THE CURVATURE INVARIANT OF A HILBERT MODULE OVER $\mathbb{C}[z_1, \dots, z_d]$

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Abstract. A notion of curvature is introduced in multivariable operator theory, that is, for commuting d tuples of operators acting on a common Hilbert space whose "rank" is finite in an appropriate sense.

The curvature invariant is a real number in the interval $[0, r]$ where r is the rank, and for good reason it is desireable to know its value. For example, there are significant and concrete consequences when it assumes either of the two extreme values 0 or r. In the few simple cases where it can be calculated directly, it turns out to be an integer. This paper addresses the general problem of computing this invariant.

Our main result is an operator-theoretic version of the Gauss-Bonnet-Chern formula of Riemannian geometry. The proof is based on an asymptotic formula which expresses the curvature of a Hilbert module as the trace of a certain self-adjoint operator. The Euler characteristic of a Hilbert module is defined in terms of the algebraic structure of an associated finitely generated module over the algebra of complex polynomials $\mathbb{C}[z_1, \dots, z_d]$, and the result is that these two numbers are the same for graded Hilbert modules. Thus the curvature of such a Hilbert module is an integer; and since there are standard tools for computing the Euler characteristic of finitely generated modules over polynomial rings, the problem of computing the curvature can be considered solved in these cases.

The problem of computing the curvature of ungraded Hilbert modules remains open.

Contents

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Introduction.

In a recent paper [1] the author showed that the simplest form of von Neumann's inequality fails for the unit ball of $\mathbb{C}^d$ when $d \geq 2$, and that consequently the traditional approach to dilation theory (based on normal dilations) is inappropriate for multivariable operator theory in dimension greater than 1. A modification of dilation theory was proposed for higher dimensions, and that modification involves a particular commuting d-tuple of operators called the d-shift in an essential way. The d-shift is not subnormal and does not satisfy von Neumann's inequality.

This reformulation of dilation theory bears a strong resemblance to the algebraic theory of finitely generated modules over polynomial rings, originating with Hilbert's work of the 1890s [18],[19]. For example, the module structure defined by the d-shift occupies the position of the rank-one free module in the algebraic theory. On the other hand, since we are working with bounded operators on Hilbert spaces (rather than linear transformations on vector spaces) there are also geometric aspects that accompany this additional structure. In particular, it is possible to define a numerical invariant (the curvature) for appropriate Hilbert modules over $\mathbb{C}[z_1, \dots, z_d]$. This is a new invariant in operator theory, analogous to the integral of the Gaussian curvature of a compact oriented Riemannian $2n$-manifold.

The curvature invariant $K(H)$ takes values in the interval $[0, r]$ where r is the rank of H. Both extremal values $K(H) = r$ and $K(H) = 0$ have significant operator-theoretic implications. We show in section 2 that for pure Hilbert modules H, the curvature invariant is maximal $K(H) = \operatorname{rank}(H)$ iff H is the free Hilbert module of rank $r = \operatorname{rank}(H)$ (the free Hilbert module of rank r is the module defined by the orthogonal direct sum of r copies of the d-shift, see Remark 1.3 below). The opposite extreme $K(H) = 0$ is closely related to the existence of "inner sequences" for the invariant subspaces of H^2. More precisely, a closed submodule $M \subseteq H^2$ is associated with an "inner sequence" iff $K(H^2/M) = 0$, H^2/M denoting the quotient Hilbert module.

If one seeks to make use of these extremal properties one obviously must calculate $K(H)$. But direct computation appears to be difficult for most of the natural examples, and in the few cases where the computations can be explicitly carried out the curvature turns out to be an integer. Thus we were led to ask if the curvature invariant can be expressed in terms of some other invariant which is a) obviously an integer and b) easier to calculate.

We establish such a formula in section 5 (Theorem B), which applies to Hilbert modules (in the category of interest) which are "graded" in the sense that the d-tuple of operators which defines the module structure should be circularly symmetric. Theorem B asserts that the curvature of such a Hilbert module agrees with the Euler characteristic of a certain finitely generated *algebraic* module that is associated with it in a natural way. Since the Euler characteristic of a finitely generated module over $\mathbb{C}[z_1, \dots, z_d]$ is relatively easy to compute using conventional algebraic methods, the problem of calculating the curvature can be considered solved for graded Hilbert modules.

The problem of calculating the curvature of *ungraded* finite rank Hilbert modules remains open (additional concrete problems are discussed in section 7).

Theorem B is proved by establishing asymptotic formulas for both the curvature and Euler characteristic of arbitrary (i.e., perhaps ungraded) Hilbert modules (Theorems C and D), the principal result being Theorem C. Theorem C is proved by showing that the curvature invariant is actually the trace of a certain self-adjoint trace class operator, and we prove an appropriate asymptotic formula for the trace of that operator in section 3. These results have been summarized in [2].

Cowen and Douglas have introduced a geometric notion of curvature for certain operators whose adjoints have "sufficiently many" eigenvectors [8]. The Cowen-Douglas curvature operator is associated with a Hermitian vector bundle over a bounded domain in $\mathbb{C}$. This bundle is constructed by organizing the eigenvector-eigenvalue information attached to the operator. Our work differs from that of Cowen and Douglas in three ways. First, we are primarily interested in the multivariable case where one is given several mutually commuting operators. Second, we concentrate on higher dimensional analogues of contractions...modules whose geometry is associated with the unit ball of $\mathbb{C}^d$. Third, we make no other geometric assumptions on the Hilbert module beyond that of being "finite rank" in an appropriate sense (more precisely, the defect operator associated with the Hibert module should be of finite rank). Cowen and Douglas have extended some of their results to multivariable cases (see [9], [10]), but the overlap between the two approaches is slight. We also point out that in [4], [22], [23] Misra, Bagchi, Pati, and Sastry have studied d-tuples of operators that are invariant under a group action. The connection between our work and the latter is not completely understood, but again the two approaches are fundamentally different. Finally, though the curvature of a Hilbert module is a global invariant, it may also be appropriate to call attention to two recent papers [13], [14] which deal with the *local* properties of short exact sequences of Hilbert modules.

We now describe our results more precisely, beginning with the definition of the curvature invariant. Let $\bar{T} = (T_1, \dots, T_d)$ be a d-tuple of mutually commuting operators acting on a common Hilbert space H. $\bar{T}$ is called a *d-contraction* if

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2$$

for all $\xi_1, \dots, \xi_d \in H$. The number d will normally be fixed, and of course we are primarily interested in the cases $d \geq 2$. Let $A = \mathbb{C}[z_1, \dots, z_d]$ be the complex unital algebra of all polynomials in d commuting variables $z_1, \dots, z_d$. A commuting d-tuple $T_1, \dots, T_d$ of operators in the algebra $\mathcal{B}(H)$ of all bounded operators on H gives rise to an A-module structure on H in the natural way,

$$f \cdot \xi = f(T_1, \dots, T_d)\xi, \qquad f \in A, \quad \xi \in H;$$

and $(T_1, \dots, T_d)$ is a d-contraction iff H is a *contractive* A-module in the following sense,

$$\|z_1\xi_1 + \cdots + z_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2$$

for all $\xi_1, \dots, \xi_d \in H$. Thus it is equivalent to speak of d-contractions or of contractive Hilbert A-modules, and we will shift from one point of view to the other when it is convenient to do so.

For every d-contraction $\bar{T} = (T_1, \dots, T_d)$ we have $0 \leq T_1T_1^* + \cdots + T_dT_d^* \leq \mathbf{1}$, and hence the "defect operator"

$$\Delta = (\mathbf{1} - T_1T_1^* - \cdots - T_dT_d^*)^{1/2} \tag{0.1}$$

is a positive operator on H of norm at most one. The *rank* of $\bar{T}$ is defined as the dimension of the range of Δ. Throughout this paper we will be primarily concerned with finite rank d-contractions (resp. finite rank contractive Hilbert A-modules).

Let H be a finite rank contractive Hilbert A-module and let $(T_1, \dots, T_d)$ be its accociated d-contraction. For every point $z = (z_1, \dots, z_d)$ in complex d-space $\mathbb{C}^d$ we form the operator

$$T(z) = \bar{z}_1 T_d + \cdots + \bar{z}_d T_d \in \mathcal{B}(H), \tag{0.2}$$

$\bar{z}_k$ denoting the complex conjugate of the complex number z_k. Notice that the operator function $z \mapsto T(z)$ is an antilinear mapping of $\mathbb{C}^d$ into $\mathcal{B}(H)$, and since $(T_1, \dots, T_d)$ is a d-contraction we have

$$\|T(z)\| \leq |z| = (|z_1|^2 + \cdots + |z_d|^2)^{1/2}$$

for all $z \in \mathbb{C}^d$. In particular, if z belongs to the open unit ball

$$B_d = \{z \in \mathbb{C}^d : |z| < 1\}$$

then $\|T(z)\| < 1$ and $\mathbf{1} - T(z)$ is invertible. Thus for every $z \in B_d$ we can define a positive operator $F(z)$ acting on the finite dimensional Hilbert space ΔH as follows,

$$F(z)\xi = \Delta(\mathbf{1} - T(z)^*)^{-1}(\mathbf{1} - T(z))^{-1}\Delta\xi, \qquad \xi \in \Delta H.$$

In order to define the curvature invariant $K(H)$ we require the boundary values of the real-valued function $z \in B_d \mapsto \operatorname{trace} F(z)$. These do not exist in a conventional sense because in all significant cases this function is unbounded. However, we show that "renormalized" boundary values do exist almost everywhere on the sphere ∂B_d with respect to the natural rotation-invariant probability measure σ on ∂B_d.

Theorem A. *For σ-almost every $\zeta \in \partial B_d$, the limit*

$$K_0(\zeta) = \lim_{r \uparrow 1}(1 - r^2) \mathit{trace}\, F(r\zeta) = 2 \cdot \lim_{r \uparrow 1}(1 - r) \mathit{trace}\, F(r\zeta)$$

exists and satisfies $0 \leq K_0(\zeta) \leq \mathit{rank}(H)$.

Section 1 is devoted to the proof of Theorem A. The curvature invariant is defined by averaging K_0 over the sphere

$$K(H) = \int_{\partial B_d} K_0(\zeta)\, d\sigma(\zeta). \tag{0.3}$$

Obviously, $K(H)$ is a real number satisfying $0 \leq K(H) \leq \operatorname{rank}(H)$.

The definition of the Euler characteristic $\chi(H)$ of a finite rank contractive A-module H is more straightforward. $\chi(H)$ depends only on the *linear algebra* of the following A-submodule of H:

$$M_H = \operatorname{span}\{f \cdot \xi : f \in A, \xi \in \Delta H\}.$$

Notice that we have not taken the closure in forming M_H. Note too that if $r = \operatorname{rank}(H)$ and $\zeta_1, \dots, \zeta_r$ is a linear basis for ΔH, then M_H is the set of "linear combinations"

$$M_H = \{f_1 \cdot \zeta_1 + \cdots + f_r \cdot \zeta_r : f_k \in A\}.$$

In particular, M_H is a finitely generated A-module.

It is a consequence of Hilbert's syzygy theorem for ungraded modules (cf. Theorem 182 of [20] or Corollary 19.8 of [17]) that M_H has a finite free resolution; that is, there is an exact sequence of A-modules

$$0 \to F_n \to \cdots \to F_2 \to F_1 \to M_H \to 0 \tag{0.4}$$

where F_k is a free module of finite rank β_k,

$$F_k = \underbrace{A \oplus \cdots \oplus A}_{\beta_k \text{times}}.$$

The alternating sum of the "Betti numbers" of this free resolution $\beta_1 - \beta_2 + \beta_3 - + \dots$ does not depend on the particular finite free resolution of M_H, hence we may define the *Euler characteristic* of H by

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k, \tag{0.5}$$

where β_k is the rank of F_k in any finite free resolution of M_H of the form (0.4).

One of the more notable results in the Riemannian geometry of surfaces is the Gauss-Bonnet theorem, which asserts that if M is a compact oriented Riemannian 2-manifold and

$$K : M \to \mathbb{R}$$

is its Gaussian curvature function, then

$$\frac{1}{2\pi} \int_M K \, dA = \beta_0 - \beta_1 + \beta_2 \tag{0.6}$$

where β_k is the kth Betti number of M. In particular, the integral of K depends only on the *topological* type of M. This remarkable theorem was generalized by Shiing-Shen Chern to compact oriented even-dimensional Riemannian manifolds in 1944 [7].

In section 5 we will establish the following result, which we view as an analogue of the Gauss-Bonnet-Chern theorem for graded Hilbert A-modules. By a *graded* Hilbert A-module we mean a pair (H, Γ) where H is a (finite rank, contractive) Hilbert A-module and $\Gamma : \mathbb{T} \to \mathcal{B}(H)$ is a strongly continuous unitary representation of the circle group such that

$$\Gamma(\lambda) T_k \Gamma(\lambda)^{-1} = \lambda T_k, \qquad k = 1, 2, \dots, d, \lambda \in \mathbb{T},$$

$T_1, \dots, T_d$ being the d-contraction associated with the module structure of H. Thus, graded Hilbert A-modules are precisely those whose underlying operator d-tuple $(T_1, \dots, T_d)$ possesses circular symmetry. Γ is called the *gauge group* of H.

Theorem B. *Let H be a graded (contractive, finite rank) Hilbert A-module for which the spectrum of the gauge group is bounded below. Then $K(H) = \chi(H)$, and in particular $K(H)$ is an integer.*

We remark that the hypothesis on the spectrum of the gauge group is equivalent to several other natural ones, see Proposition 5.4. Theorem B depends on the following asymptotic formulas for $K(H)$ and $\chi(H)$, which are valid for finite rank contractive Hilbert A-modules, graded or not. For such an H, let $(T_1, \dots, T_d)$ be its associated d-contraction and define a completely positive normal map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*.$$

Since H is contractive and finite rank, $\mathbf{1} - \phi^n(\mathbf{1})$ is a positive finite rank operator for every $n = 1, 2, \dots$.

Theorem C. *For every contractive finite rank Hilbert A-module H,*

$$K(H) = d! \lim_{n\to\infty} \frac{\text{trace } (\mathbf{1} - \phi^n(\mathbf{1}))}{n^d}.$$

Theorem D. *For every contractive finite rank Hilbert A-module H,*

$$\chi(H) = d! \lim_{n\to\infty} \frac{\text{rank } (\mathbf{1} - \phi^n(\mathbf{1}))}{n^d}.$$

Theorems C and D are proved in sections 3 and 4; taken together, they lead immediately to the general inequality $K(H) \leq \chi(H)$ (Corollary 2 of Theorem D). We have already alluded to the fact that the number $K(H)$ is actually the trace of a certain self-adjoint trace-class operator $d\Gamma$, which exists for any finite rank contractive Hilbert module. While the trace of this operator is always nonnegative, it is noteworthy that $d\Gamma$ itself is never a positive operator. Indeed, we have found it useful to think of $d\Gamma$ as a higher dimensional operator-theoretic counterpart of the differential of the Gauss map $\gamma : M \to S^2$ of an oriented 2-manifold $M \subseteq \mathbb{R}^3$. We have glossed over some details in order to make the essential point; see section 3 for a more comprehensive discussion. In any case, the formula

$$K(H) = \text{trace } d\Gamma$$

is an essential component underlying Theorems B and C.

Theorem B implies that $K(H)$ is an integer for pure finite rank graded Hilbert modules H. We do not know if it is an integer for pure ungraded Hilbert modules. In the case $\text{rank}(H) = 1$ this is equivalent to the existence of an inner sequence for every closed submodule of the free Hilbert module $H^2(\mathbb{C}^d)$ (see Theorem 2.2).

In section 7 we discuss examples illustrating various phenomena, and we pose several open problems. We also give the following applications of the material described above (the reader is referred to section 7 for a more detailed discussion of these results).

Theorem E. *Let $M \subseteq H^2$ be a closed submodule of $H^2(\mathbb{C}^d)$ which contains a nonzero polynomial. Then M has an inner sequence.*

Corollary of Theorem F. *In dimension $d \geq 2$, every graded submodule of infinite codimension in $H^2(\mathbb{C}^d)$ is an infinite rank Hilbert A-module.*

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1. Curvature invariant. The curvature invariant of a finite rank Hilbert A-module is defined as the integral of the "renormalized" boundary values of a natural function defined in the open unit ball. The purpose of this section is to establish the existence and basic properties of this boundary value function (Theorem A). Further properties of the curvature invariant are developed in sections 2 and 3.

