_\rho = \rho(b^* a)$.

Proposition 4.5. *Let $L = \Delta + [v, \cdot]$ have the form (4.3), and consider L as an operator on the Hilbert space $L^2(A, \rho)$. Then $L + L^* = 2\Delta$ and $L - L^* = 2[v, \cdot]$.*

proof. It suffices to show that $\langle \Delta(a), b \rangle_\rho = \langle a, \Delta(b) \rangle_\rho$ and $\langle [v, a], b \rangle_\rho = - \langle a, [v, b] \rangle_\rho$. But if z is any element of A satisfying $z^* = -z$ and $\rho([z, a]) = 0$ for all $a \in A$, then the operator $D(a) = [z, a]$ is a symmetric derivation which induces a skew-adjoint operator in $\mathcal{B}(L^2(A, \rho))$. Indeed,

$$\langle D(a), b \rangle_\rho = \rho(b^* D(a)) = \rho(D(b^* a)) - \rho(D(b^*) a) = 0 - \rho(D(b)^* a) = - \langle a, D(b) \rangle_\rho.$$

Therefore D^2 is a self-adjoint operator on $L^2(A, \rho)$. It follows that Δ is a self-adjoint operator on $L^2(A, \rho)$ and $[v, \cdot]$ is skew-adjoint. (4.5) follows \square

5. Modular Cohomology. The purpose of this section is to discuss certain cohomological issues so as to provide a context for the following section.

Definition 5.1. *A modular algebra is a pair (A, ρ) consisting of a unital $*$ -algebra A and a faithful state ρ on A with the following property: for every element $a \in A$ there is an element $\delta(a) \in A$ such that*

$$(5.1.1) \quad \rho(ab) = \rho(b\delta(a)), \quad b \in A.$$

Remarks. By a faithful state on A we mean a linear functional ρ satisfying $\rho(\mathbf{1}) = 1$ and $\rho(a^* a) > 0$ for every $a \neq 0 \in A$. Since ρ is faithful the bilinear form

is nondegenerate, and hence the element $\delta(a)$ defined by (5.1.1) is *unique*.

Straightforward calculations show that δ is an automorphism of the algebra structure of A for which $\rho \circ \delta = \rho$, and from the property $\rho(x^*) = \overline{\rho(x)}$ one readily deduces

$$(5.2) \quad \delta(a)^* = \delta^{-1}(a^*),$$

δ^{-1} denoting the inverse automorphism. δ is called the *modular automorphism* of (A, ρ) . We deviate from the traditional notation Δ for the modular automorphism associated with a faithful normal state of a von Neumann algebra in order to reserve Δ for the Laplacian of a momentum space.

Example 5.3. For our immediate purposes we have the case where A is the $*$ -algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices and ρ is a faithful state. If τ is the normalized trace on $M_n(\mathbb{C})$ then we can define an operator $h \in M_n(\mathbb{C})$ by $\rho(a) = \tau(ha)$, $a \in A$. h is a self-adjoint matrix with strictly positive spectrum and δ is the inner automorphism $\delta(a) = hah^{-1}$.

Example 5.4. Let M be an arbitrary von Neumann algebra and let ρ be a faithful normal state of M . Let σ^ρ be the modular automorphism group of ρ . We define A to be the set of all elements $a \in M$ whose spectrum relative to this group is compact; equivalently, A consists of all elements a for which there is a function $f = f_a \in L^1(\mathbb{R})$ whose Fourier transform has compact support and for which

$$\int_{-\infty}^{+\infty} f(t) \sigma_t^\rho(a) dt = 0.$$

A is a unital $*$ -algebra for which $\sigma_t^\rho(A) = A$ for every $t \in \mathbb{R}$. For each $a \in A$, the function $t \in \mathbb{R} \mapsto \sigma_t^\rho(a)$ extends uniquely to an entire function

$$z \in \mathbb{C} \mapsto \sigma_z^\rho(a) \in A.$$

$\{\sigma_z^\rho : z \in \mathbb{C}\}$ defines a group of automorphisms of A parameterized by the additive group of complex numbers, satisfying $\sigma_z^\rho(a)^* = \sigma_{\bar{z}}^\rho(a^*)$. The KMS condition as formulated in [23, 8.12.2] implies that the automorphism $\delta = \sigma_i$ satisfies

$$\rho(ab) = \rho(b\delta(a)), \quad a, b \in A.$$

Therefore (A, ρ) is a modular algebra.

Example 5.4 shows that the considerations of this section are appropriate in situations where there is no trace whatsoever, such as that in which one has an E_0 -semigroup acting on a factor of type *III*. We intend to take up these more general issues in a subsequent paper.

We first sketch a generalization of Connes' cyclic cohomology that is appropriate for modular algebras (A, ρ) . By a *modular cochain* of dimension $n \geq 0$ we mean a multilinear functional of $n + 1$ variables $\phi : A^{n+1} \rightarrow \mathbb{C}$ satisfying

$$(5.5) \quad \phi(\sigma_0^0, \dots, \sigma_1^0, \dots, \sigma_n^0) + (-1)^{n+1} \phi(\sigma_{-1}^0, \dots, \sigma_n^0, \sigma_0^0) + \dots + (-1)^{n-1} \phi(\sigma_0^0, \dots, \sigma_{n-1}^0, \sigma_n^0) = 0$$

The formula (5.5) is ambiguous in the case $n = 0$, and in that case we intend that ϕ should satisfy $\phi(a) = \phi(\delta^{-1}(a))$. $C^n(A)$ denotes the vector space of n -dimensional cochains. For every $\phi \in C^n(A)$ we define a coboundary $b_\delta\phi$ by

$$(5.6) \quad b_\delta\phi(a^0, \dots, a^{n+1}) = \sum_{k=0}^n (-1)^k \phi(a^0, \dots, a^k a^{k+1}, \dots, a^{n+1}) \\ + (-1)^{n+1} \phi(\delta^{-1}(a^{n+1})a^0, a^1, \dots, a^n).$$

One can show that the operator b_δ maps $C^n(A)$ into $C^{n+1}(A)$ and satisfies $b_\delta^2 = 0$. The cohomology of the resulting complex is called *modular cohomology*. A multilinear form $\phi : A^{n+1} \rightarrow \mathbb{C}$ satisfying (5.5) and $b_\delta\phi = 0$ is called a modular cocycle. ϕ is called *exact* if there is an $n - 1$ cochain $\psi \in C^{n-1}(A)$ such that $\phi = b_\delta\psi$.

Notice that if ρ is a trace on A then δ is the identity automorphism and we have cyclic cohomology ([9, pp. 182–190]). In the case where ρ is a faithful state on a matrix algebra the resulting modular cohomology is trivial; nevertheless these considerations (especially for dimensions zero, one and two) will be relevant for our purposes below.

A zero cochain is a linear functional $\phi \in A'$ satisfying $\phi \circ \delta = \phi$. Because ρ is faithful there is a unique operator $p \in A$ for which $\phi(a) = \rho(ap)$, and the condition $\phi \circ \delta = \phi$ is equivalent to $\delta(p) = p$; i.e., p commutes with the “density matrix” of ρ .

A zero cocycle is simply a scalar multiple of ρ , $\phi(a) = \lambda\rho(a)$ for some $\lambda \in \mathbb{C}$. We will require the following simple characterization of exactness for one-cochains.

Proposition 5.7. *Let $\phi \in C^1(A)$. Then ϕ is exact iff it has the form*

$$(5.7.1) \quad \phi(a, x) = \rho(a[v, x]),$$

where v is an element of A satisfying $\rho([v, a]) = 0$ for every $a \in A$.

proof. Suppose first that $p \in A$ satisfies $\rho([p, a]) = 0$ for all $a \in A$ (equivalently, p commutes with the density matrix of ρ) and consider the form $\phi(a, x) = \rho(a[p, x])$. Let ψ be the linear functional $\psi(z) = -\rho(zp)$. Since $\delta(p) = p$ we have

$$\psi(\delta(z)) = -\rho(\delta(z)p) = -\rho \circ \delta(zp) = -\rho(zp) = \psi(z),$$

and hence $\psi \in C^0(A)$. The coboundary of ψ is given by

$$b_\delta\psi(a, x) = \psi(ax) - \psi(\delta^{-1}(x)a) = -\rho(axp) + \rho(\delta^{-1}(x)ap) \\ = \rho(apx) - \rho(axp) = \rho(a[p, x]),$$

and hence $\phi = b_\delta\psi$.

Conversely, suppose ϕ is exact and choose $\psi \in C^0(A)$ such that $\phi(a, x) = \psi(ax) - \psi(\delta^{-1}(x)a)$. Since ρ is faithful there is an element $p \in A$ for which $\psi(z) = -\rho(zp)$. One has $\delta(p) = p$, and the preceding argument can be reversed to obtain formula (5.7.1) \square

6. Exactness. A central issue in this work has been to appropriately characterize the semielliptic operators $L \in \mathcal{D}(A, \rho)$ that can be decomposed as in (4.3)

$$L = \Delta + [v, \cdot]$$

as a first order perturbation of the Laplacian of some momentum space $(P, \langle \cdot, \cdot \rangle)$. In this section we introduce the notion of *exact* differential operator and give several characterizations of exactness (Theorem 6.9). In section 7 we show that the exact semielliptic operators are precisely those that can be so decomposed.

Throughout this section, (A, ρ) will denote a matrix algebra A endowed with a faithful state ρ . Let $L \in \mathcal{D}(A, \rho)$ be a differential operator. We define ω_L as the following bilinear form on A ,

$$\omega_L(x, y) = \rho(xL(y)) - \rho(L(x)y), \quad x, y \in A.$$

ω_L will occupy a central role in the remainder of this paper.

Any operator $L \in \mathcal{L}(A)$ can be regarded as an operator on the Hilbert space $L^2(A, \rho)$ and as such it has an adjoint $L^* \in \mathcal{B}(L^2(A, \rho))$. If L is a *symmetric* operator (i.e., $L(x^*) = L(x)^*$ for every $x \in A$) then L^* is uniquely defined by

$$(6.1) \quad \rho(L(x)y) = \rho(xL^*(y)), \quad x, y \in A.$$

Notice that (6.1) is false if L is not symmetric, but it can be repaired in that case by replacing $L(x)$ on the left side with $L(x^*)^*$.

We are interested in operators that belong to $\mathcal{D}(A, \rho)$. For such an L we find that $L^*(\mathbf{1}) = 0$ by setting $y = \mathbf{1}$ in (6.1) and using $\rho \circ L = 0$. Similarly, $\rho \circ L^* = 0$ follows from the condition $L(\mathbf{1}) = 0$. However, the adjoint of an operator in $\mathcal{D}(A, \rho)$ need *not* be symmetric. That information is contained in the form ω_L as follows.

Proposition 6.2. *Let $L \in \mathcal{D}(A, \rho)$. Then $L^* \in \mathcal{D}(A, \rho)$ if and only if ω_L is a modular one-cochain.*

proof. The condition on ω_L means

$$(6.3) \quad \omega_L(x, y) = -\omega_L(\delta^{-1}(y), x), \quad x, y \in A.$$

The left side of (6.3) is $\rho(xL(y)) - \rho(L(x)y)$ while the right side is

$$\begin{aligned} -\rho(\delta^{-1}(y)L(x)) + \rho(L(\delta^{-1}(y))x) &= -\rho(L(x)y) + \rho(\delta^{-1}(y)L^*(x)) \\ &= -\rho(L(x)y) + \rho(L^*(x)y). \end{aligned}$$

Setting these two expressions equal we find that (6.3) is equivalent to

$$(6.4) \quad \rho(xL(y)) = \rho(L^*(x)y).$$

The left side of (6.4) can be rewritten as follows,

$$\begin{aligned} \rho(xL(y)) &= \overline{\rho(L(y)^*x^*)} = \overline{\langle x^*, L(y) \rangle_\rho} = \overline{\langle L^*(x^*), y \rangle_\rho} \\ &= \overline{\rho(y^*L^*(x^*))} = \rho(L^*(x^*)^*y). \end{aligned}$$

Thus (6.4) is equivalent to the formula $L^*(x^*)^* = L^*(x)$, i.e., that L^* should be a symmetric operator \square

Definition 6.5. *An operator $L \in \mathcal{D}(A, \rho)$ is called exact if there is a modular zero-cochain $\phi \in C^0(A)$ such that $\omega_L = b_\delta \phi$.*

Remarks. Since the coboundary operator b_δ maps cochains to cochains, we see from Proposition 6.2 that for any exact operator $L \in \mathcal{D}(A, \rho)$ we have $L^* \in \mathcal{D}(A, \rho)$. Exactness is considerably stronger, however, and the remainder of this section is devoted to giving several more concrete characterizations of exact differential operators.

We have already pointed out that the modular cohomology of a matrix algebra is trivial. Hence L is exact iff ω_L is a modular one-cocycle. Equivalently, an operator $L \in \mathcal{D}(A, \rho)$ is exact iff the following two conditions are satisfied

$$(6.6.1) \quad L^* \in \mathcal{D}(A, \rho)$$

$$(6.6.2) \quad b_\delta \omega_L = 0.$$

Let δ be the modular automorphism of (A, ρ) . δ induces a natural linear isomorphism $\hat{\delta} : \Omega^1 \rightarrow \Omega^1$ defined by

$$(6.7) \quad \hat{\delta}\left(\sum_j a_j dx_j\right) = \sum_j \delta(a_j) d\delta(x_j).$$

One has $\hat{\delta}(a\omega b) = \delta(a)\hat{\delta}(\omega)\delta(b)$ for $a, b \in A$, $\omega \in \Omega^1$. Notice too that the formula $\delta(a)^* = \delta^{-1}(a^*)$ for $a \in A$ implies that

$$(6.8) \quad \hat{\delta}(\omega^*) = \hat{\delta}^{-1}(\omega)^*, \quad \omega \in \Omega^1,$$

where $\hat{\delta}^{-1}$ in (6.8) denotes the inverse of $\hat{\delta} : \Omega^1 \rightarrow \Omega^1$, defined by $\hat{\delta}^{-1}(a dx) = \delta^{-1}(a) d\delta^{-1}(x)$, $a, x \in A$.

Theorem 6.9. *For any differential operator L in $\mathcal{D}(A, \rho)$, the following are equivalent.*

- (i) L is exact.
- (ii) There is an element $v \in A$ satisfying $v^* = -v$, $\delta(v) = v$, and

$$\omega_L(x, y) = \rho(x[v, y]), \quad x, y \in A.$$

- (iii) $L - L^*$ is a derivation.
- (iv) The symbol of L satisfies the KMS condition

$$\sigma_L(\omega_1 \omega_2) = \sigma_L(\omega_2 \hat{\delta}(\omega_1)), \quad \omega_1, \omega_2 \in \Omega^1.$$

For the proof, we require the following two results about symbols.

Lemma 6.10. *Let $L \in \mathcal{L}(A)$ be an arbitrary operator. The symbol of L obeys*

$$\sigma_L(a\xi) = \sigma_L(\xi \delta(a)), \quad a \in A, \xi \in \Omega^2.$$

proof. Using the homomorphism of modules $\theta_L \in \text{hom}(\Omega^2, A)$ of Lemma 2.5 and the fact that $\sigma_L = \rho \circ \theta_L$ we have

$$\begin{aligned} \sigma_L(a\xi) &= \rho(\theta_L(a\xi)) = \rho(a\theta_L(\xi)) = \rho(\theta_L(\xi)\delta(a)) \\ &= \rho(\theta_L(\xi \delta(a))) = \sigma_L(\xi \delta(a)), \end{aligned}$$

as asserted \square

Lemma 6.11. *Let L be an operator in $\mathcal{D}(A, \rho)$ and let $L^* \in \mathcal{B}(L^2(A, \rho))$ be its adjoint. Then we have*

$$\sigma_L(\omega_1 \omega_2) = \sigma_{L^*}(\omega_2 \hat{\delta}(\omega_1)),$$

for all $\omega_1, \omega_2 \in \Omega^1$.

proof. We claim first that for every $a, x, y \in A$ we have

$$(6.12) \quad \sigma_L(dy a dx) = \sigma_{L^*}(a dx d\delta(y)).$$

Indeed, using Lemma 2.5 and the remarks following it, the left side of (6.12) can be written

$$\begin{aligned} \sigma_L(dy a dx) &= \rho \circ \theta_L(dy a dx) = \rho(L(yax) - yL(ax) - L(ya)x + yL(a)x) \\ &= -\rho(yL(ax)) - \rho(L(ya)x) + \rho(yL(a)x). \end{aligned}$$

On the other hand,

$$\sigma_{L^*}(a dx d\delta(y)) = \rho(aL^*(x\delta(y)) - axL^*(\delta(y)) - aL^*(x)\delta(y) + axL^*(\mathbf{1})\delta(y)).$$

Using $L^*(\mathbf{1}) = 0$ and the formula $\rho(uL^*(v)) = \rho(L(u)v)$ the right side of the preceding formula becomes

$$\rho(L(a)x\delta(y)) - \rho(L(ax)\delta(y)) - \rho(yaL^*(x)) = \rho(yL(a)x) - \rho(yL(ax)) - \rho(L(ya)x).$$

It follows that

$$\sigma_L(dy a dx) - \sigma_{L^*}(a dx d\delta(y)) = 0,$$

as asserted in (6.12).

We may conclude that for all $\omega_2 \in \Omega^1$ we have

$$\sigma_L(dy \omega_2) = \sigma_{L^*}(\omega_2 \hat{\delta}(dy)).$$

Now for any $b, y \in A$ we have by Lemma 6.10

$$\sigma_L(b dy \omega_2) = \sigma_L(dy \omega_2 \delta(b)),$$

and by (6.12) the right side is $\sigma_{L^*}(\omega_2 \delta(b) d\delta(y))$. Thus for any ω_1 which is a finite sum of elements of the form $b dy$ we have

$$\sigma_L(\omega_1 \omega_2) = \sigma_{L^*}(\omega_2 \hat{\delta}(\omega_1)),$$

and 6.11 follows \square

Turning now to the proof of 6.9, we show (ii) \implies (i) \implies (iii) \implies (iv) \implies (ii). The first of these implications is immediate from Proposition 5.7.

(i) \implies (iii). Proposition 5.7 implies that $\omega_L(x, y) = \rho(xD(y))$ where D is a derivation. On the other hand, by (6.11) we have

$$\omega(x, y) = \rho(xL(y)) - \rho(L(x)y) = \rho(x(L(y) - L^*(y))).$$

Hence $L - L^* - D$ is plainly a derivation

(iii) \implies (iv). Since $L - L^*$ is a first order differential operator its symbol must vanish, and hence $\sigma_L - \sigma_{L^*} = \sigma_{L-L^*} = 0$. Thus by Lemma 6.11 we have

$$\sigma_L(\omega_1\omega_2) = \sigma_{L^*}(\omega_2\hat{\delta}(\omega_1)) = \sigma_L(\omega_2\hat{\delta}(\omega_1)),$$

as asserted.

(iv) \implies (ii). Assuming the KMS condition 6.9 (iv), the preceding argument can clearly be reversed to show that $\sigma_{L-L^*} = 0$, hence $D = L - L^*$ is a first order differential operator. Since $L(\mathbf{1}) = L^*(\mathbf{1}) = 0$ we have $D(\mathbf{1}) = 0$ and thus D is a derivation.