Let H be a Hilbert A-module with canonical operators $T_1, \dots, T_d$. For every $z \in \mathbb{C}^d$ we define the operator $T(z) \in \mathcal{B}(H)$ as in (0.2),

$$T(z) = \bar{z}_1 T_1 + \dots + \bar{z}_d T_d.$$

We have already pointed out that $\|T(z)\| \leq |z|$, and hence $\mathbf{1} - T(z)$ is invertible for all z in the open unit ball B_d. Thus we can define an operator-valued function $F : B_d \to \mathcal{B}(\Delta H)$ as follows:

$$F(z)\xi = \Delta(\mathbf{1} - T(z)^*)^{-1}(\mathbf{1} - T(z))^{-1}\Delta\xi, \qquad \xi \in \Delta H \tag{1.1}$$

where Δ is the defect operator associated with H, $\Delta = (\mathbf{1} - T_1T_1^* - \dots - T_dT_d^*)^{1/2}$.

Assuming that $\operatorname{rank}(H) < \infty$, for every z $F(z)$ is a positive operator acting on a finite dimensional Hilbert space and we may consider the numerical function $z \in B_d \mapsto \operatorname{trace} F(z)$. We show in Theorem A below that this function has "renormalized" boundary values

$$K_0(z) = \lim_{r \to 1}(1 - r^2)\operatorname{trace} F(rz)$$

for almost every point $z \in \partial B_d$ relative to the natural measure $d\sigma$ on ∂B_d. Once this is established we can define the curvature invariant $K(H)$ by integrating K_0 over ∂B_d. The key formula behind Theorem A is the following.

Theorem 1.2. *Let $F : B_d \to \mathcal{B}(\Delta H)$ be the function (1.1). There is a Hilbert space E and an operator-valued holomorphic function $\Phi : B_d \to \mathcal{B}(E, \Delta H)$ such that*

$$(1 - |z|^2)F(z) = \mathbf{1} - \Phi(z)\Phi(z)^*, \qquad z \in B_d.$$

The multiplication operator associated with Φ maps $H^2 \otimes E$ into $H^2 \otimes \Delta H$ and has norm at most 1.

Remark 1.3: Free Hilbert modules and their multipliers. The statement of Theorem 1.2 requires clarification. We take this opportunity to discuss basic terminology and collect a number of observations about multipliers for later use.

Consider the Hilbert A-module $H^2 = H^2(\mathbb{C}^d)$ [1]. H^2 can be defined most quickly as the symmetric Fock space over a d-dimensional Hilbert space $Z \cong \mathbb{C}^d$, and the canonical operators $S_1, \dots, S_d$ of H^2 are defined by symmetric tensoring with a fixed orthonormal basis $e_1, \dots, e_d$ for Z. $(S_1, \dots, S_d)$ is called the d-shift, and H^2 is called the *free Hilbert module of rank one.*

Let H be an arbitrary contractive Hilbert A-module, and let r be a positive integer or $\infty = \aleph_0$. We will write $r \cdot H$ for the direct sum of r copies of the Hilbert module H, and of course $r \cdot H$ is a Hilbert A-module in a natural way. If C is a Hilbert space of dimension r, then we can make the tensor product of Hilbert spaces $H \otimes C$ into a Hilbert A module by defining the action of a polynomial f on an element $\xi \otimes \zeta$ ($\xi \in H$, $\zeta \in C$) by $f \cdot (\xi \otimes \zeta) = f \cdot \xi \otimes \zeta$, and extending in the obvious way by linearity. The Hilbert A-modules $H \otimes C$ and $r \cdot H$ are *isomorphic* in the sense that there is a unitary operator from one to the other which intertwines the respective actions of polynomials. When $H = H^2$ we refer to both $r \cdot H^2$ and $H^2 \otimes C$ as the *free Hilbert A-module of rank r*.

Perhaps this terminology requires justification, and for that discussion it is better to consider H^2 as the completion of the polynomial algebra $A = \mathbb{C}[z_1, \dots, z_d]$ in a natural Hilbert space norm. This norm derives from an inner product on A having certain maximality properties, and the space $H^2(\mathbb{C}^d)$ generalizes to higher dimensions the familiar Hardy space in one complex dimension (see [1], section 1). Every element of H^2 can be realized as a holomorphic function defined in the open unit ball $B_d \subseteq \mathbb{C}^d$, and the Hilbert module action of a polynomial on an element of H^2 corresponds to pointwise multiplication of complex functions defined on B_d.

There is a well-established notion of *free module* in commutative algebra, and free modules of finite rank (over, say, the polynomial algebra $A = \mathbb{C}[z_1, \dots, z_d]$) can be characterized by various universal properties. They are concretely defined as finite direct sums of copies of A, with the obvious action of polynomials on elements of the direct sum. The most basic universal property of free modules is that every finitely generated A-module is isomorphic to a quotient F/K where F is a free module of finite rank (which can be taken as the minimal number of generators) and $K \subseteq F$ is a submodule. This universal property actually characterizes free modules provided that one imposes a natural condition of "minimality".

Now if H is a contractive finite rank Hilbert module over $A = \mathbb{C}[z_1, \dots, z_d]$ and $K \subseteq H$ is a closed subspace of H which is invariant under the action of the given operators on H, then the quotient H/K is a Hilbert space and the A-module structure of H can be promoted naturally to obtain a contractive A-module structure on H/K. It is quite easy to show that $\operatorname{rank}(H/K) \leq \operatorname{rank}(H)$. In particular, if $F = H^2 \oplus \cdots \oplus H^2$ is a finite direct sum of copies of the Hilbert module H^2 and $K \subseteq F$ is any closed submodule, then F/K *is a finite rank contractive Hilbert module.*

It is significant that finite direct sums $F = H^2 \oplus \cdots \oplus H^2$ of the basic Hilbert module H^2 have precisely the above universal property, in the category of "pure" Hilbert modules of finite rank. This observation depends on a known result in multivariable dilation theory (which actually extends appropriately to noncommuting operators). For our purposes it is convenient to reformulate Theorem 4.4 of [1] in the following way.

1.4 Dilation Theorem. *Let H be a (contractive) Hilbert A-module, and let $H^2 \otimes \overline{\Delta H}$ be the associated free Hilbert A-module. There is a unique bounded linear operator $L : H^2 \otimes \overline{\Delta H} \to H$ satisfying*

$$L(f \otimes \zeta) = f \cdot \Delta\zeta, \qquad f \in A, \zeta \in \overline{\Delta H}.$$

L is a homomorphism of Hilbert A-modules, and we have

$$LL^* = \mathbf{1} - \lim_{n\to\infty} \phi^n(\mathbf{1})$$

where ϕ is the completely positive map on $\mathcal{B}(H)$ associated with the natural operators $T_1, \dots, T_d$ of H, $\phi(X) = T_1 X T_1^* + \cdots + T_d X T_d^*$.

Purity. Notice that since $\phi(\mathbf{1}) = T_1T_1^*+\cdots+T_dT_d^* \le \mathbf{1}$ we have $\|\phi\| = \|\phi(\mathbf{1})\| \le 1$, hence the sequence of positive operators $\mathbf{1} \ge \phi(\mathbf{1}) \ge \phi^2(\mathbf{1}) \ge \dots$ converges strongly to a positive limit $\lim_n \phi^n(\mathbf{1})$. A Hilbert module H is called *pure* if $\lim_n \phi^n(\mathbf{1}) = 0$. It is quite easy to see that free Hilbert modules are pure, that closed submodules of pure Hilbert modules are pure, and the same is true of quotients of pure Hilbert modules.

Notice that when the given Hilbert module H of Theorem 1.4 is pure the operator $L : F \to H$ is a coisometry $LL^* = \mathbf{1}_H$; hence L implements an isomorphism of the quotient $F/\ker L$ and H. We deduce the following result, which justifies our use of the term free Hilbert module for finite direct sums of the basic Hilbert module H^2.

Corollary. *Let H be a pure Hilbert module of rank $r = 1, 2, \dots, \infty$, and let $F = r \cdot H^2$ be the free Hilbert module of rank r. Then there is a closed submodule $K \subseteq F$ such that H is unitarily equivalent to F/K.*

This result implies that in order to understand the structure of pure Hilbert modules of finite rank, one should focus attention on free Hilbert modules of finte rank, their (closed) submodules, and their quotients.

Multipliers. Elements of free Hilbert modules, and homomorphisms from one free Hilbert module to another, can be "evaluated" at points in the open unit ball B_d in $\mathbb{C}^d$. We now describe these evaluation maps, and we briefly discuss the relation between module homorphisms and multipliers.

Let E be a separable Hilbert space and consider the free Hilbert A-module $F = H^2 \otimes E$ of rank $r = \dim E$. One thinks of elements of $H^2 \otimes E$ as E-valued holomorphic functions defined on B_d in the following way. Let $\{u_z : z \in B_d\}$ be the family of holomorphic functions defined on B_d by

$$u_z(w) = (1 - \langle w, z\rangle)^{-1}, \qquad w \in B_d.$$

Each function u_z can be identified with an element of H^2 in a natural way and its H^2 norm is given by

$$\|u_z\| = (1 - |z|^2)^{-1/2}, \qquad z \in B_d$$

see [1], Proposition 1.12. Since H^2 is spanned by $\{u_z : z \in B_d\}$, $H^2 \otimes E$ is spanned by $\{u_z \otimes \zeta : z \in B_d, \zeta \in E\}$.

Using these elements we may evaluate an element $\xi \in H^2 \otimes E$ at a point $z \in B_d$ to obtain an vector $\xi(z) \in E$ by way of the Riesz lemma,

$$\langle \xi(z), \zeta\rangle_E = \langle \xi, u_z \otimes \zeta\rangle_{H^2}, \qquad \zeta \in E,$$

and the obvious estimate shows that $\|\xi(z)\| \le \|\xi\|(1 - |z|^2)^{-1/2}$. $z \mapsto \xi(z)$ is obviously a holomorphic E-valued function defined on the ball B_d. Writing A for the algebra $\mathbb{C}[z_1,\dots,z_d]$ of all complex polynomials in d variables, note that the A-module structure of $H^2 \otimes E$ is conveniently expressed in terms of the values of the function $\xi(\cdot)$ as follows,

$$(f \cdot \xi)(z) = f(z)\xi(z), \qquad f \in A, \xi \in H^2 \otimes E, z \in B_d.$$

Similarly, any bounded homomorphism of free modules can be evaluated at points in B_d to obtain a holomorphic operator-valued function. In more detail, let E_1, E_2 be separable Hilbert spaces and let $\Phi : H^2 \otimes E_1 \to H^2 \otimes E_2$ be a bounded linear operator satisfying

$$\Phi(f \cdot \xi) = f \cdot \Phi(\xi), \qquad f \in A, \xi \in H^2 \otimes E_1.$$

Then we have the elementary estimate

$$|\langle \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle| \leq \frac{\|\Phi\| \cdot \|\zeta_1\| \cdot \|\zeta_2\|}{\sqrt{1 - |z|^2}},$$

for $\zeta_k \in E_k$ and $z \in B_d$, so by another application of the Riesz lemma there is a unique operator-valued function $z \in B_d \mapsto \Phi(z) \in \mathcal{B}(E_1, E_2)$ satisfying

$$\langle \Phi(z)\zeta_1, \zeta_2 \rangle = \langle \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle, \qquad \zeta_k \in E_k, z \in B_d.$$

The multiplication operator defined by the function $\Phi(\cdot)$ agrees with the original operator Φ in the sense that $\Phi(\xi)(z) = \Phi(z)\xi(z)$ for every $z \in B_d$, $\xi \in H^2 \otimes E_1$, and we refer to the function $\Phi(\cdot)$ as the *multiplier* associated with the homomorphism of A-modules $\Phi : H^2 \otimes E_1 \to H^2 \otimes E_2$.

Further connections between the homomorphism and its multiplier are summarized as follows.

(1.3a) $\sup_{|z|<1} \|\Phi(z)\| \leq \|\Phi\|$,

(1.3b) the adjoint $\Phi^* \in \mathcal{B}(H^2 \otimes E_2, H^2 \otimes E_1)$ of the operator Φ is related to the operator function $z \in B_d \mapsto \Phi(z)^* \in \mathcal{B}(E_2, E_1)$ as follows,

$$\Phi^*(u_z \otimes \zeta) = u_z \otimes \Phi(z)^*\zeta, \qquad z \in B_d, \quad \zeta \in E_2.$$

We sketch the proof of these facts for the convenience of the reader. For (1.3b), fix $f \in H^2$, $\zeta_k \in E_k$, $k = 1, 2$, and $z \in B_d$. Then we have

$$\begin{aligned}\langle f \otimes \zeta_1, \Phi^*(u_z \otimes \zeta_2) \rangle &= \langle \Phi(f \otimes \zeta_1), u_z \otimes \zeta_2 \rangle = \langle f \cdot \Phi(1 \otimes \zeta_1), u_z \otimes \zeta_2 \rangle \\ &= \langle f(z)\Phi(z)\zeta_1, \zeta_2 \rangle = f(z) \langle \Phi(z)\zeta_1, \zeta_2 \rangle = \langle f, u_z \rangle \langle \zeta_1, \Phi(z)^*\zeta_2 \rangle \\ &= \langle f \otimes \zeta_1, u_z \otimes \Phi(z)^*\zeta_2 \rangle.\end{aligned}$$

Since $H^2 \otimes E_1$ is spanned by vectors of the form $f \otimes \zeta_1$, (1.3b) follows.

To prove (1.3a) it suffices to show that for every $\zeta_k \in E_k$, $k = 1, 2$ with $\|\zeta_k\| \leq 1$ we have $|\langle \Phi(z)\zeta_1, \zeta_2 \rangle| \leq \|\Phi\|$, and for that, write

$$(1 - |z|^2)^{-1} |\langle \Phi(z)\zeta_1, \zeta_2 \rangle| = \|u_z\|^2 |\langle \zeta_1, \Phi(z)^*\zeta_2 \rangle| = |\langle u_z \otimes \zeta_1, u_z \otimes \Phi(z)^*\zeta_2 \rangle|.$$

By the formula (1.3b) just established, the right side is

$$|\langle u_z \otimes \zeta_1, \Phi^*(u_z \otimes \zeta_2) \rangle| \leq \|u_z\|^2 \|\zeta_1\| \|\zeta_2\| \|\Phi^*\| \leq \|u_z\|^2 \|\Phi\| = (1 - |z|^2)^{-1} \|\Phi\|,$$

from which the assertion of (1.3a) follows.

Remark. Experience with one-dimensional operator theory might lead one to expect that the inequality of (1.3.a) is actually equality. However, the failure of von Neumann's inequality for the ball B_d in dimension $d \geq 2$ (cf. [1], Theorem 3.3) implies that this is not so. Considering the simplest case in which both spaces $E_1 = E_2 = \mathbb{C}$ consist of scalars, it was shown in [1] that in dimension $d \geq 2$ there are bounded holomorphic functions defined on the open unit ball which are not associated with bounded homomorphisms of H^2 into itself. Indeed, explicit examples are given of continuous functions defined on the closed unit ball $f : \overline{B_d} \to \mathbb{C}$ which are holomorphic in the interior B_d but which do not belong to H^2; for such functions f the multiplier condition $f \cdot H^2 \subseteq H^2$ must fail.

We now turn attention to the proof of Theorem 1.2 and Theorem A.

Definition 1.5: Factorable Operators. *Let H be a Hilbert A-module. A positive operator $X \in \mathcal{B}(H)$ is said to be factorable if there is a free Hilbert A-module $F = H^2 \otimes E$ and a bounded homomorphism $L : F \to H$ of Hilbert modules such that $X = LL^*$.*

Given a pair of factorable operators X_1, X_2, say $X_k = L_k L_k^*$ where $L_k \in \hom(F_k, H)$, then we can define $L \in \hom(F_1 \oplus F_2, H)$ by $L(\xi_1, \xi_2) = L_1\xi_1 + L_2\xi_2$ and we find that $X_1 + X_2 = LL^*$. Thus the set of factorable operators is a subcone of the positive cone in $\mathcal{B}(H)$. Lemma 1.4 implies that in general, $\mathbf{1} - \lim_n \phi^n(\mathbf{1})$ is factorable, and in particular for a pure Hilbert A-module $\mathbf{1}$ is factorable. We are particularly concerned with factorable operators on free Hilbert A-modules and require the following characterization which, among other things, implies that the cone of factorable operators on a pure Hilbert A-module is closed in the weak operator topology.

Proposition 1.6. *Let $\phi(B) = T_1 B T_1^* + \dots + T_d B T_d^*$ be the completely positive map of $\mathcal{B}(H)$ associated with a Hilbert A-module H. For every positive operator X on H, the following are equivalent.*

(1) *X is factorable.*
(2) *$\phi(X) \le X$ and the sequence of positive operators $X \ge \phi(X) \ge \phi^2(X) \ge \dots$ decreases to 0.*

For pure Hilbert modules H, (2) can be replaced with

(2)′ *$\phi(X) \le X$.*

proof of (1) $\implies$ (2). This direction is straightforward; letting L be a homomorphism of some free Hilbert A-module F into H, we may consider the natural operators $S_1, \dots, S_d$ of F and the associated operator map $\sigma(B) = S_1 B S_1^* + \dots + S_d B S_d^*$, $B \in \mathcal{B}(F)$. Then $\sigma(\mathbf{1}_F)$ is a projection, and since free modules are pure we also have $\sigma^n(\mathbf{1}_F) \downarrow 0$. Thus $\phi(LL^*) = \sum_k T_k LL^* T_k^* = \sum_k L S_k S_k^* L^* = L\sigma(\mathbf{1}_F)L \le LL^*$. Similarly, $\phi^n(LL^*) = L\sigma^n(\mathbf{1}_F)L^* \downarrow 0$, showing that $X = LL^*$ satisfies (2).

proof of (2) $\implies$ (1). Let $X \ge 0$ satisfy (2) and consider the closed subspace $K \subseteq H$ obtained by closing the range of the positive operator $X^{1/2}$. We will make K into a *pure* Hilbert A-module as follows.

We claim first that there is a unique d-contraction $\tilde{T}_1, \dots, \tilde{T}_d$ acting on K such that

$$T_k X^{1/2} = X^{1/2} \tilde{T}_k, \qquad k = 1, 2, \dots, d.$$

Indeed, the uniqueness of $\tilde{T}_1, \dots, \tilde{T}_d$ is clear from the fact that K is the closure of the range of $X^{1/2}$, hence the restriction of $X^{1/2}$ to K has trivial kernel.

In order to construct the operators $\tilde{T}_k$ it is easier to work with adjoints, and we will define operators $A_k = \tilde{T}_k^*$ as follows. Fix $k = 1, \dots, d$ and $\xi \in F$. Then

$$\|X^{1/2} T_k^* \xi\|^2 \le \sum_{k=1}^d \|X^{1/2} T_k^* \xi\|^2 = \sum_{k=1}^d \langle T_k X T_k^* \xi, \xi \rangle = \langle \phi(X)\xi, \xi \rangle \le \langle X\xi, \xi \rangle,$$

hence $\|X^{1/2} T_k^* \xi\| \le \|X^{1/2}\xi\|^2$. Thus there is a unique contraction $A_k \in \mathcal{B}(K)$ such that

$$A_k X^{1/2} = X^{1/2} T_k^*, \qquad k = 1, \dots, d. \tag{1.7}$$

As in the previous estimate,the hypothesis (2) together with (1.7) implies

$$\sum_{k=1}^{d} \|A_k X^{1/2}\xi\|^2 \leq \langle \phi(X)\xi, \xi\rangle \leq \langle X\xi, \xi\rangle = \|X^{1/2}\xi\|^2, \qquad \xi \in F,$$

and hence $A_1^*A_1 + \cdots + A_d^*A_d \leq \mathbf{1}_K$. Since the T_k^* mutually commute, (1.7) implies that the A_k must mutually commute, and hence $\tilde{T}_k = A_k^*$, $k = 1, \ldots, d$ defines a d-contraction acting on K.

Next, we claim that $(\tilde{T}_1, \ldots, \tilde{T}_d)$ is a pure d-contraction in the sense that if $\tilde{\phi} : \mathcal{B}(K) \to \mathcal{B}(K)$ is the map defined by

$$\tilde{\phi}(A) = \sum_{k=1}^{d} \tilde{T}_k A \tilde{T}_k^*,$$

then $\tilde{\phi}^n(\mathbf{1}_K) \downarrow 0$ as $n \to \infty$. Since $\{\tilde{\phi}^n(\mathbf{1}_K) : n \geq 0\}$ is a uniformly bounded sequence of positive operators, the claim will follow if we show that

$$\lim_{n\to\infty} \left\langle \tilde{\phi}^n(\mathbf{1}_K)\eta, \eta \right\rangle = 0$$

for all η in the dense linear manifold $X^{1/2}H$ of K. But for η of the form $\eta = X^{1/2}\xi$, $\xi \in H$, we have

$$\left\langle \tilde{\phi}^n(\mathbf{1}_K)X^{1/2}\xi, X^{1/2}\xi \right\rangle = \left\langle X^{1/2}\tilde{\phi}^n(\mathbf{1}_K)X^{1/2}\xi, \xi \right\rangle.$$

Since $X^{1/2}\tilde{T}_k = T_k X^{1/2}$ for all k it follows that $X^{1/2}\tilde{\phi}^n(\mathbf{1}_K)X^{1/2} = \phi^n(X)$ for every $n = 0, 1, 2, \ldots$, hence $\left\langle \tilde{\phi}^n(\mathbf{1}_K)X^{1/2}\xi, X^{1/2}\xi \right\rangle = \langle \phi^n(X)\xi, \xi\rangle$ and the right side decreases to zero as $n \to \infty$ by hypothesis (2).

Using the operators $\tilde{T}_1, \ldots, \tilde{T}_d \in \mathcal{B}(K)$ we make K into a pure Hilbert A-module; moreover, if we consider $X^{1/2}$ as an operator from K to H then it becomes a homomorphism of Hilbert A-modules. By Lemma 1.4 there is a free Hilbert A-module F and an operator $L_0 \in \hom(F, K)$ such that

$$L_0 L_0^* = \mathbf{1}_K - \lim_{n\to\infty} \tilde{\phi}^n(\mathbf{1}_K) = \mathbf{1}_K.$$

Hence the composition $L = X^{1/2}L_0$ belongs to $\hom(F, H)$. Finally, since $L_0 L_0^* = \mathbf{1}_K$ we have

$$LL^* = X^{1/2}L_0L_0^*X^{1/2} = X,$$

proving that X is factorable. ∎

Lemma 1.8. *let H be a (contractive) Hilbert A-module and Let $L : H^2 \otimes \overline{\Delta H} \to H$ be the operator of Lemma 1.4. There is a free Hilbert A-module F and a homomorphism $\Phi \in \hom(F, H^2 \otimes \overline{\Delta H})$ such that*

$$L^*L + \Phi\Phi^* = \mathbf{1}_{H^2 \otimes \overline{\Delta H}}.$$

proof. We have to show that the positive operator $\mathbf{1} - L^*L \in \mathcal{B}(H^2 \otimes \overline{\Delta H})$ is factorable. To that end, we will show that $\mathbf{1} - L^*L$ is the limit in the weak operator topology of a sequence of positive operators X_n satisfying $\sigma(X_n) \leq X_n$ for every n, σ denoting the completely positive operator mapping associated with $H^2 \otimes \overline{\Delta H}$. Since the set of all positive operators X satisfying $\sigma(X) \leq X$ is weakly closed it will follow that $\sigma(\mathbf{1} - L^*L) \leq \mathbf{1} - L^*L$; and since the underlying Hilbert module $H^2 \otimes \overline{\Delta H}$ is free and therefore pure, an application of Proposition 1.6 will complete the proof.

We first create some room by noting that $H^2 \otimes \overline{\Delta H}$ is a submodule of the larger free Hilbert A-module $H^2 \otimes H$, and we can extend the definition of L to the larger module $H^2 \otimes H$ by the same formula $L(f \otimes \xi) = f\Delta\xi$, $f \in A$, $\xi \in H$. Notice that the extended L vanishes on the orthocomplement of $H^2 \otimes \overline{\Delta H}$.

Fix a real number r, $0 < r < 1$. The d-tuple $(rT_1, \dots, rT_d)$ is obviously a d-contraction acting on H, and since $r < 1$ it is pure. Let

$$\Delta_r = (\mathbf{1} - r^2(T_1T_1^* + \cdots + T_dT_d^*))^{1/2}$$

be the associated defect operator and let $L_r : H^2 \otimes H \to H$ be the linear operator defined as in Lemma 1.4 by

$$L_r(f \otimes \xi) = f(rT_1, \dots, rT_d)\Delta_r\xi, \qquad f \in A, \xi \in H.$$

L_r is a homomorphism of $H^2 \otimes H$ into the Hilbert A-module structure of H defined by $(rT_1, \dots, rT_d)$, and by Lemma 1.4 L_r is a coisometry, $L_rL_r^* = \mathbf{1}_H$. Thus $P_r = \mathbf{1}_{H^2 \otimes H} - L_r^*L_r$ is the projection of $H^2 \otimes H$ onto the kernel of L_r, an invariant projection for the canonical operators $S_1, \dots, S_d$ of $H_2 \otimes H$. From the equation $P_rS_kP_r = S_kP_r$, $k = 1, \dots, d$ it follows that

$$\sigma(P_r) = P_r\sigma(P_r)P_r \leq P_r\sigma(\mathbf{1})P_r,$$

hence $\sigma(P_r) \leq P_r$.

Now let Q be the projection of $H^2 \otimes H$ onto the submodule $H^2 \otimes \Delta H$. Since Q commutes with $S_1, \dots, S_d$ it defines a homomorphism of the A-module $H^2 \otimes H$ onto the A-module $H^2 \otimes \Delta H$. It follows that the net of operators

$$X_r = QP_r \restriction_{H^2 \otimes \Delta H} \in \mathcal{B}(H^2 \otimes \Delta H)$$

satisfies $\sigma(X_r) \leq X_r$ for every $r < 1$, since $\sigma(QP_rQ) = Q\sigma(P_r)Q \leq QP_rQ$.

We claim that X_r converges weakly to $\mathbf{1} - L^*L$ as $r \uparrow 1$. Indeed, since $X_rQ = Q - QL_r^*L_rQ$, it suffices to show that the restriction of L_r to $H^2 \otimes \Delta H$ converges strongly to L. Since the operators L_r are uniformly bounded, it suffices to show that for every polynomial f and $\zeta \in \Delta H$ we have

$$\|L_r(f \otimes \zeta) - L(f \otimes \zeta)\| \to 0, \qquad \text{as } r \uparrow 1. \tag{1.9}$$

Now $L_r(f \otimes \zeta) = f(rT_1, \dots, rT_d)\Delta_r\zeta$. The operators $\Delta_r^2 = \mathbf{1} - r^2(T_1T_2^* + \dots T_dT_d^*)$ decrease to $\Delta^2 = \mathbf{1} - (T_1T_1^* + \cdots + T_dT_d^*)$ as r increases to 1. Since the square root function is operator monotone on positive operators it follows that Δ_r decreases to Δ, and thus Δ_r converges strongly to Δ. Since $f(rT_1, \dots, rT_d)$ converges to

$f(T_1, \dots, T_d)$ in the operator norm as $r \uparrow 1$, we conclude that $f(rT_1, \dots, rT_d)\Delta_r\zeta$ converges to $f(T_1, \dots, T_d)\Delta\zeta$ and Lemma 1.8 follows. ∎

proof of Theorem 1.2. Fix $\alpha \in B_d$, $\zeta_1, \zeta_2 \in \Delta H$. From (1.1) we can write

$$\langle F(\alpha)\zeta_1, \zeta_2\rangle = \left\langle (\mathbf{1} - T(\alpha))^{-1}\Delta\zeta_1, (\mathbf{1} - T(\alpha))^{-1}\Delta\zeta_2 \right\rangle. \tag{1.10}$$

Consider the operator $L : H^2 \otimes \Delta H \to H$ given by $L(f \otimes \zeta) = f \cdot \Delta\zeta$. Notice that for the element $u_\alpha \in H^2$ defined by

$$u_\alpha(z) = (1 - \langle z, \alpha\rangle)^{-1}, \qquad z \in B_d$$

we have

$$L(u_\alpha \otimes \zeta) = (\mathbf{1} - T(\alpha))^{-1}\Delta\zeta. \tag{1.11}$$

Indeed, the sequence of polynomials $f_n \in H^2$ defined by

$$f_n(z) = \sum_{k=0}^{n} \langle z, \alpha\rangle^k$$

converges in the H^2-norm to u_α since

$$\|u_\alpha - f_n\|^2 = \sum_{k=n+1}^{\infty} |\alpha|^{2k} \to 0$$

as $n \to \infty$. Since

$$L(f_n \otimes \zeta) = f_n \cdot \Delta\zeta = \sum_{k=0}^{n} T(\alpha)^k \Delta\zeta,$$

formula (1.11) follows by taking the limit as $n \to \infty$.

From (1.11) we find that

$$\langle F(\alpha)\zeta_1, \zeta_2\rangle = \langle L(u_\alpha \otimes \zeta_1), L(u_\alpha \otimes \zeta_2)\rangle = \langle L^*Lu_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2\rangle.$$

By Lemma 1.8 we have $L^*L = \mathbf{1} - \Phi\Phi^*$, and using the formula $\Phi^*(u_\alpha \otimes \zeta) = u_\alpha \otimes \Phi(\alpha)^*\zeta$ of (1.3b) we can write

$$\begin{aligned}
\langle F(\alpha)\zeta_1, \zeta_2\rangle &= \langle (\mathbf{1} - \Phi\Phi^*)u_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2\rangle \\
&= \langle u_\alpha \otimes \zeta_1, u_\alpha \otimes \zeta_2\rangle - \langle u_\alpha \otimes \Phi(\alpha)^*\zeta_1, u_\alpha \otimes \Phi(\alpha)^*\zeta_2\rangle \\
&= \|u_\alpha\|^2(\langle \zeta_1, \zeta_2\rangle - \langle \Phi(\alpha)^*\zeta_1, \Phi(\alpha)^*\zeta_2\rangle) \\
&= (1 - |\alpha|^2)^{-1} \langle (\mathbf{1} - \Phi(\alpha)\Phi(\alpha)^*\zeta_1, \zeta_2\rangle.
\end{aligned}$$

Theorem 1.2 follows after multiplying through by $1 - |\alpha|^2$. ∎

Theorem A. *Let H be a Hilbert A-module of finite positive rank, let $F : B_d \to \mathcal{B}(\Delta H)$ be the operator function defined by (1.1), and let σ denote normalized measure on the sphere ∂B_d. Then for σ-almost every $z \in \partial B_d$, the limit*

$$K_0(z) = \lim_{r \uparrow 1} (1 - r^2)\, trace\, F(r \cdot z)$$

exists and satisfies

$$0 \le K_0(z) \le rank\, H.$$

proof. By Theorem 1.2, there is a separable Hilbert space E and a homomorphism of free Hilbert A-modules $\Phi : H^2 \otimes E \to H^2 \otimes \Delta H$, $\|\Phi\| \le 1$, whose associated multiplier $z \in B_d \mapsto \Phi(z) \in \mathcal{B}(E, \Delta H)$ satisfies

$$(1 - r^2) F(rz) = \mathbf{1} - \Phi(rz)\Phi(rz)^*, \qquad z \in \partial B_d, \quad 0 < r < 1.$$

Since $\Phi(\cdot)$ is a bounded holomorphic operator-valued function defined in the open unit ball B_d it has a radial limit function

$$\lim_{r \to 1} \|\Phi(rz) - \tilde{\Phi}(z)\| = 0. \tag{1.12}$$

almost everywhere $d\sigma(z)$ on the boundary ∂B_d relative to the operator norm. This can be seen as follows. Since ΔH is finite dimensional every bounded operator $A : E \to \Delta H$ is a Hilbert-Schmidt operator, and we have

$$\|A\|^2 \le \operatorname{trace} A^* A \le \dim \Delta H \cdot \|A\|^2.$$

Consider the separable Hilbert space $\mathcal{L}^2(E, \Delta H)$ of all such Hilbert-Schmidt operators. We may consider $\Phi : z \in B_d \mapsto \Phi(z) \in \mathcal{L}^2(E, \Delta H)$ as a bounded vector-valued holomorphic function. Hence there is a Borel set $N \subseteq \partial B_d$ of σ-measure zero such that for all $z \in \partial B_d$ the limit

$$\lim_{r \to 1} \Phi(rz) = \tilde{\Phi}(z)$$

exists in the norm of $\mathcal{L}^2(E, \Delta H)$ (for example, one verifies this by making use of the radial maximal function (see 5.4.11 of [30])). (1.12) follows.