Choose an operator $v \in A$ such that $D(x) = [v, x]$, $x \in A$. We claim that $v + v^*$ is a scalar. To see this we use the fact that $D = L - L^*$ is a skew-adjoint operator on $L^2(A, \rho)$ which satisfies $\rho \circ D = 0$. Setting $\overline{D}(x) = D(x^*)^*$ we find that \overline{D} and D^* are related by

$$\rho(\overline{D}(a)b) = \rho(aD^*(b)).$$

Since $D^* = -D$ and since $\rho \circ D = 0$, we have

$$\rho(aD^*(b)) = -\rho(aD(b)) = -\rho(D(ab)) + \rho(D(a)b) = \rho(D(a)b).$$

Hence $\overline{D} = D$. Using the fact that $D = [v, \cdot]$ and $\overline{D} = [-v^*, \cdot]$ we conclude that $[v + v^*, a] = 0$ for every $a \in A$, hence $v + v^*$ is a scalar.

By replacing v with $v - \lambda\mathbf{1}$ for a suitable real number λ , we can assume that $v^* = -v$; and the condition $\rho \circ [v, \cdot] = 0$ implies that $\delta(v) = v$, completing the proof of Theorem 6.9 \square

Corollary 6.13. *Let $(P, \langle \cdot, \cdot \rangle)$ be a momentum space with Laplacian Δ and let $L \in \mathcal{D}(A, \rho)$ have the form*

$$L = \Delta + [v, \cdot]$$

where $v^* = -v$, $\delta(v) = v$. Then L is exact.

proof. By Proposition 4.5, $L - L^* = 2[v, \cdot]$ is a derivation and thus L satisfies condition (iii) of Theorem 6.9 \square

Remarks. We emphasize that the symbol σ_L of an operator $L \in \mathcal{D}(A, \rho)$ does not normally give rise to a modular cocycle. More precisely, let $\phi : A^3 \rightarrow \mathbb{C}$ be the trilinear form

$$\phi(a, x, y) = \sigma_L(a \, dx \, dy).$$

A straightforward computation shows that

$$b_\delta \phi = 0.$$

However, ϕ does not satisfy the functional equation for two-cochains

$$\phi(a, x, y) = \phi(\delta^{-1}(y), a, x)$$

except in rather special circumstances. We omit further discussion of the latter since it does not come to bear on the sequel.

For an semielliptic operator $L \in \mathcal{D}(A, \rho)$, the form $\omega_L(x, y) = \rho(xL(y)) - \rho(L(x)y)$ plays a role closely analogous to the ‘‘driving force’’ of a classical mechanical system. Indeed, in what follows, the condition 6.9 (ii) for exactness,

$$\omega_L(x, y) = -\rho(x[y, v])$$

will occupy a position parallel to the hypothesis of classical mechanics

$$F = -\text{grad } V$$

that the driving force should be conservative

Example 6.14. We conclude this section by describing some elementary examples of semielliptic operators that are not exact. Let (A, τ) be the pair consisting of a full matrix algebra $M_n(\mathbb{C})$, $n \geq 3$, with normalized trace τ . Let α be a $*$ -automorphism of A and consider the operator

$$(6.15) \quad L(x) = \alpha(x) - x, \quad x \in A.$$

One may verify directly that L is the generator of the semigroup

$$\phi_t = e^{-t} \exp(t\alpha), \quad t \geq 0$$

and that (ϕ, τ) is a Markov semigroup. Thus, L is a semielliptic operator in $\mathcal{D}(A, \tau)$. The following result implies that operators of the form (6.15) are typically not exact.

Proposition. *Assume that there is an abelian $*$ -subalgebra $N \subseteq A$ such that $\alpha(N) = N$ and $\alpha^2 \upharpoonright_N$ is not the identity map of N . Then L is not exact.*

proof. Let $\omega_L : A \times A \rightarrow \mathbb{C}$ be the form

$$\omega_L(x, y) = \tau(xL(y)) - \tau(L(x)y).$$

We will show that there is no derivation D of A for which $\omega_L(x, y) = \tau(xD(y))$, $x, y \in A$. This means that L fails to satisfy condition (ii) of Theorem 6.9, hence L is not exact.

Using the fact that every automorphism of A preserves the trace, we have

$$\begin{aligned} \omega_L(x, y) &= \tau(x(\alpha(y) - y)) - (\alpha(x) - x)y = \tau(x\alpha(y) - \alpha(x)y) \\ &= \tau(x(\alpha(y) - \alpha^{-1}(y))). \end{aligned}$$

Thus we have to show that the operator $\alpha - \alpha^{-1}$ is not a derivation.

But $\alpha - \alpha^{-1}$ leaves the abelian $*$ -subalgebra N invariant, so if it is a derivation it must be 0 on N . This implies that $\alpha(x) = \alpha^{-1}(x)$ for every $x \in N$, contradicting the hypothesis on α \square

7. Classification of semielliptic operators.

We now take up the classification of semielliptic operators in $\mathcal{D}(A, \rho)$. This is based on the classification of *metrics*. A will denote a full matrix algebra.

Definition 7.1. *A metric on A is a linear functional $g : \Omega^2 \rightarrow \mathbb{C}$ satisfying*

$$g(\omega^*\omega) \geq 0, \quad \omega \in \Omega^1.$$

A metric gives rise to a positive semidefinite inner product $(\cdot, \cdot) : \Omega^1 \times \Omega^1 \rightarrow \mathbb{C}$ by way of

$$(\omega_1, \omega_2) = g(\omega_2^*\omega_1), \quad \omega_1, \omega_2 \in \Omega^1$$

and we have

$$(7.2) \quad (a\omega_1, \omega_2) = (\omega_1, a^*\omega_2).$$

Conversely, any (semidefinite) inner product (\cdot, \cdot) satisfying (7.2) arises in this way from a unique metric $g : \Omega^2 \rightarrow \mathbb{C}$. We have found it more convenient to work with

metrics *qua* linear functionals rather than with metrics *qua* inner products. Notice, for example, that the symbol of any semielliptic operator L defines a metric g by way of $g = -\sigma_L$.

Metrics can be concretely presented as follows. Let $z_1, z_2, \dots, z_r \in A$ and set

$$(7.3) \quad g(a \, dx \, dy) = \sum_{k=1}^r \rho(a[z_k^*, x][z_k, y]).$$

It is easily seen that g is a metric. Notice that this metric satisfies

$$(7.4) \quad g(a\xi) = g(\xi \delta(a)), \quad a \in A, \xi \in \Omega^2,$$

δ denoting the modular automorphism of ρ . Conversely, it can be shown that any metric satisfying (7.4) can be expressed in the form (7.3) for some set of elements z_1, z_2, \dots, z_r in A . Note too that the z_k do not need to be self-adjoint or skew-adjoint, and they need not satisfy $\delta(z_k) = z_k$.

In this section we consider metrics g which satisfy the KMS condition of Theorem 6.9 (iv),

$$(7.5) \quad g(\omega_1 \omega_2) = g(\omega_2 \hat{\delta}(\omega_1)), \quad \omega_1, \omega_2 \in \Omega^1.$$

We prove that a KMS metric can be decomposed into a sum of the form (7.3) in which the operators z_1, \dots, z_r satisfy the additional conditions $z_k^* = z_k$ and $\delta(z_k) = z_k$ for all k , and that moreover there is a unique momentum space associated with g .

Theorem 7.6. *Let g be a nonzero metric which satisfies the KMS condition (7.5). There is a linearly independent set of self-adjoint operators $x_1, \dots, x_n \in A$ such that $\delta(x_k) = x_k$ and $\rho(x_k) = 0$ for every k , and for which*

$$g(a \, dx \, dy) = \sum_{k=1}^n \rho(a[x_k, x][x_k, y])$$

for every $a, x, y \in A$.

If x'_1, \dots, x'_m is another finite set satisfying all of these conditions then $m = n$ and there is a real orthogonal $n \times n$ matrix (u_{ij}) such that

$$x'_k = \sum_{j=1}^n u_{kj} x_j, \quad 1 \leq k \leq n.$$

Remarks. Before giving the proof of Theorem 7.6 we discuss some immediate consequences.

Notice that by replacing x_k with $p_k = \sqrt{-1} x_k$ we obtain a linearly independent set of skew-adjoint elements $\{p_1, \dots, p_n\}$ satisfying $\rho([p_k, a]) = \rho(p_k) = 0$ for every $a \in A$ such that

$$g(a \, dx \, dy) = - \sum_{k=1}^n \rho(a[p_k, x][p_k, y]),$$

and for which a similar uniqueness holds. Thus the real vector space P spanned by $\{p_1, \dots, p_n\}$ is independent of the particular choice of p_1, \dots, p_n , as is the inner product defined on P by

$$\langle p_i, p_j \rangle = \delta_{ij}.$$

We conclude that *every KMS metric g is associated with a unique momentum space $(P, \langle \cdot, \cdot \rangle)$.*

This association $g \rightsquigarrow (P, \langle \cdot, \cdot \rangle)$ of a momentum space with a KMS metric is in fact bijective. Indeed, if we let Δ be the Laplacian of $(P, \langle \cdot, \cdot \rangle)$ then from (2.10) we find that $2g = -\sigma_\Delta$ is determined by Δ , and therefore by $(P, \langle \cdot, \cdot \rangle)$. Thus Theorem 7.6 implies the following

Corollary 1. *The above association defines a bijective correspondence between the set of KMS metrics and the set of momentum spaces in (A, ρ) .*

Indeed, Theorem 7.6 can be restated in the following equivalent (and somewhat more invariant) way.