Thus for $z \in \partial B_d \setminus N$ we see from Theorem 1.2 that

$$\lim_{r \to 1} \|(1 - r^2) F(rz) - (\mathbf{1} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*)\| = \lim_{r \to 1} \| - \Phi(rz)\Phi(rz)^* + \tilde{\Phi}(z)\tilde{\Phi}(z)\| = 0,$$

and after applying the trace we obtain

$$\lim_{r \to 1} (1 - r^2) \operatorname{trace} F(rz) = \operatorname{trace} (\mathbf{1}_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*)$$

almost everywhere ($d\sigma$) on ∂B_d. In particular, the limit function $K_0(\cdot)$ is expressed in terms of $\tilde{\Phi}(\cdot)$ almost everywhere on ∂B_d as follows

$$K_0(z) = \operatorname{trace} (\mathbf{1}_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*) = \operatorname{rank} H - \operatorname{trace} \tilde{\Phi}(z)\tilde{\Phi}(z)^*. \tag{1.13}$$

Since $\|\tilde{\Phi}(z)\| \le 1$ we have $0 \le \mathbf{1}_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^* \le \mathbf{1}_{\Delta H}$ and hence

$$0 \le K_0(z) \le \operatorname{trace} (\mathbf{1}_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*) \le \operatorname{trace} \mathbf{1}_{\Delta H} = \operatorname{rank}(H)$$

for almost every $z \in \partial B_d$. ∎

The curvature invariant of H is defined by averaging $K_0(\cdot)$ over the sphere

$$K(H) = \int_{\partial B_d} K_0(z)\, d\sigma(z). \tag{1.14}$$

2. Extremal properties of $K(H)$.

Let H be a contractive, finite rank Hilbert module over $A = \mathbb{C}[z_1, \dots, z_d]$, and let

$$K(H) = \int_{\partial B_d} K_0(z)\, d\sigma(z)$$

be its curvature invariant. From Theorem A we have $0 \leq K(H) \leq \operatorname{rank} H$. In this section we will show that the curvature invariant is sufficiently sensitive to detect exactly when H is a free module in the following sense.

Theorem 2.1. *Suppose in addition that H is pure. Then $K(H) = \operatorname{rank} H$ iff H is isomorphic to the free Hilbert module $H^2 \otimes \Delta H$ of rank $r = \operatorname{rank} H$.*

We will also show that the other extreme value $K(H) = 0$ has the following significance for the structure of the invariant subspaces of H^2.

Theorem 2.2. *Let $M \subseteq H^2$ be a proper closed submodule of the rank-one free Hilbert module. There exists an inner sequence for M iff $K(H^2/M) = 0$, where H^2/M is the quotient Hilbert module.*

Remark. The notion of inner sequence for an invariant subspace $M \subseteq H^2$ will be introduced below (see Definition 2.6 and the discussion following it).

proof of Theorem 2.1. Suppose first that H is isomorphic to a free Hilbert module $H^2 \otimes C$ of rank $r = \dim C$. In this case the curvature invariant is easily computed directly and it is found to be $\dim C$; we include a sketch of this calculation for completeness.

The canonical operators of $H^2 \otimes C$ are given by $T_k = S_k \otimes \mathbf{1}_C$, $k = 1, \dots, d$, where $S_1, \dots, S_d \in \mathcal{B}(H^2)$ is the d-shift, and the defect operator is $\Delta = [1] \otimes \mathbf{1}_C$, where $[1]$ denotes the projection onto the one-dimensional subspace of H^2 spanned by the constant function 1. In particular the range of Δ is identified with C. For $z \in B_d$ let u_z be the element of H^2 defined by the holomorphic function

$$u_z(w) = (1 - \langle w, z \rangle)^{-1}, \qquad w \in B_d.$$

Then for $z \in B_d$ we have

$$(\mathbf{1}_{H^2} - \sum_{k=1}^{d} \bar{z}_k S_k)^{-1} 1 = u_z,$$

hence for $\zeta \in C$,

$$(\mathbf{1}_{H^2 \otimes C} - \sum_{k=1}^{d} \bar{z}_k T_k)^{-1} (1 \otimes \zeta) = u_z \otimes \zeta.$$

Letting $z \in B_d \mapsto F(z) \in \mathcal{B}(C)$ be the function of (1.1), it follows that

$$\operatorname{trace} F(z) = \|u_z\|^2 \dim C = (1 - |z|^2)^{-1} \dim C.$$

Thus $(1 - |z|^2) \operatorname{trace} F(z) \equiv \dim C$ is constant over the unit ball, hence $K_0(\cdot) \equiv \dim C$, and finally

$$K(H^2 \otimes C) = \int_{\partial B_d} K_0(z)\, d\sigma(z) = \dim C,$$

as asserted.

Conversely, suppose that H is a pure Hilbert module for which $K(H) = \operatorname{rank} H$. We will show that H is isomorphic to the free Hilbert module $H^2 \otimes \Delta H$. Let $L : H^2 \otimes \Delta H \to H$ be the dilation homomorphism of Lemma 1.4, $L(f \otimes \zeta) = f \cdot \Delta\zeta$, $f \in A$, $\zeta \in \Delta H$. Since H is assumed pure we see from Lemma 1.4 that L^* is an isometry,

$$LL^* = \mathbf{1}_H - \lim_{n\to\infty} \phi^n(\mathbf{1}_H) = \mathbf{1}_H$$

and we have to show that L is also an isometry. To that end, let $\Phi : F \to H^2 \otimes \Delta H$ be the homomorphism of Lemma 1.8,

$$L^*L + \Phi\Phi^* = \mathbf{1}_{H^2\otimes\Delta H}.$$

We will show that $\Phi = 0$.

Since $0 \le K_0(z) \le \operatorname{rank} H$ almost everywhere $d\sigma(z)$ on ∂B_d and since

$$K(H) = \int_{\partial B_d} K_0(z)\, d\sigma(z) = \operatorname{rank} H,$$

we must have $K_0(z) = \operatorname{rank} H = \operatorname{trace}(\mathbf{1}_{\Delta H})$ almost everywhere on ∂B_d. On the other hand, letting $\tilde\Phi(\cdot) : \partial B_d \to \mathcal{B}(\Delta H)$ be the boundary value function associated with the multiplier of Φ, we see from (1.13) that

$$K_0(z) = \operatorname{trace}(\mathbf{1}_{\Delta H} - \tilde\Phi(z)\tilde\Phi(z)^*),$$

almost everywhere on ∂B_d. Since the trace is faithful, we conclude from

$$\operatorname{trace}(\mathbf{1}_{\Delta H} - \tilde\Phi(z)\tilde\Phi(z)^*) = K_0(z) = \operatorname{trace}(\mathbf{1}_{\Delta H})$$

that $\tilde\Phi(z)$ must vanish almost everywhere $d\sigma(z)$ on ∂B_d. Since the multiplier of Φ is uniquely determined by its boundary values it must vanish identically throughout B_d, hence $\Phi = 0$. ∎

The curvature invariant also detects "inner sequences". More precisely, let $M \subseteq H^2$ be a proper closed submodule of H^2 and let P_M be the projection of H^2 onto M. We first point out that there is a (finite or infinite) sequence $\phi_1, \phi_2, \dots$ of holomorphic functions defined on B_d, which define multipliers of H^2 (i.e., $\phi_k \cdot H^2 \subseteq H^2$), whose associated multiplication operators $M_{\phi_k} \in \mathcal{B}(H^2)$ satisfy

$$M_{\phi_1}M_{\phi_1}^* + M_{\phi_2}M_{\phi_2}^* + \cdots = P_M. \tag{2.3}$$

To see this, note that if $S_1, \dots, S_d$ denotes the natural operators of H^2 then we have

$$S_1P_MS_1^* + \cdots + S_dP_MS_d^* \le P_M. \tag{2.4}$$

Indeed, since $S_kM \subseteq M$ we must have $S_kP_MS_k \le P_M$, and since

$$\|S_1P_MS_1^* + \cdots + S_dP_MS_d^*\| \le \|S_1S_1^* + \cdots + S_dS_d^*\| \le 1,$$

(2.4) follows. By Proposition 1.6 P_M is factorable; i.e., there is a free Hilbert module $H^2 \otimes E$ and a homomorphism of Hilbert modules $\Phi : H^2 \otimes E \to H^2$ such that $P_M = \Phi\Phi^*$. Let $e_1, e_2, \dots$ be an orthonormal basis for E and define $\phi_k \in H^2$ by $\phi_k = \Phi(1 \otimes e_k)$, $k = 1, 2, \dots$. Since Φ is a homomorphism of Hilbert modules of norm at most 1 we find that the ϕ_k are in H^∞ and in fact they define multipliers of H^2, $\phi_k \cdot H^2 \subseteq H^2$, for which equation (2.3) holds.

For definiteness of notation, we can assume that the sequence $\phi_1, \phi_2, \dots$ is infinite by adding harmless zero functions if it is not.

Now let $\phi_1, \phi_2, \dots$ be any sequence of multipliers of H^2 satisfying (2.3). Notice that

$$\sup_{|z|<1} \sum_{n=1}^{\infty} |\phi_n(z)|^2 \leq 1. \tag{2.5}$$

Indeed, if $\{u_\alpha : \alpha \in B_d\}$ denotes the family of functions in H^2 defined in Remark 1.3, then $v_\alpha = (1 - |\alpha|^2)^{1/2} u_\alpha$ is a unit vector in H^2 which is an eigenvector for the adjoint of any multiplication operator associated with a multiplier of H^2; thus for the operators M_{ϕ_n} we have $M_{\phi_n}^* v_\alpha = \overline{\phi_n(\alpha)} v_\alpha$. Using (2.3) we find that

$$\sum_{n=1}^{\infty} |\phi_n(\alpha)|^2 = \sum_{n=1}^{\infty} \|M_{\phi_n}^* v_\alpha\|^2 = \sum_{n=1}^{\infty} \left\langle M_{\phi_n} M_{\phi_n}^* v_\alpha, v_\alpha \right\rangle = \langle P_M v_\alpha, v_\alpha \rangle \leq 1,$$

and (2.5) follows.

Therefore, the boundary functions $\tilde{\phi}_n : \partial B_d \to \mathbb{C}$ defined almost everywhere by $\tilde{\phi}_n(z) = \lim_{r \to 1} \phi_n(rz)$ must also satisfy (2.5)

$$\sum_{n=1}^{\infty} |\tilde{\phi}_n(z)|^2 \leq 1, \qquad z \in \partial B_d$$

almost everywhere with respect to the natural normalized measure σ on ∂B_d.

Definition 2.6. *Let M be a closed invariant subspace of H^2 and let $\phi_1, \phi_2, \dots$ be a finite or infinite sequence of multipliers satisfying equation (2.3). $\{\phi_1, \phi_2, \dots\}$ is called an inner sequence for M if for almost every $z \in \partial B_d$ relative to the natural measure, the boundary values $\{\tilde{\phi}_1, \tilde{\phi}_2, \dots\}$ satisfy*

$$|\tilde{\phi}_1(z)|^2 + |\tilde{\phi}_2(z)|^2 + \dots = 1.$$

We have seen above that every invariant subspace $M \subseteq H^2$ is associated with a sequence $\{\phi_1, \phi_2, \dots\}$ which satisfies (2.3). However, we do not consider that a satisfactory higher dimensional analogue of Beurling's theorem because we do not know if it is possible to find such a sequence that is also an *inner sequence*. It is not hard to see that for a fixed M, if some sequence satisfying (2.3) is an inner sequence then every such sequence is an inner sequence (the proof is omitted since we do not require this result in the sequel).

Of course, in dimension $d = 1$ Beurling's theorem implies that there is a single multiplier ϕ satisfying equation (2.3), $M_\phi M_\phi^* = P_M$; and since the unilateral shift can be extended to a unitary operator (the bilateral shift) this very formula implies that ϕ is inner: $|\phi(z)| = 1$ for almost every z on the unit circle. However, in dimension $d \geq 2$ one can no longer satisfy (2.3) with a single function, and in fact the sequences of (2.3) are typically infinite. Moreover, the natural operators of H^2 do not form a subnormal d-tuple. For these reasons, arguments that are effective in the one-dimensional case break down in dimension $d \geq 2$. Thus we do not know if invariant subspaces of H^2 are associated with inner sequences, and that is one of the significant open problems in this theory. Theorem 2.2 shows the relevance of the curvature invariant for this problem, and we turn now to its proof.

proof of Theorem 2.2. Let F be the direct sum of an infinite number of copies of H^2 and define $\Phi \in \hom(F, H^2)$ by

$$\Phi(f_1, f_2, \dots) = \sum_{n=1}^{\infty} \phi_n \cdot f_n.$$

Then $\Phi\Phi^* = P_M$. Letting $L : H^2 \to H^2/M$ be the natural projection of H^2 onto the quotient Hilbert module, then L is a homomorphism of Hilbert modules satisfying $LL^* = \mathbf{1}_{H^2/M}$, and the kernel of L is $\mathbf{1}_{H^2} - P_M$. Indeed L defines the natural dilation of H^2/M as in Lemma 1.4, and it is clear from its construction that Φ satisfies the formula of Lemma 1.8

$$L^*L + \Phi\Phi^* = \mathbf{1}_{H^2}.$$

Writing

$$K(H^2/M) = \int_{\partial B_d} K_0(z)\, d\sigma(z),$$

we see from formula (1.13) that in this case

$$K_0(z) = 1 - \tilde{\Phi}(z)\tilde{\Phi}(z)^* = 1 - \sum_{n=1}^{\infty} |\tilde{\phi}_n(z)|^2,$$

and therefore $K(H^2/M) = 0$ iff $\sum_n |\tilde{\phi}_n(z)|^2 = 1$ almost everywhere on ∂B_d. ∎

In Theorem E of section 7 we will combine Theorem 2.2 with later results to establish the existence of inner sequences in many cases. The general problem remains open, and is discussed in section 7.

3. Asymptotics of $K(H)$: curvature operator, stability.

Let us recall a convenient description of the Gaussian curvature of a compact oriented Riemannian 2-manifold M. It is not necessary to do so, but for simplicity we will assume that $M \subseteq \mathbb{R}^3$ can be embedded in $\mathbb{R}^3$ in such a way that it inherits the usual metric structure of $\mathbb{R}^3$. After choosing one of the two orientations of M (as a nondegenerate 2-form) we normalize it in the obvious way to obtain a continuous field of *unit* normal vectors at every point of M.

For every point p of M one can translate the normal vector at p to the origin of $\mathbb{R}^3$ (without changing its direction), and the endpoint of that translated vector is a point $\gamma(p)$ on the unit sphere S^2. This defines the Gauss map

$$\gamma : M \to S^2 \tag{3.1}$$

of M to the sphere. Now fix $p \in M$. The tangent plane T_pM is obviously parallel to the corresponding tangent plane $T_{\gamma(p)}S^2$ of the sphere (they have the same normal vector) and hence both are cosets of the same 2-dimensional subspace $V \subseteq \mathbb{R}^3$:

$$T_pM = p + V, \qquad T_{\gamma(p)}S^2 = \gamma(p) + V.$$

Thus the differential $d\gamma(p)$ defines a linear operator on the two-dimensional vector space V, and the Gaussian curvature $K(p)$ of M at p is defined as the determinant of this operator $K(p) = \det d\gamma(p)$. $K(p)$ does not depend on the choice of orientation. The Gauss-Bonnet theorem asserts that the average value of $K(\cdot)$ is the alternating sum of the Betti numbers of M

$$\frac{1}{2\pi}\int_M K(p) = \beta_0 - \beta_1 + \beta_2.$$

In this section we define a curvature *operator* associated with any finite rank Hilbert A-module H. This operator can be thought of as a quantized (higher-dimensional) analogue of the differential of the Gauss map $\gamma : M \to S^2$. We show that it belongs to the trace class, that its trace agrees with the curvature invariant $K(H)$ of section 1, and in Theorem C below we establish a key asymptotic formula for $K(H)$.

Let H be a finite rank contractive A-module and let $L : H^2 \otimes \Delta H \to H$ be the dilation map of Lemma 1.4. Along with the free Hilbert module $H^2 \otimes \Delta H$ we must also work with the subnormal Hardy module $H^2(\partial B_d) \otimes \Delta H$, defined as the closure in the norm of $L^2(\partial B_d) \otimes \Delta H$ of the space of restrictions to ∂B_d of all holomorphic polynomials $f : \mathbb{C}^d \to \Delta H$. There is a natural way of extending functions in $H^2(\partial B_d) \otimes \Delta H$ holomorphically to the interior of the unit ball B_d [30]. Theorem 4.3 of [1] implies that these two spaces of ΔH-valued holomorphic functions on B_d are related as follows

$$H^2 \otimes \Delta H \subseteq H^2(\partial B_d) \otimes \Delta H. \tag{3.2}$$

The inclusion map (3.2) is an isometry in dimension $d = 1$, and is a compact operator of norm 1 when $d \geq 2$. Significantly, it is never a Hilbert-Schmidt operator (see Remark 3.9 below).

The Hilbert module structure of $H^2(\partial B_d) \otimes \Delta H$ is defined by the natural multiplication operators $Z_1, \dots, Z_d$,

$$Z_k : f(z_1, \dots, z_d) \mapsto z_k f(z_1, \dots, z_d), \qquad k = 1, 2, \dots, d.$$

By way of contrast with the d-shift, $Z_1, \dots, Z_d$ is a *subnormal* d-contraction satisfying

$$Z_1^*Z_1 + \cdots + Z_d^*Z_d = \mathbf{1}, \qquad Z_1Z_1^* + \cdots + Z_dZ_d^* = \mathbf{1} - \tilde{E}_0,$$

$\tilde{E}_0$ denoting the rank$H$-dimensional projection onto the constant ΔH-valued functions in $H^2(\partial B_d) \otimes \Delta H$.

Let $b : H^2 \otimes \Delta H \to H^2(\partial B_d) \otimes \Delta H$ denote the inclusion map of (3.2). We define a linear map $\Gamma : \mathcal{B}(H^2 \otimes \Delta H) \to \mathcal{B}(H^2(\partial B_d) \otimes \Delta H)$ as follows

$$\Gamma(X) = bXb^*. \tag{3.3}$$

Remark 3.4. We first record some simple observations about the operator mapping Γ. It is obvious that Γ is a normal completely positive linear map. Γ is also an order isomorphism because b is injective. Indeed, if $\Gamma(X) \geq 0$ then $\langle X\xi, \xi\rangle \geq 0$ for every ξ in the range $b^*(H^2(\partial B_d) \otimes \Delta H)$, and $b^*(H^2(\partial B_d) \otimes \Delta H)$ is dense in $H^2 \otimes \Delta H$ because b has trivial kernel. A similar argument shows that Γ is in fact a complete order isomorphism. However, in dimension $d \geq 2$ the range of Γ is a linear space of compact operators which is norm-dense in $\mathcal{K}(H^2(\partial B_d) \otimes \Delta H)$.

The operator mapping of central importance for defining the curvature operator is not Γ but rather its "differential", defined as follows for arbitrary finite rank contractive Hilbert A-modules H.

Definition 3.5. *Let $Z_1, \dots, Z_d$ be the canonical operators of the Hardy module $H^2(\partial B_d) \otimes \Delta H$. The linear map $d\Gamma : \mathcal{B}(H^2 \otimes \Delta H) \to \mathcal{B}(H^2(\partial B_d) \otimes \Delta H)$ is defined as follows*

$$d\Gamma(X) = \Gamma(X) - \sum_{k=1}^{d} Z_k \Gamma(X) Z_k^*.$$

The curvature operator of H is defined as the self-adjoint operator

$$d\Gamma(L^*L) \in \mathcal{B}(H^2(\partial B_d) \otimes \Delta H),$$

where $L : H^2 \otimes \Delta H \to H$ is the dilation map $L(f \otimes \zeta) = f \cdot \Delta\zeta$, $f \in H^2$, $\zeta \in \Delta H$.

Remarks. Notice that L^*L is a positive operator on $H^2 \otimes \Delta H$, that $\Gamma(L^*L)$ is a positive compact operator on the Hardy module $H^2(\partial B_d) \otimes \Delta H$ (at least in dimension $d \geq 2$), and that the curvature operator $d\Gamma(L^*L)$ is a self-adjoint compact operator which is neither positive nor negative.

We have found it useful to think of the operator $\Gamma(L^*L)$ as a higher-dimensional "quantized" analogue of the Gauss map $\gamma : M \to S^2$ of (3.1), and of the curvature operator $d\Gamma(L^*L)$ as its "differential". Of course, this is only an analogy. But we will also find that $d\Gamma(L^*L)$ belongs to the trace class, and

$$\operatorname{trace} d\Gamma(L^*L) = K(H).$$

We have already suggested an analogy between the term $K(H)$ on the right and the average Gaussian curvature of, say, a surface

$$\frac{1}{2\pi} \int_M K = \frac{1}{2\pi} \int_M \det d\gamma(p).$$

On the other hand, $K(H)$ is defined in section 1 as the integral of the trace (not the determinant) of an operator-valued function, and thus these analogies must not be carried to extremes.

We also remark that the curvature operator can be defined in somewhat more concrete terms as follows. Let $T(z)$ denote the operator function of $z \in \mathbb{C}^d$ defined in (0.2). $T(z)$ is invertible for $|z| < 1$, and hence every vector $\xi \in H$ gives rise to a vector-valued holomorphic function $\hat{\xi} : B_d \to \Delta H$ defined on the ball by way of

$$\hat{\xi}(z) = \Delta(\mathbf{1} - T(z)^*)^{-1}\xi, \qquad z \in B_d.$$

It is a fact that the function $\hat{\xi}$ belongs to the Hardy module $H^2(\partial F) \otimes \Delta H$, and thus we have defined a linear mapping $B : \xi \in H \mapsto \hat{\xi} \in H^2(\partial F) \otimes \Delta H$. Indeed, the reader can verify that B is related to b and L by $B = bL^*$, and hence the curvature operator of Definition 3.5 is given by

$$d\Gamma(L^*L) = BB^* - \sum_{k=1}^{d} Z_k BB^* Z_k^*.$$

We will not require this formula nor the operator B in what follows.

We take this opportunity to introduce a sequence of polynomials that will be used repeatedly in the sequel. Let $q_0, q_1, \dots \in \mathbb{Q}[x]$ be the sequence of polynomials which are normalized so that $q_k(0) = 1$, and which are defined recursively by $q_0(x) = 1$ and

$$q_k(x) - q_k(x-1) = q_{k-1}(x), \qquad k \geq 1. \tag{3.6}$$

One finds that for $k \geq 1$,

$$q_k(x) = \frac{(x+1)(x+2)\dots(x+k)}{k!}. \tag{3.7}$$

When $x = n$ is a positive integer, $q_k(n)$ is the binomial coefficient $\binom{n+k}{k}$, and more generally $q_k(\mathbb{Z}) \subseteq \mathbb{Z}$, $k = 0, 1, 2, \dots$.

We now work out the basic properties of the operator mapping

$$d\Gamma : \mathcal{B}(H^2 \otimes \Delta H) \to \mathcal{B}(H^2(\partial B_d) \otimes \Delta H).$$

The essential properties of the inclusion map $b : H^2 \otimes \Delta H \to H^2(\partial B_d) \otimes \Delta H$ are summarized as follows. We will write E_n, $n = 0, 1, 2, \dots$ for the projection of $H^2 \otimes \Delta H$ onto its subspace of homogeneous (vector-valued) polynomials of degree n, and we have

$$\begin{aligned} \operatorname{trace} E_n &= \dim\{f \in H^2 : f(\lambda z) = \lambda^n f(z), \lambda \in \mathbb{C}, z \in \mathbb{C}^d\} \cdot \dim \Delta H \\ &= q_{d-1}(n) \cdot \operatorname{rank} H, \end{aligned}$$

(see Appendix A of [1]).

Let $\tilde{E}_n$ be the corresponding sequence of projections acting on the Hardy module $H^2(\partial B_d) \otimes \Delta H$. We will also write N and $\tilde{N}$ for the respective number operators on $H^2 \otimes \Delta H$ and $H^2(\partial B_d) \otimes \Delta H$,

$$N = \sum_{n=0}^{\infty} n E_n, \quad \tilde{N} = \sum_{n=0}^{\infty} n \tilde{E}_n.$$

Proposition 3.8. *Let $b : H^2 \otimes \Delta H \to H^2(\partial B_d) \otimes \Delta H$ be the natural inclusion. Then*

1. $bE_n = \tilde{E}_n b, \qquad n = 0, 1, 2, \dots$.
2. $b \in \hom(H^2 \otimes \Delta H, H^2(\partial B_d) \otimes \Delta H)$.
3. $b^* b = q_{d-1}(N)^{-1} = \sum_{n=0}^{\infty} \frac{1}{q_{d-1}(n)} E_n$.

proof. Properties (1) and (2) are immediate from the definition of b. Property (3) follows from a direct comparison of the norms in H^2 and the Hardy module $H^2(\partial B_d)$. Indeed, if $f,g\in H^2$ are both homogeneous polynomials of degree n of the specific form

$$f(z)=\langle z,\alpha\rangle^n\,,\quad g(z)=\langle z,\beta\rangle^n\,,\qquad \alpha,\beta\in\mathbb{C}^d,$$

then $\langle f,g\rangle_{H^2}=\langle\beta,\alpha\rangle^n$, whereas if we consider f,g as elements of $H^2(\partial B_d)$ then we have

$$\langle bf,bg\rangle=\langle f,g\rangle_{H^2(\partial B_d)}=q_{d-1}(n)^{-1}\langle\beta,\alpha\rangle^n\,,$$

see Proposition 1.4.9 of [30]. Since E_nH^2 is spanned by such f,g we find that for all $f,g\in E_nH^2$,

$$\langle bf,bg\rangle=q_{d-1}(n)^{-1}\langle f,g\rangle_{H^2}\,.$$

Thus

$$E_nb^*bE_n=q_{d-1}(n)^{-1}E_n=q_{d-1}(N)^{-1}E_n,$$

and (3) follows for the one-dimensional case $\Delta H=\mathbb{C}$ because b^*b commutes with E_n and $\sum_n E_n=\mathbf{1}$.

If we now tensor both H^2 and $H^2(\partial B_d)$ with the finite dimensional space ΔH then we obtain (3) in general after noting that $\dim(K_1\otimes K_2)=\dim K_1\cdot\dim K_2$ for finite dimensional vector spaces K_1,K_2. ∎

Remark 3.9. In the one-variable case $d=1$, $q_{d-1}(x)$ is the constant polynomial 1, and hence 3.8 (3) asserts the familiar fact that b is a unitary operator; i.e., there is no difference between H^2 and the Hardy module $H^2(S^1)$ in dimension 1.

In dimension $d\geq 2$ however, $q_{d-1}(x)$ is a polynomial of degree $d-1$ and hence

$$b^*b=q_{d-1}(N)^{-1}$$

is a positive compact operator. Significantly, the operator b^*b is never trace class. Indeed, the computations of Appendix A of [1] imply that $b^*b\in\mathcal{L}^p$ iff $p>\frac{d}{d-1}>1$.

We need to know which operators $X\in\mathcal{B}(H^2\otimes\Delta H)$ have trace-class "differentials" $d\Gamma(X)$ and the following result provides this information, including an asymptotic formula for the trace of $d\Gamma(X)$.

Theorem 3.10. *For every operator X in the complex linear span of the cone*

$$\mathcal{C}=\{X\in\mathcal{B}(H^2\otimes\Delta H): d\Gamma(X)\geq 0\},$$

$d\Gamma(X)$ is a trace-class operator and

$$\mathit{trace}\, d\Gamma(X)=\mathit{rank}\, H\;\cdot\;\lim_{n\to\infty}\frac{\mathit{trace}\,(XE_n)}{\mathit{trace}\,E_n},$$

where $E_0,E_1,\dots$ is the sequence of spectral projections of the number operator of $H^2\otimes\Delta H$.

Theorem 3.10 depends on a general identity, which we establish first.

Lemma 3.11. *For every $X \in \mathcal{B}(H^2 \otimes \Delta H)$ and $n = 0, 1, 2, \dots$ we have*

$$\mathit{trace}\,(d\Gamma(X)\tilde{P}_n) = \mathit{rank}\,H \cdot \frac{\mathit{trace}\,(XE_n)}{\mathit{trace}\,E_n},$$

where $\tilde{P}_n = \tilde{E}_0 + \tilde{E}_1 + \cdots + \tilde{E}_n$, $\{\tilde{E}_n\}$ being the spectral projections of the number operator of the Hardy module $H^2(\partial B_d) \otimes \Delta H$.

Remark. Notice that all of the operators $E_n, XE_n, d\Gamma(X)\tilde{P}_n$ appearing in Lemma 3.11 are of finite rank. Note too that traces on the right refer to the Hilbert space $H^2 \otimes \Delta H$, while the trace on the left refers to the Hilbert space $H^2(\partial B_d) \otimes \Delta H$.

proof of Lemma 3.11. Let $\dot{E}_n$ be the projection of H^2 onto its space of homogeneous polynomials of degree n. Then $E_n = \dot{E}_n \otimes \mathbf{1}_{\Delta H}$, and hence

$$\operatorname{trace} E_n = \operatorname{trace} \dot{E}_n \cdot \dim \Delta H = q_{d-1}(n) \cdot \operatorname{rank} H,$$

for all $n = 0, 1, \dots$ where $q_{d-1}(x)$ is the polynomial of (3.6) (see Appendix A of [1]). Thus we have to show that

$$\operatorname{trace}\,(d\Gamma(X)\tilde{P}_n) = \frac{\operatorname{trace}\,(XE_n)}{q_{d-1}(n)}, \qquad n = 0, 1, \dots. \tag{3.12}$$

For that, fix n. Let $T_k = S_k \otimes \mathbf{1}_{\Delta H}$, $k = 1, \dots, d$ be the canonical operators of the free module $H^2 \otimes \Delta H$ and let $\phi(X) = T_1 X T_1^* + \cdots + T_d X T_d^*$ be the associated completely positive operator mapping. We can write

$$X - \phi^{n+1}(X) = \sum_{k=0}^{n} \phi^k(X - \phi(X))$$

and, since the range of the operator $\phi^{n+1}(X)$ is contained in the orthocomplement of the space of homogeneous polynomials of degree n, we have $\phi^{n+1}(X)E_n = 0$. Thus

$$XE_n = \sum_{k=0}^{n} \phi^k(X - \phi(X))E_n$$

and taking the trace we obtain

$$\operatorname{trace}\,(XE_n) = \sum_{k=0}^{n} \operatorname{trace}\,(\phi^k(X - \phi(X))E_n) = \operatorname{trace}\,((X - \phi(X)) \sum_{k=0}^{n} \phi_*^k(E_n))$$

where ϕ_* is the pre-adjoint of ϕ, defined on trace class operators B by

$$\phi_*(B) = \sum_{k=1}^{d} T_k^* B T_k.$$

Let $P_n = E_0 + E_1 + \cdots + E_n$. We now establish the critical formula

$$\sum_{k=0}^{n} \phi_*^k(E_n) = q_{d-1}(n) q_{d-1}(N)^{-1} P_n. \tag{3.13}$$

Indeed, the relation $T_k E_l = E_{l+1} T_k$, $k = 1, \dots, d$ implies that for all $0 \le k \le n$ and all $i_1, \dots, i_k \in \{1, 2, \dots, d\}$ we have

$$T_{i_1}^* \dots T_{i_k}^* E_n = E_{n-k} T_{i_1}^* \dots T_{i_k}^*.$$

Hence

$$\phi_*^k(E_n) = E_{n-k} \phi_*^k(\mathbf{1}).$$

Now the operators $\phi_*^k(\mathbf{1})$ were explicitly computed in Limma 7.9 of [1] for the rank-one case $F = H^2$, and it was found that

$$\phi_*^k(\mathbf{1}_{H^2}) = \sum_{p=0}^{\infty} g_k(p) \dot{E}_p, \tag{3.14}$$

where $g_k(x)$ is the rational function

$$g_k(x) = \frac{(x+k+1)(x+k+2)\dots(x+k+d-1)}{(x+1)(x+2)\dots(x+d-1)},$$

if $d \ge 2$ and $g_k(x) = 1$ if $d = 1$. Thus in all cases we have

$$g_k(x) = \frac{q_{d-1}(x+k)}{q_{d-1}(x)}$$

and hence (3.14) becomes

$$\phi_*^k(\mathbf{1}_{H^2}) = \sum_{p=0}^{\infty} \frac{q_{d-1}(k+p)}{q_{d-1}(p)} \dot{E}_p.$$

The result is obtained for $H^2 \otimes \Delta H$ by replacing $\mathbf{1}_{H^2}$ with $\mathbf{1}_{H^2} \otimes \mathbf{1}_{\Delta H}$, and by replacing $\dot{E}_p$ with $E_p = \dot{E}_p \otimes \mathbf{1}_{\Delta H}$. We conclude that

$$\phi_*^k(\mathbf{1}_F) = \sum_{p=0}^{\infty} \frac{q_{d-1}(k+p)}{q_{d-1}(p)} E_p.$$

Thus

$$E_{n-k} \phi_*^k(\mathbf{1}_F) = \frac{q_{d-1}(n)}{q_{d-1}(n-k)} E_{n-k},$$

and we find that

$$\begin{aligned} \sum_{k=0}^{n} E_{n-k} \phi_*^k(\mathbf{1}_F) &= q_{d-1}(n) \sum_{k=0}^{n} \frac{1}{q_{d-1}(n-k)} E_{n-k} \\ &= q_{d-1}(n) \sum_{l=0}^{n} \frac{1}{q_{d-1}(l)} E_l = q_{d-1}(n) q_{d-1}(N)^{-1} P_n, \end{aligned}$$

where $P_n = E_0 + E_1 + \dots + E_n$ as asserted in (3.13).