Theorem 7.6A. *For every KMS metric $g : \Omega^2 \rightarrow \mathbb{C}$ there is a unique momentum space $(P, \langle \cdot, \cdot \rangle)$ such that $2g = -\sigma_\Delta$, Δ denoting the Laplacian of $(P, \langle \cdot, \cdot \rangle)$.*

Two metrics g_1, g_2 are said to be *equivalent* if there is a $*$ -automorphism $\alpha : A \rightarrow A$ such that $g_2(\xi) = g_1(\hat{\alpha}(\xi))$, $\xi \in \Omega^2$, $\hat{\alpha}$ denoting the induced mapping of Ω^2 :

$$\hat{\alpha}(a \, dx \, dy) = \alpha(a) \, d\alpha(x) \, d\alpha(y).$$

The above remarks lead immediately to the following classification of KMS metrics in terms of their momentum spaces.

Corollary 2. *Let g_1, g_2 be two KMS metrics with respective momentum spaces $(P_1, \langle \cdot, \cdot \rangle_1)$ and $(P_2, \langle \cdot, \cdot \rangle_2)$. g_1 and g_2 are equivalent iff there is a unitary operator $u \in A$ such that*

$$\begin{aligned} uP_1u^* &= P_2, & \text{and} \\ \langle upu^*, uqu^* \rangle_2 &= \langle p, q \rangle_1, & p, q \in P_1. \end{aligned}$$

proof. Assuming that g_1 and g_2 are equivalent, let α be a $*$ -automorphism of A such that $g_2 = g_1 \circ \hat{\alpha}$. We may find a unitary operator $u \in A$ such that $\alpha(x) = uxu^*$. We can obviously define a momentum space $(uP_1u^*, \langle \cdot, \cdot \rangle')$ where $\langle upu^*, uqu^* \rangle' = \langle p, q \rangle_1$, $p, q \in P_1$; and $(uP_1u^*, \langle \cdot, \cdot \rangle')$ gives rise to the metric $g_1 \circ \hat{\alpha} = g_2$. Hence by the uniqueness assertion of Theorem 7.6 we must have $(P_2, \langle \cdot, \cdot \rangle) = (uP_1u^*, \langle \cdot, \cdot \rangle')$.

The opposite implication is equally straightforward \square

proof of Theorem 7.6. Since $g(\omega^*\omega) \geq 0$ for every $\omega \in \Omega^1$, a standard construction leads to a finite dimensional Hilbert space H_g and a complex linear map $\theta : \Omega^1 \rightarrow H_g$ satisfying

$$(7.7.1) \quad \theta_g(\Omega^1) = H_g, \quad \text{and}$$

$$(7.7.2) \quad \langle \theta_g(\omega)^*, \theta_g(\eta) \rangle = \langle \theta_g(\omega), \theta_g(\eta) \rangle, \quad \omega, \eta \in \Omega^1$$

Let a be an element of A . The formula $g(\omega_2^* a \omega_1) = g((a^* \omega_2)^* \omega_1)$ implies that

$$\langle \theta(a \omega_1), \theta(\omega_2) \rangle = \langle \theta(\omega_1), \theta(a^* \omega_2) \rangle \quad \omega_i \in \Omega^1.$$

It follows that there is a unique $*$ -representation $\pi : A \rightarrow \mathcal{B}(H_g)$ defined by

$$(7.8) \quad \pi(a)\theta(\omega) = \theta(a\omega), \quad \omega \in \Omega^1.$$

$\pi(\mathbf{1}) = \mathbf{1}$ because $\mathbf{1} \cdot \omega = \omega$ for every $\omega \in \Omega^1$.

Next, we show that there is a natural conjugation J of H_g , that is, an antilinear operator on H_g satisfying $J^2 = \mathbf{1}$ and $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$, for all $\xi, \eta \in H_g$. For that we require some information about the behavior of g with respect to the (complex) modular automorphism group of the distinguished state on A , $\rho(a) = \text{trace}(ha)$. For $z \in \mathbb{C}$ there is an automorphism σ_z of the algebra structure of A defined by

$$\sigma_z(a) = h^{-iz} a h^{iz}.$$

We have $\sigma_z \sigma_w = \sigma_{z+w}$ and $\sigma_z(a^*) = \sigma_{\bar{z}}(a)^*$. For fixed $a \in A$, $z \mapsto \sigma_z(a)$ defines an entire function from \mathbb{C} to the Banach space A which is uniformly bounded on horizontal strips $-M \leq \Im(z) \leq M$, $0 < M < \infty$. For real z , σ_z is a $*$ -automorphism of A .

Each automorphism σ_z determines a natural automorphism $\hat{\sigma}_z$ of the differential algebra Ω^* , and in particular $\hat{\sigma}_z$ acts on Ω^1 and Ω^2 by

$$\begin{aligned} \hat{\sigma}_z(a dx) &= \sigma_z(a) d\sigma_z(x), \\ \hat{\sigma}_z(a dx dy) &= \sigma_z(a) d\sigma_z(x) d\sigma_z(y). \end{aligned}$$

If $z = t$ is real then $\hat{\sigma}_t(\omega^*) = \hat{\sigma}_t(\omega)^*$ for all ω . The modular automorphism δ introduced in §5 is given by $\delta(a) = \sigma_i(a)$, $a \in A$.

We now define a second involution $\#$ of Ω^1 using the natural square root of the modular automorphism

$$\delta^{1/2}(a) = \sigma_{i/2}(a) = h^{1/2} a h^{-1/2}$$

as follows:

$$(7.9) \quad (a dx)^\# = \delta^{1/2}((a dx)^*) = -d\delta^{1/2}(x^*) \delta^{1/2}(a^*).$$

Since $\delta^{1/2}(x^*) = \delta^{-1/2}(x)^*$ we see that $\omega \mapsto \omega^\#$ defines an involution in Ω^1 .

Lemma 7.10. *Every KMS metric $g : \Omega^2 \rightarrow \mathbb{C}$ satisfies*

- (i) $g(\hat{\sigma}_z(\xi)) = g(\xi), \quad \xi \in \Omega^1, z \in \mathbb{C}$
- (ii) $g(\omega_2^* \omega_1) = g((\omega_1^\#)^* \omega_2^\#) \quad \omega_1, \omega_2 \in \Omega^1,$

We have $(a\omega)^\# = \omega^\# \delta^{1/2}(a^*)$ and $(\omega b)^\# = \delta^{1/2}(b^*) \omega^\#$, for every $a, b \in A$, $\omega \in \Omega^1$.

For (i), notice first that $g \circ \hat{\delta} = g$. Indeed, for $\omega_1, \omega_2 \in \Omega^1$ we can apply the KMS condition twice to obtain

$$g(\hat{\delta}(\omega_1\omega_2)) = g(\hat{\delta}(\omega_1)\hat{\delta}(\omega_2)) = g(\omega_2\hat{\delta}(\omega_1)) = g(\omega_1\omega_2),$$

and $g \circ \hat{\delta} = g$ follows because Ω^2 is spanned by products of the form $\omega_1\omega_2$, $\omega_k \in \Omega^1$.

Fix an element $\xi \in \Omega^2$ and consider the entire function

$$f(z) = g(\hat{\sigma}_z(\xi)), \quad z \in \mathbb{C}.$$

By the preceding remarks f is entire and bounded on the horizontal strip $0 \leq \Im(z) \leq 1$. We claim that $f(z+i) = f(z)$ for every $z \in \mathbb{C}$. Indeed, since $\sigma_{z+i} = \sigma_i\sigma_z = \delta\sigma_z$ and since $g \circ \hat{\delta} = g$ we have

$$f(z+i) = g(\hat{\delta}(\hat{\sigma}_z(\xi))) = g(\hat{\sigma}_z(\xi)) = f(z).$$

Thus f is a bounded entire function which, by Liouville's theorem, must be a constant.

For (ii) we write

$$g((\omega_1^\#)^*\omega_2^\#) = g((\delta^{1/2}(\omega_1^*))^*\delta^{1/2}(\omega_2^*)) = g(\delta^{-1/2}(\omega_1)\delta^{1/2}(\omega_2^*)).$$

By what was just proved we can use $g \circ \hat{\delta}^{1/2} = g$ on the right hand term to obtain

$$g(\omega_1\delta(\omega_2^*)) = g(\omega_2^*\omega_1),$$

and (ii) follows. The formula $(a\omega)^\# = \omega^\#\delta^{1/2}(a^*)$ follows directly from the definition of $\#$. \square

Lemma 7.10 (ii) implies that

$$\langle \theta(\omega_1), \theta(\omega_2) \rangle = \langle \theta(\omega_2^\#), \theta(\omega_1^\#) \rangle,$$

and hence we can define a unique antilinear isometry J of H_g by

$$(7.11) \quad J\theta(\omega) = \theta(\omega^\#), \quad \omega \in \Omega^1.$$

Since $\omega^{\#\#} = \omega$ it follows that $J^2 = \mathbf{1}$, and hence J is a conjugation of H_g .