Now from Proposition 3.8 (3) we have $q_{d-1}(N)^{-1} = b^*b$, and 3.8 (1) implies that $b^*bP_n = b^*\tilde{P}_n b$. Thus we conclude from (3.13) that

$$\begin{aligned} \operatorname{trace}(XE_n) &= q_{d-1}(n) \cdot \operatorname{trace}((X - \phi(X))b^*bP_n) \\ &= q_{d-1}(n) \cdot \operatorname{trace}((X - \phi(X))b^*\tilde{P}_n b) \\ &= q_{d-1}(n) \cdot \operatorname{trace}(b(X - \phi(X))b^*\tilde{P}_n). \end{aligned} \tag{3.15}$$

Finally, writing $\psi(B) = Z_1BZ_1^* + \cdots + Z_dBZ_d^*$ for the natural completely positive map of $\mathcal{B}(H^2(\partial B_d) \otimes \Delta H)$ associated with its A-module structure we have

$$\begin{aligned} b(X - \phi(X))b^* = bXb^* - b\phi(X)b^* &= bXb^* - \psi(bXb^*) \\ &= \Gamma(X) - \psi(\Gamma(X)) = d\Gamma(X). \end{aligned}$$

Thus (3.15) becomes

$$\operatorname{trace}(XE_n) = q_{d-1}(n) \cdot \operatorname{trace}(d\Gamma(X)\tilde{P}_n),$$

and (3.12) follows. ∎

proof of Theorem 3.10. It suffices to show that for any operator X in $\mathcal{B}(H^2 \otimes \Delta H)$ for which $d\Gamma(X) \geq 0$, we must have trace $d\Gamma(X) < \infty$ as well as the limit formula of 3.10. From Lemma 3.11 we have

$$\operatorname{trace}(d\Gamma(X)\tilde{P}_n) = \operatorname{rank} H \cdot \frac{\operatorname{trace}(XE_n)}{\operatorname{trace} E_n}, \qquad n = 0, 1, 2, \dots. \tag{3.16}$$

We claim first that $X \geq 0$. To see that, let $T_1, \dots, T_d$ and $Z_1, \dots, Z_d$ be the canonical operators of $H^2 \otimes \Delta H$ and $H^2(\partial B_d) \otimes \Delta H$ respectively, and note that by Proposition 3.7 we have $bT_k = Z_k b$, $k = 1, \dots, d$. Hence

$$0 \leq d\Gamma(X) = bXb^* - \sum_{k=1}^d Z_k bXb^* Z_k^* = bXb^* - b(\sum_{k=1}^d T_k X T_k^*)b^* = \Gamma(X - \sum_{k=1}^d T_k X T_k^*).$$

Since Γ is an order isomorphism the latter implies $X - \sum_k T_k X T_k^* \geq 0$, or

$$X - \phi(X) \geq 0, \tag{3.17}$$

ϕ being the completely positive map of $\mathcal{B}(H^2 \otimes \Delta H)$, $\phi(A) = T_1AT_1^* + \cdots + T_dAT_d^*$.

Free Hilbert A-modules are pure, hence $\phi^n(\mathbf{1}_{H^2 \otimes \Delta H}) \downarrow 0$ as $n \to \infty$. It follows that for every positive operator $A \in \mathcal{B}(F)$ we have $0 \leq \phi^n(A) \leq \|A\|\phi^n(\mathbf{1})$, and hence $\phi^n(A) \to 0$ in the strong operator topology, as $n \to \infty$. By taking linear combinations we find that $\lim_{n\to\infty} \phi^n(A) = 0$ in the strong operator topology for every $A \in \mathcal{B}(H^2 \otimes \Delta H)$.

Returning now to equation (3.17) we find that

$$X - \phi^{n+1}(X) = \sum_{k=1}^n \phi^k(X - \phi(X)) \geq 0$$

for every $n = 1, 2, \dots$ and since $\phi^{n+1}(X)$ must tend strongly to 0 by the preceding paragraph, we conclude that $X \geq 0$ by taking the limit on n.

Since X is a positive operator and $\rho_n(A) = \operatorname{trace}(AE_n)/\operatorname{trace} E_n$ is a state of $\mathcal{B}(H^2 \otimes \Delta H)$ we have

$$0 \leq \frac{\operatorname{trace}(XE_n)}{\operatorname{trace} E_n} \leq \|X\|$$

for every n. Since the projections $\tilde{P}_n$ increase to $\mathbf{1}_{\partial F}$ with increasing n we conclude from (3.16) that

$$\operatorname{trace}(d\Gamma(X)) = \sup_{n \geq 0} \operatorname{trace}(d\Gamma(X)\tilde{P}_n) \leq \operatorname{rank} H \cdot \|X\| < \infty.$$

Moreover, since in this case

$$\operatorname{trace}(d\Gamma(X)) = \lim_{n\to\infty} \operatorname{trace}(d\Gamma(X)\tilde{P}_n),$$

we may infer the limit formula of Theorem 3.9 directly from (3.16) as well. ∎

In view of Theorem 3.9 and the fact that for every factorable operator X on $H^2 \otimes \Delta H$ we have $d\Gamma(X) \geq 0$ (see Proposition 1.6), the following lemma shows how to compute the trace of $d\Gamma(X)$ in the most important cases.

Lemma 3.18. *Let $F = H^2 \otimes E$ be a free Hilbert A-module, and let $\Phi : F \to H^2 \otimes \Delta H$ be a homomrphism of Hilbert A-modules. Considering Φ as a multiplier $z \in B_d \mapsto \Phi(z) \in \mathcal{B}(E, \Delta H)$ with boundary value function $\tilde{\Phi} : \partial B_d \to \mathcal{B}(E, \Delta H)$ we have*

$$\mathit{trace}\, d\Gamma(\Phi\Phi^*) = \int_{\partial B_d} \mathit{trace}\,(\tilde{\Phi}(z)\tilde{\Phi}(z)^*)\, d\sigma(z).$$

Remark. Note that for σ-almost every $z \in \partial B_d$, $\tilde{\Phi}(z)\tilde{\Phi}(z)^*$ is a positive operator in $\mathcal{B}(\Delta H)$, and since ΔH is finite dimensional the right side is well defined and dominated by $\|\Phi\|^2 \cdot \operatorname{rank} H$.

proof of Lemma 3.18. Consider the linear operator $A : E \to H^2(B_d; \Delta H)$ defined by $A\zeta = b(\Phi(1 \otimes \zeta))$, $\zeta \in E$. We claim first that

$$d\Gamma(\Phi\Phi^*) = AA^*. \tag{3.19}$$

Indeed, since $b\Phi \in \hom(F, H^2(\partial B_d) \otimes \Delta H)$ we have

$$\sum_{k=1}^{d} Z_k b\Phi\Phi^* b^* Z_k^* = \sum_{k=1}^{d} Z_k (b\Phi)(b\Phi)^* Z_k^* = b\Phi(\sum_{k=1}^{d} T_k T_k^*)(b\Phi)^*,$$

where $T_1, \dots, T_d$ are the canonical operators of $F = H^2 \otimes E$, and hence

$$d\Gamma(\Phi\Phi^*) = b\Phi(b\Phi)^* - \sum_{k=1}^{d} Z_k b\Phi\Phi^* b^* Z_k^* = b\Phi(\mathbf{1}_F - \sum_{k=1}^{d} T_k T_k^*)(b\Phi)^*.$$

The operator $\mathbf{1}_F - \sum_k T_k T_k^*$ is the projection of $F = H^2 \otimes E$ onto its space of E-valued constant functions and, denoting by $[1]$ the projection of H^2 onto the one dimensional space of constants $\mathbb{C} \cdot 1$, the preceding formula becomes

$$d\Gamma(\Phi\Phi^*) = b\Phi([1] \otimes \mathbf{1}_E)(b\Phi)^* = AA^*,$$

as asserted in (3.19).

Now fix an orthonormal basis $e_1, e_2, \dots$ for E. By formula (3.19) we can evaluate the trace of $d\Gamma(\Phi\Phi^*)$ in terms of the vector functions $Ae_n \in H^2(\partial B_d) \otimes \Delta H$ as follows,

$$\begin{aligned} \text{trace}\, d\Gamma(\Phi\Phi^*) &= \text{trace}_{H^2(\partial B_d)\otimes \Delta H}(AA^*) = \text{trace}_E(A^*A) \\ &= \sum_n \langle A^*Ae_n, e_n\rangle = \sum_n \|Ae_n\|^2_{H^2(\partial B_d)\otimes\Delta H}. \end{aligned} \tag{3.20}$$

Turning now to the term on the right in Lemma 3.18, we first consider $Ae_n = b(\Phi(1 \otimes e_n))$ as a function from the open ball B_d to ΔH. In terms of the multiplier $\Phi(\cdot)$ of Φ we have

$$Ae_n(z) = b\Phi(1 \otimes e_n)(z) = \Phi(z)e_n$$

and hence the boundary values $\tilde{Ae_n}$ of Ae_n are given by $\tilde{Ae_n}(z) = \tilde{\Phi}(z)e_n$ for σ-almost every $z \in \partial B_d$. Thus for such $z \in \partial B_d$ we have

$$\text{trace}_{\Delta H}(\tilde\Phi(z)\tilde\Phi(z)^*) = \text{trace}_E(\tilde\Phi(z)^*\tilde\Phi(z)) = \sum_n \|\tilde\Phi(z)e_n\|^2 = \sum_n \|\tilde{Ae_n}(z)\|^2.$$

Integrating the latter over the sphere we obtain

$$\int_{\partial B_d} \text{trace}_{\Delta H}(\tilde\Phi(z)\tilde\Phi(z)^*)\, d\sigma = \sum_n \int_{\partial B_d} \|\tilde{Ae_n}(z)\|^2\, d\sigma(z) = \sum_n \|\tilde{Ae_n}\|^2_{H^2(\partial B_d)\otimes\Delta H}$$

and from (3.20) we see that this coincides with trace $d\Gamma(\Phi\Phi^*)$. ∎

We now establish the required asymptotic formula for $K(H)$.

Theorem C. *For every finite rank Hilbert A-module H, the curvature operator $d\Gamma(L^*L)$ belongs to the trace class $\mathcal{L}^1(H^2(\partial B_d) \otimes \Delta H)$, and we have*

$$K(H) = \mathit{trace}\, d\Gamma(L^*L) = d! \lim_{n\to\infty} \frac{\mathit{trace}\,(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{n^d}$$

where $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is the canonical completely positive map associated with the A-module structure of H.

Let $\Delta = (\mathbf{1} - \phi(\mathbf{1}))^{1/2}$. We will actually prove a slightly stronger assertion, namely

$$K(H) = \text{trace}\, d\Gamma(L^*L) = (d-1)! \lim_{n\to\infty} \frac{\text{trace}\,(\phi^n(\Delta^2))}{n^{d-1}}. \tag{3.21}$$

We first point out that it suffices to prove (3.21). For that, let $a_k = \operatorname{trace}\phi^k(\Delta^2)$, $k = 0, 1, 2, \dots$. Since

$$\mathbf{1} - \phi^{n+1}(\mathbf{1}) = \sum_{k=0}^{n} \phi^k(\mathbf{1} - \phi(\mathbf{1})) = \sum_{k=0}^{n} \phi^k(\Delta^2)$$

and since for every $r = 1, 2, \dots$ the polynomial q_r of (3.7) obeys

$$q_r(n) = \frac{(n+1)\dots(n+r)}{r!} \sim \frac{n^r}{r!},$$

we have

$$d!\frac{\operatorname{trace}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{n^d} \sim \frac{\operatorname{trace}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{q_d(n)} = \frac{a_0 + a_1 + \dots + a_n}{q_d(n)}$$

while

$$(d-1)!\frac{\operatorname{trace}\phi^n(\Delta^2)}{n^{d-1}} \sim \frac{\operatorname{trace}\phi^n(\Delta^2)}{q_{d-1}(n)} = \frac{a_n}{q_{d-1}(n)}$$

Thus the following elementary lemma allows one to deduce Theorem C from (3.21).

Lemma 3.22. *Let $d = 1, 2, \dots$ and let $a_0, a_1, \dots$ be a sequence of real numbers such that*

$$\lim_{n\to\infty} \frac{a_n}{q_{d-1}(n)} = L \in \mathbb{R}.$$

Then

$$\lim_{n\to\infty} \frac{a_0 + a_1 + \dots + a_n}{q_d(n)} = L.$$

proof of Lemma 3.22. Choose $\epsilon > 0$. By hypothesis, there is an $n_0 \in \mathbb{N}$ such that

$$(L - \epsilon)q_{d-1}(k) \le a_k \le (L + \epsilon)q_{d-1}(k), \qquad k \ge n_0. \tag{3.23}$$

By the recursion formula (3.6) we have

$$\sum_{k=n_0}^{n} q_{d-1}(k) = \sum_{k=n_0}^{n} (q_d(k) - q_d(k-1)) = q_d(n) - q_d(n_0 - 1).$$

Thus if we sum (3.23) from n_0 to n and divide through by $q_d(n)$ we obtain

$$(L - \epsilon)(1 - \frac{q_d(n_0 - 1)}{q_d(n)}) \le \frac{a_{n_0} + \dots + a_n}{q_d(n)} \le (L + \epsilon)(1 - \frac{q_d(n_0 - 1)}{q_d(n)}).$$

Since $q_d(n) \to \infty$ as $n \to \infty$, the latter inequality implies

$$L - \epsilon \le \liminf_{n\to\infty} \frac{a_0 + \dots + a_n}{q_d(n)} \le \limsup_{n\to\infty} \frac{a_0 + \dots + a_n}{q_d(n)} \le L + \epsilon,$$

and since ϵ is arbitrary, Lemma 3.23 follows. ∎

proof of Theorem C. Let $L : H^2 \otimes \Delta H \to H$ be the dilation map $L(f \otimes \zeta) = f \cdot \Delta\zeta$, $f \in H^2$, $\zeta \in \Delta H$. We claim that for every $n = 0, 1, \dots$

$$\operatorname{trace} \phi^n(\Delta^2) = \operatorname{trace}(L^* L E_n), \tag{3.24}$$

$E_n \in \mathcal{B}(H^2 \otimes \Delta H)$ being the projection onto the space of homogeneous polynomials of degree n. Indeed, from Lemma 1.4 we have

$$LL^* = \mathbf{1} - \lim_{n\to\infty} \phi^n(\mathbf{1}) = \mathbf{1} - \phi^\infty(\mathbf{1}),$$

and since $\phi(\phi^\infty(\mathbf{1})) = \phi^\infty(\mathbf{1})$ we can write

$$\Delta^2 = \mathbf{1} - \phi(\mathbf{1}) = (\mathbf{1} - \phi^\infty(\mathbf{1})) - \phi(\mathbf{1} - \phi^\infty(\mathbf{1})) = LL^* - \phi(LL^*).$$

Thus

$$\phi^n(\Delta^2) = \phi^n(LL^*) - \phi^{n+1}(LL^*). \tag{3.25}$$

Consider the free Hilbert module $F = H^2 \otimes \Delta H$ and its associated completely positive map $\phi_F : \mathcal{B}(F) \to \mathcal{B}(F)$. Since $L \in \hom(F, H)$ we have

$$\phi^k(LL^*) = L\phi_F^k(\mathbf{1}_F)L^*$$

for every $k = 0, 1, \dots$. Moreover,

$$\phi_F^n(\mathbf{1}_F) - \phi_F^{n+1}(\mathbf{1}_F) = \phi_F^n(\mathbf{1}_F - \phi_F(\mathbf{1}_F)) = \phi_F^n(E_0) = E_n,$$

so that (3.25) implies

$$\phi^n(\Delta^2) = LE_nL^*.$$

The formula (3.24) follows immediately since

$$\operatorname{trace}_H(LE_nL^*) = \operatorname{trace}_F(L^*LE_n).$$

By Lemma 1.8 there is a free module $\tilde{F}$ and $\Phi \in \hom(\tilde{F}, F)$ such that

$$L^*L = \mathbf{1}_F - \Phi\Phi^*.$$

Since both $d\Gamma(\mathbf{1}_F)$ and $d\Gamma(\Phi\Phi^*)$ are positive operators by Proposition 1.6 (indeed $d\Gamma(\mathbf{1}_F)$ is the projection of $H^2(\partial B_d) \otimes \Delta H$ onto its subspace of constant functions), it follows from Theorem 3.9 that the curvature operator $d\Gamma(L^*L)$ is trace class and, in view of (3.24), satisfies

$$\operatorname{trace} d\Gamma(L^*L) = \lim_{n\to\infty} \frac{\operatorname{trace}(L^*LE_n)}{q_{d-1}(n)} = \lim_{n\to\infty} \frac{\operatorname{trace}\phi^n(\Delta^2)}{q_{d-1}(n)}. \tag{3.26}$$

Finally, we use Lemma 3.18 together with $L^*L = \mathbf{1}_F - \Phi\Phi^*$ to evaluate the left side of (3.26) and we find that

$$\operatorname{trace} d\Gamma(L^*L) = \int_{\partial B_d} \operatorname{trace}(\mathbf{1}_{\Delta H} - \tilde{\Phi}(z)\tilde{\Phi}(z)^*)\, d\sigma(z).$$

Formula (1.13) shows that the term on the right is $K(H)$. ∎

Remark 3.27. Closed submodules of finite-rank Hilbert A-modules need not have finite rank (see section 7, Corollary of Theorem F). However, if H_0 is a submodule of a finite rank Hilbert module H which is of finite *codimension* in H, then H_0 is of finite rank. Indeed, if P_0 is the projection of H onto H_0, then

$$\operatorname{rank}(H_0) = \operatorname{rank}(\mathbf{1}_{H_0} - \phi_{H_0}(\mathbf{1}_{H_0})) = \operatorname{rank}(P_0 - \phi_H(P_0)).$$

Since $P_0 - \phi_H(P_0) = (\mathbf{1}_H - \phi_H(\mathbf{1}_H)) - P_0^\perp + \phi_H(P_0^\perp)$, we have

$$\operatorname{rank}(H_0) \leq \operatorname{rank}(H) + \operatorname{rank}(P_0^\perp) + \operatorname{rank}(\phi_H(P_0^\perp)) < \infty.$$

On the other hand, given a submodule $H_0 \subseteq H$ with $\dim(H/H_0) < \infty$, the defect operator Δ_0 of H_0 is not conveniently related to the defect operator Δ of H and thus there is no obvious way of relating $K(H_0)$ to $K(H)$. Nevertheless, Theorem C implies the following.

Corollary 1: stability of Curvature. *Let H be a finite rank contractive Hilbert A-module and let H_0 be a closed submodule such that $\dim(H/H_0) < \infty$. Then $K(H_0) = K(H)$. In particular, $K(F) = 0$ for any finite dimensional Hilbert A-module F, and for any H as above we have*

$$K(H \oplus F) = K(H).$$

proof. By estimating as in Remark 3.27 we have

$$\begin{aligned}\operatorname{trace}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0})) \leq& \operatorname{trace}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) + \\ & \operatorname{trace} P_0^\perp + \operatorname{trace}(\phi_H^{n+1}(P_0^\perp)),\end{aligned}$$

P_0 denoting the projection of H on H_0. Similarly,

$$\begin{aligned}\operatorname{trace}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) \leq& \operatorname{trace}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0})) + \\ & \operatorname{trace} P_0^\perp + \operatorname{trace}(\phi_H^{n+1}(P_0^\perp)),\end{aligned}$$

Thus we have the inequality
(3.28)
$$|\operatorname{trace}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) - \operatorname{trace}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0}))| \leq \operatorname{trace} P_0^\perp + \operatorname{trace}(\phi_H^{n+1}(P_0^\perp)).$$

One estimates the right side as follows. Note that

$$\left\langle \phi^{n+1}(P_0^\perp)\xi, \xi \right\rangle = \sum_{i_1, \dots, i_{n+1}=1}^{d} \left\langle P_0^\perp T_{i_{n+1}}^* \dots T_1^* \xi, T_{i_{n+1}}^* \dots T_1^* \xi \right\rangle$$

vanishes iff ξ belongs to the kernel of every operator of the form $P_0^\perp f(T_1, \dots, T_d)^*$ where $f \in E_{n+1}H^2$ is a homogeneous polynomial of degree $n+1$. Hence the range of the positive finite rank operator $\phi^{n+1}(P_0^\perp)$ is the orthocomplement of all such vectors ξ, and is therefore spanned linearly by the ranges of all operators $f(T_1,\dots,T_d)P_0^\perp$, $f \in E_{n+1}H^2$, i.e.,

$$\operatorname{span}\{f\cdot\zeta : f \in E_{n+1}H^2, \quad \zeta \in P_0^\perp H\}.$$

It follows that

$$\operatorname{trace}(\phi^{n+1}(P_0^\perp)) \le \dim(E_{n+1}H^2)\cdot \operatorname{trace} P_0^\perp = q_{d-1}(n+1)\operatorname{trace} P_0^\perp.$$

Thus (3.28) implies that

$$\left|\frac{\operatorname{trace}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H))}{n^d} - \frac{\operatorname{trace}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0}))}{n^d}\right|$$

is at most

$$\operatorname{trace}(P_0^\perp)\frac{1+q_{d-1}(n+1)}{n^d}.$$

Since $q_{d-1}(x)$ is a polynomial of degree $d-1$, the latter tends to zero as $n\to\infty$, and the conclusion $|K(H)-K(H_0)| = 0$ follows from Theorem C after taking the limit on n. ∎

We also point out the following application to invariant subspaces of the d-shift $S_1,\dots,S_d$ acting on H^2. In dimension $d = 1$ the invariant subspaces of the simple unilateral shift define submodules which are isomorphic to H^2 itself, and in particular they all have rank one. In higher dimensions, on the other hand, we can never have that behavior for submodules of finite codimension.

Corollary 2. *Suppose that $d \ge 2$, and let M be a proper closed submodule of H^2 of finite codimension. Then rank$(M) > 1$.*

proof. Since H^2 is a free Hilbert A-module of rank 1 we have $K(H^2) = 1$ (this computation was done in the proof of Theorem 2.1). By Corollary 1 above we must have $K(M) = 1$ as well. By Lemma 7.14 of [1], no proper submodule of H^2 can be a free Hilbert module in dimension $d > 1$, hence the first extremal property of $K(M)$ (Theorem 2.1) implies that we must have $\operatorname{rank}(M) > K(M) = 1$. ∎

Remark. Of course, the ranks of finite codimensional submodules of H^2 must be finite by Remark 3.27, and they can be arbitrarily large.

4. Euler characteristic: asymptotics of $\chi(H)$, stability.

Throughout this section, H will denote a *finite rank* Hilbert A-module. We will work not with H itself but with the following linear submanifold of H

$$M_H = \operatorname{span}\{f\cdot\Delta\xi : f \in A, \xi \in H\}.$$

The definition and basic properties of the Euler characteristic are independent of any topology associated with the Hilbert space H, and depend solely on the linear algebra of M_H. As we have pointed out in the introduction, M_H is a finitely generated A-module, and has finite free resolutions in the category of finitely generated A-modules

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow M_H \longrightarrow 0,$$

each F_k being a sum of β_k copies of the rank-one module A. The alternating sum of the ranks $\beta_1 - \beta_2 + \beta_3 - + \dots$ does not depend on the particular free resolution of M_H, and we define the *Euler characteristic* of H by

$$\chi(H) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k. \tag{4.1}$$

The main result of this section is an asymptotic formula (Theorem D) which expresses $\chi(H)$ in terms of the sequence of defect operators $\mathbf{1} - \phi^{n+1}(\mathbf{1})$, $n = 1, 2, \dots$, where ϕ is the completely positive map on $\mathcal{B}(H)$ associated with the canonical operators $T_1, \dots, T_d$ of H,

$$\phi(A) = T_1 A T_1^* + \dots + T_d A T_d^*.$$

The Hilbert polynomial is an invariant associated with finitely generated graded modules over polynomial rings $k[x_1, \dots, x_d]$, k being an arbitrary field. We require something related to the Hilbert polynomial, which exists in greater generality than the former, but whose existence can be deduced rather easily from Hilbert's original work [18], [19]. While this polynomial is quite fundamental (indeed, its existence might be described as *the* fundamental result of multivariable linear algebra), it is less familiar to analysts than it is to algebraists.

We define this polynomial in a way suited to our needs, and in particular we will make use of the sequence of polynomials $q_0, q_1, \dots \in \mathbb{Q}[x]$ of (3.6) and (3.7).

Theorem 4.2. *Let V be a vector space over a field k, let $T_1, \dots, T_d$ be a commuting set of linear operators on V, and make V into a $k[x_1, \dots, x_d]$-module by setting $f \cdot \xi = f(T_1, \dots, T_d)\xi$, $f \in k[x_1, \dots, x_d]$, $\xi \in V$.*

Let G be a finite dimensional subspace of V and define finite dimensional subspaces $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ by

$$M_n = \mathit{span}\{f \cdot \xi : f \in k[x_1, \dots, x_d], \quad \deg f \leq n, \quad \xi \in G\}.$$

Then there are integers $c_0, c_1, \dots, c_d \in \mathbb{Z}$ and $N \geq 1$ such that for all $n \geq N$ we have

$$\dim M_n = c_0 q_0(n) + c_1 q_1(n) + \dots + c_d q_d(n).$$

In particular, the dimension function $n \mapsto \dim M_n$ is a polynomial for sufficiently large n.

proof. We may obviously assume that $V = \cup_n M_n$, and hence V is a finitely generated $k[x_1, \dots, x_d]$-module. The fact that the function $n \mapsto \dim M_n$ is a polynomial of degree at most d for sufficiently large n follows from the result in section 8.4.5 of [21]; and the specific form of this polynomial follows from the discussion in [21], section 8.4.4. ∎

Remark 4.3. We emphasize that the dimension function $n \mapsto \dim M_n$ is generally *not* a polynomial for all $n \in \mathbb{N}$, but only for sufficiently large $n \in \mathbb{N}$.

We also point out for the interested reader that one can give a relatively simple direct proof of Theorem 4.2 by an inductive argument on the number d of operators, along lines similar to the proof of Theorem 4.11 of [17].

Suppose now that G is a finite dimensional subspace of V which *generates* V as a $k[x_1, \dots, x_d]$-module

$$V = \operatorname{span}\{f \cdot \xi : f \in k[x_1, \dots, x_d], \quad \xi \in G\}.$$

The polynomial

$$p(x) = c_0 q_0(x) + c_1 q_1(x) + \cdots + c_d q_d(x)$$

defined by theorem 4.2 obviously depends on the generator G; however, its top coefficient c_d does not. In order to discuss that, it is convenient to broaden the context somewhat. Let M be a module over the polynomial ring $k[x_1, \dots, x_d]$. A *filtration* of M is an increasing sequence $M_1 \subseteq M_2 \subseteq \dots$ of finite dimensional linear subspaces of M such that

$$\begin{aligned} M &= \cup_n M_n \qquad \text{and} \\ x_k M_n &\subseteq M_{n+1}, \qquad k = 1, 2, \dots, d, \quad n \geq 1. \end{aligned}$$

The filtration $\{M_n\}$ is called *proper* if there is an n_0 such that

$$M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_n, \qquad n \geq n_0. \tag{4.4}$$

Proposition 4.5. *Let $\{M_n\}$ be a proper filtration of M. Then the limit*

$$c = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d}$$

exists and defines a nonnegative integer $c = c(M)$ which is the same for all proper filtrations.

proof. Let $\{M_n\}$ be a proper filtration, choose n_0 satisfying (4.4), and let G be the generating subspace $G = M_{n_0}$. One finds that for $n = 0, 1, 2, \dots$

$$M_{n_0+n} = \operatorname{span}\{f \cdot \xi : \deg f \leq n, \quad \xi \in M_{n_0}\}$$

and hence there is a polynomial $p(x) \in \mathbb{Q}[x]$ of the form stipulated in Theorem 4.2 such that $\dim M_{n_0+n} = p(n)$ for sufficiently large n. Writing

$$p(x) = c_0 q_0(x) + c_1 q_1(x) + \cdots + c_d q_d(x)$$

and noting that q_k is a polynomial of degree k with leading coefficient $1/k!$, we find that

$$c_d = d! \lim_{n \to \infty} \frac{p(n)}{n^d} = d! \lim_{n \to \infty} \frac{\dim M_{n_0+n}}{n^d} = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d},$$

as asserted.

Now let $\{M'_n\}$ be another proper filtration. Since $M = \cup_n M'_n$ and M_{n_0} is finite dimensional, there is an $n_1 \in \mathbb{N}$ such that $M_{n_0} \subseteq M'_{n_1}$. Since $\{M'_n\}$ is also proper we can increase n_1 if necessary to arrange the condition of (4.4) on M'_n for all $n \geq n_1$, and hence

$$M'_{n_1+n} = \operatorname{span}\{f \cdot \xi : \deg f \leq n, \quad \xi \in M'_{n_1}\}.$$

Letting $p'(x) = c_0'q_0(x) + c_1'q_1(x) + \cdots + c_d'q_d(x)$ be the polynomial satisfying

$$\dim M'_{n_1+n} = p'(n)$$

for sufficiently large n, the preceding argument shows that

$$c_d' = d! \lim_{n\to\infty} \frac{\dim M_n'}{n^d}.$$

On the other hand, the inclusion $M_{n_0} \subseteq M'_{n_1}$, together with the condition (4.4) on both $\{M_n\}$ and $\{M_n'\}$, implies

$$\begin{aligned} M_{n_0+n} &= \operatorname{span}\{f\cdot\xi : \deg f \le n, \quad \xi \in M_{n_0}\} \\ &\subseteq \operatorname{span}\{f\cdot\eta : \deg f \le n, \quad \eta \in M'_{n_1}\} = M'_{n_1+n}. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty}\frac{\dim M_n}{n^d} = \lim_{n\to\infty}\frac{\dim M_{n_0+n}}{n^d} \le \lim_{n\to\infty}\frac{\dim M'_{n_1+n}}{n^d} = \lim_{n\to\infty}\frac{\dim M'_n}{n^d},$$

from which we conclude that $c_d \le c_d'$. By symmetry we also have $c_d' \le c_d$. ∎

The following two results together constitute a variant of the Artin-Rees lemma of commutative algebra (cf. [33], page II-9). Since the result we require is formulated differently than the Artin-Rees lemma (normally a statement about the behavior of *decreasing* filtrations associated with ideals and their relation to submodules), and since we have been unable to locate an appropriate reference, we have included complete proofs.

With any filtration $\{M_n\}$ of a $k[x_1,\dots,x_d]$-module M there is an associated $\mathbb{Z}$-graded module $\bar M$, which is defined as the (algebraic) direct sum of finite dimensional vector spaces

$$\bar M = \sum_{n\in\mathbb{Z}} \bar M_n,$$

where $\bar M_n = M_n/M_{n-1}$ for each $n \in \mathbb{Z}$, and where for nonpositive values of n, M_n is taken as $\{0\}$. The $k[x_1,\dots,x_d]$-module structure on $\bar M$ is defined by the commuting d-tuple of "shift" operators $T_1,\dots,T_d$, where T_k is defined on each summand $\bar M_n$ by

$$T_k : \xi + M_{n-1} \in M_n/M_{n-1} \mapsto x_k\xi + M_n \in M_{n+1}/M_n.$$

Remark 4.6. For our purposes, the essential feature of this construction is that for every $n \ge 1$, the following are equivalent

(1) $M_{n+1} = M_n + x_1M_n + \cdots + x_dM_d$
(2) $\bar M_{n+1} = T_1\bar M_n + \cdots + T_d\bar M_n$.