We now define a unital $*$ -antirepresentation $\overset{\circ}{\pi} : A \rightarrow \mathcal{B}(H_g)$ by

$$\overset{\circ}{\pi}(a) = J\pi(a^*)J.$$

From the definitions of π and J we find that

$$(7.12) \quad \overset{\circ}{\pi}(a)\theta(\omega) = \theta(\omega\delta^{1/2}(a)), \quad a \in A, \omega \in \Omega^1,$$

and hence $\pi(a)\overset{\circ}{\pi}(b) = \overset{\circ}{\pi}(b)\pi(a)$ for every $a, b \in A$. Thus we have a $*$ -representation of the C^* -algebra $A \otimes A^o$ (A^o denoting the C^* -algebra opposite to A) by way of

$$\pi \otimes \overset{\circ}{\pi} : a \otimes b \in A \otimes A^o \mapsto \pi(a)\overset{\circ}{\pi}(b).$$

Finally, the action of the modular group $\hat{\sigma}_t : \Omega^1 \rightarrow \Omega^1$ can be implemented by a one-parameter group of unitary operators $U = \{U_t : t \in \mathbb{R}\}$, which is defined on H_g as follows:

$$U_t\theta(\omega) = \theta(\hat{\sigma}_t(\omega)), \quad t \in \mathbb{R}, \omega \in \Omega^1.$$

Indeed, for $\omega_1, \omega_2 \in \Omega^1$,

$$\begin{aligned} \langle \theta(\hat{\sigma}_t(\omega_1)), \theta(\hat{\sigma}_t(\omega_2)) \rangle &= g((\hat{\sigma}_t(\omega_2))^*\hat{\sigma}_t(\omega_1)) = g((\hat{\sigma}_t(\omega_2^*)\hat{\sigma}_t(\omega_1))) \\ &= g(\hat{\sigma}_t(\omega_2^*\omega_1)) = g(\omega_2^*\omega_1) = \langle \theta(\omega_1), \theta(\omega_2) \rangle, \end{aligned}$$

and thus each U_t is well-defined by the above formula. U obeys the group property and is strongly continuous.

Lemma 7.13. *We may assume that H_g is the Hilbert space A^n consisting of all n -tuples $\bar{z} = (z_1, \dots, z_n)$, $z_k \in A$, with inner product*

$$\langle \bar{z}, \bar{w} \rangle = \sum_{k=1}^n \rho(w_k^* z_k),$$

and that there is a set of n real numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that the quadruple $\pi, \overset{\circ}{\pi}, U, J$ acts in the following way

$$(7.13.1) \quad \pi(a)\bar{z} = (az_1, \dots, az_n)$$

$$(7.13.2) \quad \overset{\circ}{\pi}(b)\bar{z} = (z_1\delta^{1/2}(b), \dots, z_n\delta^{1/2}(b)).$$

$$(7.13.3) \quad U_t\bar{z} = (e^{i\lambda_1 t}\sigma_t(z_1), \dots, e^{i\lambda_n t}\sigma_t(z_n)).$$

$$(7.13.4) \quad J\bar{z} = (\delta^{1/2}(z_1^*), \dots, \delta^{1/2}(z_n^*)),$$

Remark. We point out that in the course of the argument below, we will eventually prove that $\lambda_1 = \dots = \lambda_n = 0$.

proof. Since $A \otimes A^\circ$ is another matrix algebra the commutant of the set of operators $\pi(A) \cup \overset{\circ}{\pi}(A^\circ)$ is a type I_n factor for some $n = 1, 2, \dots$, and the representation $\pi \otimes \overset{\circ}{\pi}$ is unitarily equivalent to a direct sum of n copies of any irreducible representation of $A \otimes A^\circ$.

The standard representation $\lambda : A \rightarrow \mathcal{B}(L^2(A, \rho))$ is defined by

$$\lambda(a)z = az, \quad z \in A$$

and there is a conjugation J_0 of $L^2(A, \rho)$ defined by

$$J_0(z) = \delta^{1/2}(z^*).$$

One has $J_0\lambda(A)J_0 = \lambda(A)'$. The corresponding antirepresentation is defined by

$$\overset{\circ}{\lambda}(b) = J_0\lambda(b^*)J_0,$$

and one computes that

$$\overset{\circ}{\lambda}(b)z = z\delta^{1/2}(b).$$

Thus $\lambda \otimes \overset{\circ}{\lambda}$ defines an irreducible representation of $A \otimes A^\circ$ on the Hilbert space $L^2(A, \rho)$. Hence $\pi \otimes \overset{\circ}{\pi}$ is unitarily equivalent to a direct sum of n copies of $\lambda \otimes \overset{\circ}{\lambda}$.

After this change in coordinates, we may assume that π and $\overset{\circ}{\pi}$ act on A^n as specified in (7.13.1) and (7.13.2).

We show next that we can also achieve (7.13.3). Let $\overset{\circ}{U}_t$ be the unitary group which acts on A^n by way of

$$\overset{\circ}{U}_t \bar{z} = (\sigma_t(z_1), \dots, \sigma_t(z_n)), \quad t \in \mathbb{R}, z_k \in A$$

Since both U_t and $\overset{\circ}{U}_t$ induce the same action on $\pi(A)$ and $\overset{\circ}{\pi}(A)$, namely

$$\begin{aligned} U_t \pi(a) U_t^* &= \overset{\circ}{U}_t \pi(a) \overset{\circ}{U}_t^* = \pi(\sigma_t(a)), \\ U_t \overset{\circ}{\pi}(a) U_t^* &= \overset{\circ}{U}_t \overset{\circ}{\pi}(a) \overset{\circ}{U}_t^* = \overset{\circ}{\pi}(\sigma_t(a)) \end{aligned}$$

it follows that $W_t = U_t \overset{\circ}{U}_t^*$ commutes with all operators in $\pi(A) \cup \overset{\circ}{\pi}(A)$, and we have $U_t = W_t \overset{\circ}{U}_t$.

Notice that $W = \{W_t : t \in \mathbb{R}\}$ is a one-parameter unitary group in the commutant of $\pi(A) \cup \overset{\circ}{\pi}(A)$. Indeed, since $\sigma_t(a) = h^{-it} a h^{it}$ is an inner automorphism of A for each t it follows that $\overset{\circ}{U}_t$ belongs to the von Neumann algebra generated by $\pi(A) \cup \overset{\circ}{\pi}(A)$. Thus

$$W_{s+t} \overset{\circ}{U}_{s+t} = U_{s+t} = U_s U_t = W_s \overset{\circ}{U}_s W_t \overset{\circ}{U}_t = W_s W_t \overset{\circ}{U}_s \overset{\circ}{U}_t = W_s W_t \overset{\circ}{U}_{s+t},$$

so that $W_s W_t = W_{s+t}$. By the spectral theorem there are mutually orthogonal minimal projections E_1, \dots, E_n in $(\pi(A) \cup \overset{\circ}{\pi}(A))'$ and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$W_t = \sum_{k=1}^n e^{i\lambda_k t} E_k.$$

Thus there is a unitary operator V in the commutant of $\pi(A) \cup \overset{\circ}{\pi}(A)$ which brings the unitary group $W = \{W_t : t \in \mathbb{R}\}$ into diagonal form relative to the coordinates given in (7.13.3). Conjugation by this operator V does not change π or $\overset{\circ}{\pi}$, and thus we have achieved (7.13.1), (7.13.2) and (7.13.3) simultaneously.

Let J_n be the natural conjugation of A^n , defined by

$$J_n(\bar{z}) = (\delta^{1/2}(z_1^*), \dots, \delta^{1/2}(z_n^*)),$$

and let J be the conjugation defined by (7.11). It remains to show that there is a unitary operator W on A^n such that

$$(7.14) \quad W \in (\pi(A) \cup \overset{\circ}{\pi}(A) \cup \{U_t : t \in \mathbb{R}\})', \quad \text{and}$$

$$(7.15) \quad J = W J_n W^{-1}.$$

Once we have such a W , we may use it to bring J into the form (7.13.4) without disturbing what has already been achieved in (7.13.1), (7.13.2) and (7.13.3).

W is defined as follows. The standard conjugation J_n of A^n satisfies

$$\begin{aligned} J_n(\pi(a^*)\bar{z}) &= J_n(a^* z_1, \dots, a^* z_n) = (\delta^{1/2}(z_1^* a), \dots, \delta^{1/2}(z_n^* a)) \\ &= (\delta^{1/2}(z_1) \delta^{1/2}(*a), \dots, \delta^{1/2}(z_n) \delta^{1/2}(*a)) = \overset{\circ}{\pi}(a) J_n(\bar{z}), \end{aligned}$$

and hence

$$(7.16) \quad J = \overset{\circ}{\pi}(a^*) J_n \overset{\circ}{\pi}(a)$$

Since $J_n^2 = \mathbf{1}$ we also have

$$(7.17) \quad J_n \overset{\circ}{\pi}(a) J_n = \pi(a^*).$$

Since J also satisfies (7.16) and (7.17) it follows that the unitary operator $V = J J_n$ must commute with both operator algebras $\pi(A)$ and $\overset{\circ}{\pi}(A)$, and we have $J = V J_n$.

Notice that V also commutes with the unitary group $\{U_t : t \in \mathbb{R}\}$. Indeed, from the definitions of U_t and J we have

$$J U_t \theta(\omega) = J \theta(\hat{\sigma}_t \omega) = \theta((\hat{\sigma}_t \omega)^\#).$$

But for real t we have

$$(\hat{\sigma}_t \omega)^\# = \hat{\sigma}_{i/2}((\hat{\sigma}_t \omega)^*) = \hat{\sigma}_{i/2} \hat{\sigma}_t(\omega^*) = \hat{\sigma}_t(\hat{\sigma}_{i/2}(\omega^*)) = \hat{\sigma}_t(\omega^\#).$$

Hence the right side of the previous equation is

$$\theta(\hat{\sigma}_t(\omega^\#)) = U_t \theta(\omega^\#) = U_t J \theta(\omega).$$

We conclude that $J U_t = U_t J$. On the other hand, from the representation of U_t achieved in (7.13.3) we find that

$$\begin{aligned} J_n U_t \bar{z} &= J_n (e^{i\lambda_1 t} \sigma_t(z_1), \dots, e^{i\lambda_n t} \sigma_t(z_n)) \\ &= (\sigma_{i/2}(e^{-i\lambda_1 t} \sigma_t(z_1^*)), \dots, \sigma_{i/2}(e^{-i\lambda_n t} \sigma_t(z_n^*))) \\ &= (\sigma_t((e^{i\lambda_1 t} \sigma_t(z_1))^\#), \dots, \sigma_t((e^{i\lambda_n t} \sigma_t(z_n))^\#)) = U_t J_n \bar{z}. \end{aligned}$$

Thus U_t commutes with both J and J_n , and therefore it must commute with $V = J J_n$.

We now show that any “reasonable” square root W of V has the properties (7.14) and (7.15). Indeed, let

$$V = \sum_{k=1}^r \lambda_k E_k$$

be the spectral decomposition of V , where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of V and E_1, \dots, E_r are its minimal spectral projections. Set

$$W = \sum_{k=1}^r \lambda_k^{1/2} E_k,$$

where $\lambda_k^{1/2}$ denotes either square root of λ_k . It is clear that W commutes with both $\pi(A)$ and $\overset{\circ}{\pi}(A)$, and W commutes with each U_t because it belongs to the C^* -algebra generated by the operator $V \in \{U_t : t \in \mathbb{R}\}'$. Since $(U J_n)^2 = J^2 = \mathbf{1}$ it follows that $J_n U J_n = U^{-1}$. Now the operator mapping

$$\alpha : T \mapsto J_n T^* J_n$$

is a $*$ -antiautomorphism of the von Neumann algebra $\mathcal{B}(A^n)$ which carries the abelian C^* -algebra $C^*(U)$ generated by U onto itself in such a way that $\alpha(U) = U$. It follows that $\alpha(T) = T$ for all $T \in C^*(U)$. In particular $\alpha(W) = W$, i.e., $J_n W^{-1} J_n = W$, and therefore $U = W^2 = W J_n W^{-1} J_n$. Finally, we see that $J = U J_n = W J_n W^{-1}$, as required \square

We now describe θ in terms of the coordinates of Lemma 7.13

Lemma 7.18. *Considering A^n as a bimodule over A in the usual way,*

$$a\bar{z}b = (az_1b, \dots, az_nb), \quad \bar{z} \in A^n,$$

we have $\theta(a\omega b) = a\theta(\omega)b$, $a, b \in A$, $\omega \in \Omega^1$.

proof. We have $\theta(a\omega) = \pi(a)\theta(\omega) = a\theta(\omega)$, by (7.13.1). For the right action we have

$$\begin{aligned} \theta(\omega b) &= J\theta((\omega b)^\#) = J\theta(\delta^{1/2}(b^*)\delta^{1/2}(\omega^*)) = J\pi(\delta^{1/2}(b^*))\theta(\omega^\#) \\ &= J\pi(\delta^{-1/2}(b)^*)J\theta(\omega) = \overset{\circ}{\pi}(\delta^{-1/2}(b))\theta(\omega). \end{aligned}$$

Notice that $\overset{\circ}{\pi}(\delta^{-1/2}(b))$ gives the (untwisted) right module action of b , since by (7.13.2)

$$\overset{\circ}{\pi}(\delta^{-1/2}(b))\bar{z} = (z_1b, \dots, z_nb),$$

$\bar{z} \in A^n$. Hence $\theta(\omega b) = \theta(\omega)b$ \square

It follows immediately from Lemma 7.18 that $x \mapsto \theta(dx)$ is a derivation of A into the bimodule A^n , and hence there are derivations D_1, \dots, D_n of A such that

$$\theta(a dx) = (aD_1(x), \dots, aD_n(x)), \quad a, x \in A.$$

Lemma 7.19. *The derivations D_1, \dots, D_n are linearly independent.*

proof. Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are such that

$$\sum_{k=1}^n \lambda_k D_k(x) = 0, \quad x \in A.$$

Then for every $a, x \in A$ we have

$$\sum_{k=1}^n \lambda_k a D_k(x) = 0.$$

It follows that the linear operator $L : A^n \rightarrow A$ defined by $L(\bar{z}) = \sum_k \lambda_k z_k$ satisfies $L(\theta(a dx)) = 0$ for all $a, x \in A$; hence $L = 0$ on the range of θ . Since $\theta(\Omega^1) = H_g = A^n$, L must be 0 \square

The remainder of the proof is devoted to showing that $D_k(a)^* = -D_k(a^*)$ and $\rho \circ D_k = 0$, $1 \leq k \leq n$. This is done in two steps, the first of which is the following.

Lemma 7.20. *The derivations D_1, \dots, D_n have the form $D_k(a) = [x_k h^{-1/2}, a]$, where $x_k = x_k^*$ and h is the density matrix of the state ρ .*

proof. Applying (7.13.3) to the n -tuple $\bar{z} = (D_1(x), \dots, D_n(x))$ and using $J\theta(dx) = \theta((dx)^\#) = -\theta(d\delta^{1/2}(x^*))$ we obtain

$$-D_k(\delta^{1/2}(x^*)) = \delta^{1/2}(D_k(x)^*), \quad 1 \leq k \leq n.$$

After choosing elements $z_k \in A$ so that $D_k(x) = [z_k, x]$ and unravelling the preceding formula, one finds that for all $x \in A$ one has

$$[\delta^{1/2}(x^*), [z_k, x]] = [z_k, \delta^{1/2}(x^*)]$$

Thus $z_k - \delta^{1/2}(z_k^*)$ commutes with everything in A , and must be a scalar multiple of $\mathbf{1}$ for every $1 \leq k \leq n$.

Now if $z \in A$ and $\lambda \in \mathbb{C}$ are such that $z - \delta^{1/2}(z^*) = \lambda \mathbf{1}$, then by applying the normalized trace τ on A we find that $\lambda = \tau(z) - \overline{\tau(z)}$ is imaginary. Writing $\lambda = i\lambda_0$ where $\lambda_0 \in \mathbb{R}$ we can replace z with $z_0 = z - \frac{i\lambda_0}{2}\mathbf{1}$ and we find that $z_0 - \delta^{1/2}(z_0^*) = 0$. Noting that $\delta^{1/2}(x) = h^{1/2}xh^{-1/2}$, the latter implies that $x = z_0h^{-1/2}$ is self-adjoint, and hence

$$z = xh^{-1/2} + \frac{i\lambda_0}{2}\mathbf{1}.$$

The required representation of z_1, \dots, z_n follows \square

Now by definition of U_t we have

$$U_t(D_1(x), \dots, D_n(x)) = U_t(\theta(dx)) = \theta(d\sigma_t(x)) = (D_1(\sigma_t(x)), \dots, D_n(\sigma_t(x))),$$

while from (7.13.3) we have

$$U_t(D_1(x), \dots, D_n(x)) = (e^{i\lambda_1 t}\sigma_t(D_1(x)), \dots, e^{i\lambda_n t}\sigma_t(D_n(x))).$$

It follows that

$$\sigma_t(D_k(x)) = e^{-i\lambda_k t}D_k(\sigma_t(x)), \quad 1 \leq k \leq n, t \in \mathbb{R}.$$

By (7.20), D_k has the form $D_k(x) = [x_k h^{-1/2}, x]$, and thus

$$\sigma_t(x_k h^{-1/2}) - e^{-i\lambda_k t}x_k h^{-1/2}$$

must be a scalar. The following lemma shows that this implies that x_k must commute with h .

Lemma 7.21. *Let $\lambda \in \mathbb{R}$ and let x be a self adjoint element of A such that*

$$\sigma_t(xh^{-1/2}) - e^{i\lambda t}xh^{-1/2}$$

is a scalar for every $t \in \mathbb{R}$. Then $\lambda = 0$ and x commutes with h .

proof. For every $t \in \mathbb{R}$ let $\mu(t)$ be the complex number defined by

$$\sigma_t(xh^{-1/2}) - e^{i\lambda t}xh^{-1/2} = \mu(t)\mathbf{1}.$$

Applying the normalized trace τ of A to both sides of this equation we find that for $c = \tau(xh^{-1/2})$ we have $\mu(t) = c(1 - e^{i\lambda t})$. Thus

$$\sigma_t(xh^{-1/2} - c\mathbf{1}) = e^{i\lambda t}(xh^{-1/2} - c\mathbf{1}).$$

After multiplying through on the right by $h^{1/2}$ we find that

$$(7.22) \quad \sigma_t(xh^{-1/2} - c\mathbf{1})h^{1/2} = e^{i\lambda t}(xh^{-1/2} - c\mathbf{1})h^{1/2}, \quad t \in \mathbb{R}$$

Now $c = \tau(xh^{-1/2}) = \tau(h^{-1/4}xh^{-1/4})$ is a real number because x is a self-adjoint operator, and thus the operator $x - ch^{1/2}$ appearing in (7.22) is self-adjoint. The left side of (7.22) is therefore a self-adjoint operator for every $t \in \mathbb{R}$, and hence (7.22) implies that $\lambda = 0$ except in the trivial case where $x = ch^{1/2}$.

In either case it follows that $x - ch^{1/2}$ is fixed under the action of σ_t , $t \in \mathbb{R}$, and hence x must commute with h \square

Applying Lemma 7.21 to the derivations D_k we find that $D_k(a) = [x_k h^{-1/2}, a] = [h^{-1/4}x_k h^{-1/4}, a]$ is implemented by the self-adjoint operator $y_k = h^{-1/4}x_k h^{-1/4}$, where y_k commutes with h . By replacing y_k with $y_k - \rho(y_k)\mathbf{1}$ we can also assume that $\rho(y_k) = 0$. That completes the proof of all but the uniqueness assertion of Theorem 7.6.

For uniqueness, let x'_1, \dots, x'_n be a linearly independent set of self-adjoint operators satisfying $\delta(x'_k) = x'_k$, $\rho(x'_k) = 0$, and

$$(7.23) \quad g(a \, dx \, dy) = \sum_{k=1}^m \rho(a[x'_k, x][x'_k, y])$$

for all $a, x, y \in A$. Define a linear map $\theta' : \Omega^1 \rightarrow A^m$ by

$$(7.24) \quad \theta'(a \, dx) = (a[x'_1, x], \dots, a[x'_m, x]).$$

If we make A^m into a Hilbert space $m \cdot L^2(A, \rho)$ as in Lemma 7.13, and into a bimodule as in Lemma 7.18, then we find that

$$(7.25) \quad \theta'(a\omega b) = a\theta'(\omega)b, \quad a, b \in A, \omega \in \Omega^1.$$

We claim that $\theta'(\Omega^1) = A^m$. For that, note that (7.25) implies that the subspace $\theta'(\Omega^1)$ is invariant under left and right multiplication by elements of A , and hence the projection P onto the range of θ' must have the form $P = (p_{ij}\mathbf{1})$ where (p_{ij}) is an $m \times m$ matrix of complex scalars. If $P \neq \mathbf{1}$ then $\mathbf{1} - P$ must have a nonzero row with entries μ_1, \dots, μ_m . Since $(\mathbf{1} - P)\theta'(dx) = 0$ for every $x \in A$ it follows that

$$\mu_1[x'_1, x] + \dots + \mu_m[x'_m, x] = 0$$

for every $x \in A$ and hence $\sum \mu_k x'_k$ must be a scalar multiple $\nu\mathbf{1}$ of $\mathbf{1}$. However, since $\rho(x'_k) = 0$ and ρ is a state, we must have $\nu = 0$. Hence $\sum \mu_k x'_k = 0$, contradicting linear independence.

Now the formulas (7.23) and (7.24) imply that for all $\omega_1, \omega_2 \in \Omega^1$,

$$\langle \theta'(\omega_1), \theta'(\omega_2) \rangle = g(\omega_2^\# \omega_1) = \langle \theta(\omega_1), \theta(\omega_2) \rangle.$$

Thus we can define a unique unitary operator $W : A^n \rightarrow A^m$ by

$$W\theta(\omega) = \theta'(\omega), \quad \omega \in \Omega^1.$$

Since W is unitary and both θ and θ' are bimodule homomorphisms it follows that W must implement a $*$ isomorphism of the commutant of $\pi(A) \cup \overset{\circ}{\pi}(A)$ onto that of $\pi'(A) \cup \overset{\circ}{\pi}'(A)$. Since these commutants are factors of type I_n and I_m respectively, we conclude that $m = n$ and that W commutes with both the left and right module

actions of A on A^n . It follows that there is an $n \times n$ scalar unitary matrix (w_{ij}) such that $W = (w_{ij}\mathbf{1})$.

We claim that the scalars w_{ij} are real numbers, so that (w_{ij}) is a real orthogonal $n \times n$ matrix. Indeed, if J_n is the standard involution of A^n defined in (7.13.4) then we have

$$\begin{aligned} J_n \theta(\omega) &= \theta(\omega^\#), & \text{and} \\ J_n \theta'(\omega) &= \theta'(\omega^\#). \end{aligned}$$

It follows that

$$W J_n \theta(\omega) = W \theta(\omega^\#) = \theta'(\omega^\#) = J_n \theta'(\omega) = J_n W \theta(\omega), \quad \omega \in \Omega^1,$$

and hence $J_n W = W J_n$. Noting that the action of J_n on scalar operator matrices obeys

$$J_n(\lambda_{ij}\mathbf{1})J_n^{-1} = (\bar{\lambda}_{ij}\mathbf{1}),$$

we conclude that $\bar{w}_{ij} = w_{ij}$ for all i, j , as asserted.

Finally, since $\theta'(dx) = W\theta(dx)$, we have

$$[x'_k, x] = \sum_{j=1}^n w_{kj} [x_j, x] = \left[\sum_j w_{kj} x_j, x \right]$$

for all $x \in A$ it follows that each operator

$$x'_k - \sum_{k=1}^n w_{kj} x_j$$

must be a scalar multiple of $\mathbf{1}$. Since $\rho(x_k) = \rho(x'_k) = 0$ and ρ is a state, the scalars must be 0 and we have the asserted relation

$$x'_k = \sum_{k=1}^n w_{kj} x_j, \quad 1 \leq k \leq n.$$

That completes the proof of Theorem 7.6 \square

Theorem 7.6 is applied to the classification of semielliptic operators as follows.

Theorem 7.26. *Let $L \in \mathcal{D}(A, \rho)$ be an exact semielliptic operator. Then there is a unique momentum space $(P, \langle \cdot, \cdot \rangle)$ and a unique skew-adjoint operator v such that $\delta(v) = v$, $\rho(v) = 0$, and*

$$(7.26.1) \quad L = \Delta + [v, \cdot],$$

Δ denoting the Laplacian of $(P, \langle \cdot, \cdot \rangle)$.

Conversely, any operator L of the form (7.26.1) is an exact semielliptic operator.

proof. Since L is an exact semielliptic operator its symbol σ_L satisfies condition (iii) of Theorem 6.9, hence

defines a KMS metric on Ω^2 . By Theorem 7.6A there is a unique momentum space $(P, \langle \cdot, \cdot \rangle)$ whose Laplacian Δ satisfies

$$2g = -\sigma_\Delta.$$

The symbol of $L - \Delta$ therefore vanishes, so that $D = L - \Delta$ is a symmetric derivation for which $\rho \circ D = 0$. We can find a skew-adjoint operator $v \in A$ for which $D(x) = [v, x]$, and the condition $\rho \circ D = 0$ implies that v commutes with the density matrix of ρ . Replacing v with $v - \rho(v)\mathbf{1}$, we obtain (7.26.1).

The converse was established in Corollary 6.13.

Exact semielliptic operators are classified in terms of their momenta and potentials as follows.

Theorem 7.27. *Let L_1, L_2 be two exact semielliptic operators in $\mathcal{D}(A, \rho)$ with momentum spaces $(P_k, \langle \cdot, \cdot \rangle_k)$ and natural decompositions $L_k = \Delta_k + [v_k, \cdot]$, $k = 1, 2$. Then L_1 and L_2 are conjugate iff there is a $*$ -automorphism α of A such that $\rho \circ \alpha = \rho$, and*

$$(7.27.1) \quad \alpha(P_1) = P_2,$$

$$(7.27.2) \quad \langle \alpha(p), \alpha(q) \rangle_2 = \langle p, q \rangle_1, \quad p, q \in P_1$$

$$(7.27.3) \quad \alpha(v_1) = v_2.$$

proof. Assume that L_1 and L_2 are conjugate. Thus there is a $*$ -automorphism α of A such that $\rho \circ \alpha = \rho$ and $\alpha \circ L_1 = L_2 \circ \alpha$. Since α preserves ρ , it induces a unitary operator on $L^2(A, \rho)$ in the natural way:

$$U_\alpha : x \mapsto \alpha(x), \quad x \in A,$$

and we have $U_\alpha L_1 U_\alpha^{-1} = L_2$. Hence $U_\alpha(L_1 + L_1^*)U_\alpha^{-1} = L_2 + L_2^*$. Since $\Delta_k = L_k + L_k^*$ it follows that $\alpha \Delta_1 \alpha^{-1} = \Delta_2$. Hence α must induce an isomorphism of momentum spaces in the sense specified in (7.27.1) and (7.27.2).

Setting $D_k = [v_k, \cdot]$, we have

$$\alpha \circ D_1 \circ \alpha^{-1} = \alpha \circ (L_1 - \Delta_1) \circ \alpha^{-1} = L_2 - \Delta_2 = D_2,$$

so that for all x in A we have $\alpha([v_1, \alpha^{-1}(x)]) = [v_2, x]$. The latter implies that $\alpha(v_1) - v_2$ is a scalar; and since $\rho(v_2) = \rho(\alpha(v_1)) = \rho(v_1) = 0$, we deduce (7.27.3).

The proof of the converse is similar \square

Remarks. Of course, the $*$ automorphism α can be implemented by a unitary operator.

Notice too that the three invariants in (7.26.1)–(7.26.3) that classify exact semielliptic operators can be specified independently of one another. For example, starting with any such operator L one may scale the metric $\langle \cdot, \cdot \rangle$ on P to obtain a family of mutually non-conjugate operators, or one can scale the potential v to obtain a second family of mutually non-conjugate operators.

It also makes sense to speak of semielliptic operators that are “free” in the sense that their potential operator v is zero.

8. Applications. The preceding results lead directly to a new classification of E_0 -semigroups. For that one considers pairs (α, ω) consisting of an E_0 -semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H_\alpha)$ and a normal state ω of $\mathcal{B}(H_\alpha)$ satisfying

$$\omega \circ \alpha_t = \omega, \quad t \geq 0.$$

Two such pairs (α, ω) and (α', ω') are said to be *conjugate* if there is a $*$ -isomorphism $\theta : \mathcal{B}(H_\alpha) \rightarrow \mathcal{B}(H_{\alpha'})$ such that

$$(8.1.1) \quad \theta \circ \alpha_t = \alpha' \circ \theta, \quad \text{and}$$

$$(8.1.2) \quad \omega' \circ \theta = \omega.$$

Proposition 8.2. *Assume that θ is a $*$ -isomorphism satisfying (8.1.1) and (8.1.2) and let p_0 (resp. p'_0) be the support projection of ω (resp. ω'). Let $H_0 = p_0 H_\alpha$, $H'_0 = p'_0 H_{\alpha'}$ and let (ϕ, ρ) , (ϕ', ρ') be the Markov semigroups obtained by compressing α , α' to $\mathcal{B}(H_0)$, $\mathcal{B}(H'_0)$ as in §1:*

$$\begin{aligned} \phi_t(a) &= p_0 \alpha_t(a p_0) p_0, \\ \phi'_t(b) &= p'_0 \alpha'_t(b p_0) p'_0, \quad t \geq 0, \end{aligned}$$

where ρ , ρ' are obtained from ω , ω' by restriction. Then the restriction θ_0 of θ to $\mathcal{B}(H_0)$ implements a conjugacy of (ϕ, ρ) and (ϕ', ρ') .

proof. The argument is straightforward once one observes that the condition $\omega' \circ \theta = \omega$ implies that $\theta(p_0) = p'_0$ and $\rho' \circ \theta_0 = \rho$ \square

Definition 8.3. *An E_0 -semigroup pair (α, ω) is said to be of finite type if the support projection of ω is finite dimensional.*

If (α, ω) is of finite type and (ϕ, ρ) is its natural Markov semigroup obtained by compression as in §1 then we may consider the generator L of ϕ ,

$$(8.4) \quad \phi_t = \exp(tL), \quad t \geq 0.$$

L is a semielliptic operator in $\mathcal{D}(\mathcal{B}(H_0), \rho)$ and it is sensible to ask if L is exact.

Definition 8.5. *(α, ω) is called exact if it is of finite type and the generator L of its associated Markov semigroup is an exact semielliptic operator.*

Notice that if (α, ω) is exact then the generator L of (8.4) decomposes uniquely as in Theorem 7.6,

$$L = \Delta + [v, \cdot]$$

where Δ is the Laplacian of a unique momentum space $(P, \langle \cdot, \cdot \rangle)$ and v is a skew-adjoint operator satisfying $\rho(v) = 0$ and $\rho \circ [v, \cdot] = 0$. The triple $(P, \langle \cdot, \cdot \rangle, v)$ is called the *dynamical invariant* of (α, ω) . It is truly an invariant because of the following result.

Theorem 8.6. *If two exact E_0 -semigroup pairs (α, ω) and (α', ω') are conjugate then their dynamical invariants $(P, \langle \cdot, \cdot \rangle, v)$ and $(P', \langle \cdot, \cdot \rangle, v')$ are isomorphic in*

the sense that there is a $*$ -isomorphism $\alpha : \mathcal{B}(H_0) \rightarrow \mathcal{B}(H'_0)$ which satisfies the conditions of Theorem 7.27:

$$\begin{aligned}\omega' \circ \alpha &= \omega \\ \alpha(P) &= P' \\ \langle \alpha(p), \alpha(q) \rangle' &= \langle p, q \rangle, \quad p, q \in P \\ \alpha(v) &= v'.\end{aligned}$$

proof. Given the preceding remarks, the argument is a simple variation of what was done in the proof of Theorem 7.27 \square

Remarks. We have already pointed out in §1 that the converse of Theorem 8.6 is false in general, but that it is true if both α and α' are *minimal*. More generally, we can paraphrase a recent dilation theorem of B. V. R. Bhat [6] in our context as follows.

Theorem 8.7. *Let (ϕ, ρ) be a Markov semigroup acting on a separable Hilbert space H_0 . Then there is a minimal E_0 -semigroup α acting on a Hilbert space $H \supseteq H_0$ and a normal state ω on $\mathcal{B}(H)$ such that $\omega \circ \alpha_t = \omega$ for every $t \geq 0$ and such that (ϕ, ρ) is obtained from (α, ω) by compression as described above.*

If (ϕ', ρ') is another Markov semigroup which is conjugate to (ϕ, ρ) and which gives rise to a minimal E_0 -semigroup pair (α', ω') then (α, ω) and (α', ω') are conjugate.

Remark 8.8. Taken together, Theorems 8.6 and 8.7 imply that exact *minimal* E_0 -semigroup pairs (α, ω) are completely classified up to conjugacy by their dynamical invariants $(P, \langle \cdot, \cdot \rangle, v)$. More explicitly, suppose we start with a pair (A, ρ) consisting of a faithful state ρ on $A = \mathcal{B}(H_0)$, where H_0 is a finite dimensional Hilbert space. Choose an arbitrary dynamical invariant $(P, \langle \cdot, \cdot \rangle, v)$ in (A, ρ) and form its corresponding semielliptic operator $L \in \mathcal{D}(A, \rho)$:

$$L(x) = \Delta(x) + [v, x], \quad x \in A,$$

Δ being the Laplacian of $(P, \langle \cdot, \cdot \rangle)$. One exponentiates L to obtain $\phi_t = \exp(tL)$ and a Markov semigroup (ϕ, ρ) . Finally, one obtains a minimal E_0 -semigroup pair (α, ω) by the dilation procedure of Theorem 8.7. Such an (α, ω) is an exact pair whose dynamical invariant is isomorphic to $(P, \langle \cdot, \cdot \rangle, v)$. Every exact pair arises in this way, and two exact pairs are conjugate if and only if their corresponding dynamical invariants are isomorphic.

We infer from these remarks that the dynamical invariant $(P, \langle \cdot, \cdot \rangle, v)$ consists of infinitesimal structures that completely determine the associated flow (α, ω) . Thus dynamical invariants are analogous to the differential equations that govern the behavior of classical mechanical systems. There are obvious implications for the classification of type *I* primary histories.

Finally, it is possible to determine which triples $(P, \langle \cdot, \cdot \rangle, v)$ lead to “mixing” properties for their associated E_0 -semigroups. By that we mean that all limits of the form

$$\lim_{t \rightarrow \infty} \phi(\alpha_t(a)), \quad a \in \mathcal{B}(H_\alpha), \phi \in \mathcal{B}(H_\alpha)_*$$

should exist. These issues will be taken up elsewhere.

REFERENCES

1. Arveson, W., *Continuous analogues of Fock space*, Memoirs Amer. Math. Soc. **80 no. 3** (1989).
2. ———, *The harmonic analysis of automorphism groups*, Operator algebras and applications, Part 1, Proc. Symp. Pure Math., vol. 38, AMS, 1982. (Tannu Tuvan)
3. ———, *Continuous analogues of Fock space IV: essential states*, Acta Math. **164** (1990), 265–300.
4. ———, *An addition formula for the index of semigroups of endomorphisms of $\mathcal{B}(H)$* , Pac. J. Math. **137** (1989), 19–36.
5. ———, *Quantizing the Fredholm index*, Operator Theory: Proceedings of the 1988 GPOTS-Wabash conference (Conway, J. B. and Morrel, B. B., ed.), Pitman research notes in mathematics series, Longman, 1990.
6. Bhat, B. V. R., *Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^* -algebras*, Trans. A.M.S. (to appear).
7. ———, *On minimality of Evans-Hudson flows*, (preprint).
8. Chebotarev, A. M., Fagnola, F., *Sufficient conditions for conservativity of quantum dynamical semigroups*, J. Funct. Anal. (1993), 131–153.
9. Connes, A., *Non Commutative Geometry*, Academic Press, 1994.
10. Davies E. B., *Quantum Theory of Open Systems*, Academic Press, 1976.
11. Davies E. B., *Generators of dynamical semigroups*, J. Funct. Anal. **34** (1979), 421–432.
12. Evans, D., Quart J. Math. Oxford **28 yr 1977**, 369.
13. ———, *Quantum dynamical semigroups, symmetry groups, and locality*, Acta Appl. Math. **2** (1984), 333–352.
14. Evans, D. and Lewis, J. T., *Dilations of irreversible evolutions in algebraic quantum theory*, Comm. Dubl. Inst. Adv. Studies, Ser A **24** (1977).
15. Gorini, V., Kossakowski, A. and Sudarshan, E. C. G., *Completely positive semigroups on N -level systems*, J. Math. Phys. **17** (1976), 821–825.
16. Haag, R., *Local Quantum Physics*, Springer-Verlag, Berlin, 1992.
17. Hudson, R. L. and Parthasarathy, K. R., *Stochastic dilations of uniformly continuous completely positive semigroups*, Acta Appl. Math. **2** (1984), 353–378.
18. Kümmerer, B., *Markov dilations on W^* -algebras*, J. Funct. Anal. **63** (1985), 139–177.
19. ———, *Survey on a theory of non-commutative stationary Markov processes*, Quantum Probability and Applications III, vol. 1303, Springer Lecture notes in Mathematics, 1987, pp. 154–182.
20. Lindblad, G., *On the generators of quantum dynamical semigroups*, Comm. Math. Phys. **48** (1976), 119.
21. Mohari, A., Sinha, Kalyan B., *Stochastic dilation of minimal quantum dynamical semigroups*, Proc. Ind. Acad. Sci. **102** (1992), 159–173.
22. Parthasarathy, K. R., *An introduction to quantum stochastic calculus*, Birkhäuser Verlag, Basel, 1991.
23. Pedersen, G. K., *C^* -algebras and their automorphism groups*, Academic Press, 1979.
24. Powers, R. T., *An index theory for semigroups of endomorphisms of $\mathcal{B}(H)$ and type II factors*, Can. J. Math. **40** (1988), 86–114.
25. ———, *A non-spatial continuous semigroup of $*$ -endomorphisms of $\mathcal{B}(H)$* , Publ. RIMS (Kyoto University) **23** (1987), 1053–1069.
26. ———, *New examples of continuous spatial semigroups of endomorphisms of $\mathcal{B}(H)$* , (preprint 1994).
27. Powers, R. T. and Price, G., *Continuous spatial semigroups of $*$ -endomorphisms of $\mathcal{B}(H)$* , Trans. A. M. S. **321** (1990), 347–361.