Lemma 4.7. *Let $\{M_n\}$ be a filtration of a $k[x_1,\dots,x_d]$-module M. The following are equivalent:*

(1) *$\{M_n\}$ is proper.*
(2) *The $k[x_1,\dots,x_d]$-module $\bar M$ is finitely generated.*

proof of (1) $\implies$ (2). Find an $n_0 \in \mathbb{N}$ such that

$$M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_n$$

for all $n \geq n_0$. From Remark 4.6 we have $\bar{M}_{n+1} = T_1\bar{M}_n + \cdots + T_d\bar{M}_n$ for all $n \geq n_0$, hence $G = \bar{M}_1 + \cdots + \bar{M}_{n_0}$ is a finite dimensional generating space for $\bar{M}$.

proof of (2) $\implies$ (1). Assuming (2), we can find a finite set of homogeneous elements $\xi_k \in \bar{M}_{n_k}$, $k = 1, \dots, r$ which generate $\bar{M}$ as a $k[x_1, \dots, x_d]$-module. It follows that for $n \geq \max(n_1, \dots, n_r)$ we have

$$\bar{M}_{n+1} = T_1\bar{M}_n + \cdots + T_d\bar{M}_n.$$

For such an n, Remark 4.6 implies that

$$M_{n+1} = M_n + x_1 M_n + \cdots + x_d M_n,$$

hence $\{M_n\}$ is proper. ∎

Lemma 4.8. *Let $\{M_n\}$ be a proper filtration of a $k[x_1, \dots, x_d]$-module M, let $K \subseteq M$ be a submodule, and let $\{K_n\}$ be the filtration induced on K by*

$$K_n = K \cap M_n.$$

Then $\{K_n\}$ is a proper filtration of K.

proof. Form the graded modules

$$\bar{M} = \sum_{n\in\mathbb{Z}} M_n/M_{n-1}$$

and

$$\bar{K} = \sum_{n\in\mathbb{Z}} K_n/K_{n-1}.$$

Because of the natural isomorphism

$$\bar{K}_n = K \cap M_n/K \cap M_{n-1} \cong (K \cap M_n + M_{n-1})/M_{n-1} \subseteq M_n/M_{n-1} = \bar{M}_n,$$

$\bar{K}$ is isomorphic to a submodule of $\bar{M}$. Lemma 4.7 implies that $\bar{M}$ is finitely generated. Thus by Hilbert's basis theorem (asserting that graded submodules of finitely generated graded modules are finitely generated), it follows that $\bar{K}$ is finitely generated. Now apply Lemma 4.7 once again to conclude that $\{K_n\}$ is a proper filtration of K. ∎

We remark that the proof of Lemma 4.8 is inspired by Cartier's proof of the Artin-Rees lemma [33], p II-9.

Let M be a finitely generated $k[x_1, \dots, x_d]$-module, choose a finite dimensional subspace $G \subseteq M$ which generates M as an $k[x_1, \dots, x_d]$-module, and set

$$M_n = \operatorname{span}\{f \cdot \zeta : f \in k[x_1, \dots, x_d], \quad \deg f \leq n, \quad \zeta \in G\}.$$

Since $\{M_n\}$ is a proper filtration, Proposition 4.5 implies that the number

$$c(M) = d! \lim_{n\to\infty} \frac{\dim M_n}{n^d}$$

exists as an invariant of M independently of the choice of generator G. The following result shows that this invariant is additive on short exact sequences.

Proposition 4.9. *For every exact sequence*

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

of finitely generated $k[x_1, \dots, x_d]$-modules we have $c(L) = c(K) + c(M)$.

proof. Since $c(M)$ depends only on the isomorphism class of M, we may assume that $K \subseteq L$ is a submodule of L and $M = L/K$ is its quotient. Pick any proper filtration $\{L_n\}$ for L and let $\{\dot{L}_n\}$ and $\{K_n\}$ be the associated filtrations of L/K and K

$$\dot{L}_n = (L_n + K)/K \subseteq L/K,$$
$$K_n = K \cap M_n \subseteq K.$$

It is obvious that $\{\dot{L}_n\}$ is proper, and Lemma 4.8 implies that $\{K_n\}$ is proper as well.

Now for each $n \geq 1$ we have an exact sequence of finite dimensional vector spaces

$$0 \longrightarrow K_n \longrightarrow L_n \longrightarrow \dot{L}_n \longrightarrow 0,$$

and hence

$$\dim L_n = \dim K_n + \dim \dot{L}_n.$$

Since each of the three filtrations is proper we can multiply the preceding equation through by $d!/n^d$ and take the limit to obtain $c(L) = c(K) + c(L/K)$. ∎

Remark 4.10. The addition property of Proposition 4.9 generalizes immediately to the following assertion. For every finite exact sequence

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$$

of finitely generated $k[x_1, \dots, x_d]$-modules, we have

$$\sum_{k=0}^{n} (-1)^k c(M_k) = 0.$$

Corollary. *Let M be a finitely generated $k[x_1, \dots, x_d]$-module and let*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

be a finite free resolution of M, where

$$F_k = \beta_k \cdot k[x_1, \dots, x_d]$$

is a direct sum of β_k copies of the rank-one free module $k[x_1, \dots, x_d]$. Then

$$c(M) = \sum_{k=1}^{n} (-1)^{k+1} \beta_k.$$

proof. Remark (4.10) implies that

$$c(M) = \sum_{k=1}^{n} (-1)^{k+1} c(F_k),$$

and thus it suffices to show that if $F = \beta \cdot k[x_1, \dots, x_d]$ is a free module of rank $\beta \in \mathbb{N}$, then $c(F) = \beta$.

By the additivity property of 4.9 we have

$$c(\beta \cdot k[x_1, \dots, x_d]) = \beta \cdot c(k[x_1, \dots, x_d])$$

and thus we have to show that $c(k[x_1, \dots, x_d])$ is 1.

This follows from a computation of the dimensions of

$$\mathcal{P}_n = \{f \in k[x_1, \dots, x_d] : \deg f \leq n\}$$

and it is a classical result that

$$\dim \mathcal{P}_n = q_d(n) = \frac{(n+1)\dots(n+d)}{d!}$$

(see Appendix A of [1] for the relevant case $k = \mathbb{C}$). Thus

$$c(k[x_1, \dots, x_d]) = d! \lim_{n \to \infty} \frac{\dim \mathcal{P}_n}{n^d} = \lim_{n \to \infty} \frac{(n+1)\dots(n+d)}{n^d} = 1$$

and the corollary is established. ∎

We now deduce the main result of this section. Let H be a finite-rank Hilbert module over $A = \mathbb{C}[z_1, \dots, z_d]$, and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be its associated completely positive map $\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*$.

Theorem D.

$$\chi(H) = d! \lim_{n \to \infty} \frac{\mathit{rank}\,(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{n^d}.$$

proof. Consider the module

$$M_H = \operatorname{span}\{f \cdot \Delta\xi : f \in A, \quad \xi \in H\}$$

and its natural (proper) filtration

$$M_n = \operatorname{span}\{f \cdot \Delta\xi : \deg f \leq n, \quad \xi \in H\}, \qquad n = 1, 2, \dots.$$

In view of the definition of $\chi(H)$ in terms of free resolutions of M_H, the preceding corollary implies that

$$\chi(H) = c(M_H) = d! \lim_{n \to \infty} \frac{\dim M_n}{n^d}.$$

Thus it suffices to show that

$$\dim M_n = \operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))$$

for every $n = 1, 2, \dots$. For that, we will prove

$$M_n = (\mathbf{1} - \phi^{n+1}(\mathbf{1}))H. \tag{4.11}$$

Indeed, writing

$$\mathbf{1} - \phi^{n+1}(\mathbf{1}) = \sum_{k=0}^{n} \phi^k(\mathbf{1} - \phi(\mathbf{1})) = \sum_{k=0}^{n} \phi^k(\Delta^2), \tag{4.12}$$

we see in particular that $\mathbf{1} - \phi^{n+1}(\mathbf{1})$ is a positive finite rank operator for every n and hence

$$(\mathbf{1} - \phi^{n+1}(\mathbf{1}))H = \ker(\mathbf{1} - \phi^{n+1}(\mathbf{1}))^{\perp}.$$

The kernel of $\mathbf{1} - \phi^{n+1}(\mathbf{1})$ is easily computed. We have

$$\ker(\mathbf{1} - \phi^{n+1}(\mathbf{1})) = \{\xi \in H : \langle(\mathbf{1} - \phi^{n+1}(\mathbf{1}))\xi, \xi\rangle = 0\},$$

and by (4.12), $\langle(\mathbf{1} - \phi^{n+1}(\mathbf{1}))\xi, \xi\rangle = 0$ iff

$$\sum_{k=0}^{n} \langle\phi^k(\Delta^2)\xi, \xi\rangle = 0.$$

Since

$$\phi^k(\Delta^2) = \sum_{i_1, \dots, i_k = 1}^{d} T_{i_1} \dots T_{i_k} \Delta^2 T_{i_k}^* \dots T_{i_1}^*,$$

the latter is equivalent to

$$\sum_{k=0}^{n} \sum_{i_1, \dots, i_k = 1}^{d} \|\Delta T_{i_k}^* \dots T_{i_1}^* \xi\|^2 = 0.$$

Thus the kernel of $\mathbf{1} - \phi^{n+1}(\mathbf{1})$ is the orthocomplement of the space spanned by

$$\{T_{i_1} \dots T_{i_k} \Delta\eta : \eta \in H, \quad 1 \le i_1, \dots, i_k \le d, \quad k = 0, 1, \dots, n\},$$

namely $M_n = \operatorname{span}\{f \cdot \Delta\eta : \deg f \le n, \quad \eta \in H\}$. This shows that

$$\ker(\mathbf{1} - \phi^{n+1}(\mathbf{1})) = M_n^{\perp},$$

from which formula (4.11) is evident. ∎

Remark 4.13. We have already pointed out in Remark 3.27 that a closed submodule $H_0 \subseteq H$ of a finite rank contractive Hilbert module which is of finite codimension in H must also be of finite rank. However, given a such a submodule $H_0 \subseteq H$ the algebraic module M_{H_0} is not a submodule of M_H, nor is it conveniently related to M_H. Again, there is no direct way of relating $\chi(H)$ to $\chi(H_0)$ by way of their definitions. However from Theorem D we obtain the following stability result.

Corollary 1: stability of Euler characteristic. *Let H_0 be a closed submodule of a finite rank Hilbert A-module H such that* $\dim(H/H_0) < \infty$. *Then* $\chi(H_0) = \chi(H)$.

proof. By estimating as in Remark 3.27 we have

$$\begin{aligned}\operatorname{rank}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0})) \leq &\operatorname{rank}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) + \\ &\operatorname{rank} P_0^\perp + \operatorname{rank}(\phi_H^{n+1}(P_0^\perp)),\end{aligned}$$

P_0 denoting the projection of H on H_0. Similarly,

$$\begin{aligned}\operatorname{rank}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) \leq &\operatorname{rank}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0})) + \\ &\operatorname{rank} P_0^\perp + \operatorname{rank}(\phi_H^{n+1}(P_0^\perp)),\end{aligned}$$

Thus we have the inequality
(4.14)

$$|\operatorname{rank}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H)) - \operatorname{rank}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0}))| \leq \operatorname{rank} P_0^\perp + \operatorname{rank}(\phi_H^{n+1}(P_0^\perp)).$$

At this point one can estimate the rank of $\phi_H^{n+1}(P_0^\perp)$ exactly as we have estimated its trace in the proof of Corollary 1 of Theorem C in section 3, and one finds that

$$\operatorname{rank}(\phi^{n+1}(P_0^\perp)) \leq \dim(E_{n+1}H^2) \cdot \operatorname{rank} P_0^\perp = q_{d-1}(n+1)\operatorname{rank} P_0^\perp.$$

Thus (4.14) implies that

$$\left|\frac{\operatorname{rank}(\mathbf{1}_H - \phi_H^{n+1}(\mathbf{1}_H))}{n^d} - \frac{\operatorname{rank}(\mathbf{1}_{H_0} - \phi_{H_0}^{n+1}(\mathbf{1}_{H_0}))}{n^d}\right|$$

is at most

$$\operatorname{rank}(P_0^\perp)\frac{1 + q_{d-1}(n+1)}{n^d}.$$

Since $q_{d-1}(x)$ is a polynomial of degree $d-1$, the latter tends to zero as $n \to \infty$, and the conclusion $|\chi(H) - \chi(H_0)| = 0$ follows from Theorem D after taking the limit on n. ∎

For algebraic reasons, the Euler characteristic of a finitely generated A-module must be nonnegative ([20], Theorem 192). One also has the upper bound $\chi(H) \leq \operatorname{rank}(H)$. More significantly, we have the following inequality relating $K(H)$ to $\chi(H)$ in general, a consequence of Theorems C and D together.

Corollary 2. *For every finite rank Hilbert A-module H,*

$$0 \leq K(H) \leq \chi(H) \leq \mathit{rank}(H).$$

proof. Let M_H be the algebraic module associated with H and let $M_1 \subseteq M_2 \subseteq \dots$ be the proper filtration of it defined by

$$M_n = \operatorname{span}\{f \cdot \xi : f \in A, \quad \deg f \leq n, \quad \xi \in \Delta H\}$$

Δ denoting the square root of $\mathbf{1}_H - T_1T_1^* - \cdots - T_dT_d^*$. Clearly

$$\dim M_n \le \dim\{f \in A : \deg f \le n\} \cdot \dim \Delta H = q_d(n) \cdot \operatorname{rank}(H).$$

From the corollary of Proposition 4.9 which identifies $\chi(M_H)$ with $c(M_H)$,

$$\chi(H) = \chi(M_H) = d! \lim_{n\to\infty} \frac{\dim M_n}{n^d} \le d! \lim_{n\to\infty} \frac{q_d(n)}{n^d} \cdot \operatorname{rank}(H) = \operatorname{rank}(H),$$

and the inequality $\chi(H) \le \operatorname{rank}(H)$ follows.

Since the trace of a positive operator A is dominated by $\|A\| \cdot \operatorname{rank}(A)$, Theorems C and D together imply that

$$K(H) = d! \lim_{n\to\infty} \frac{\operatorname{trace}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{n^d} \le d! \lim_{n\to\infty} \frac{\operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))}{n^d} = \chi(H).$$

∎

The inequality of Corollary 2 is useful; a significant application is given in Theorem E of section 7.

5. Graded Hilbert modules: Gauss-Bonnet-Chern formula.

In this section we prove an analogue of the Gauss-Bonnet-Chern theorem for Hilbert A-modules. The most general setting in which one might hope for such a result is the class of finite rank *pure* Hilbert A-modules. These are the Hilbert A-modules which are isomorphic to quotients F/M of finite rank free modules $F = H^2 \otimes \mathbb{C}^r$ by closed submodules M. However, in Proposition 7.4 we give examples of submodules $M \subseteq H^2$ for which $K(H^2/M) < \chi(H^2/M)$. In this section we establish the result (Theorem B) under the additional hypothesis that H is *graded*. Examples are obtained by taking $H = F/M$ where F is free of finite rank and M is a closed submodule generated by a set of *homogeneous* polynomials (perhaps of different degrees). In particular, one can associate such a module H with any algebraic variety in complex projective space $\mathbb{P}^{d-1}$ (see section 7).

By a *graded* Hilbert space we mean a pair H, Γ where H is a (separable) Hilbert space and $\Gamma : \mathbb{T} \to \mathcal{B}(H)$ is a strongly continuous unitary representation of the circle group $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Γ is called the *gauge group* of H. Alternately, one may think of the structure H, Γ as a $\mathbb{Z}$-graded Hilbert space by considering the spectral subspaces $\{H_n : n \in \mathbb{Z}\}$ of Γ,

$$H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}\}.$$

The spectral subspaces give rise to an orthogonal decomposition

$$H = \cdots \oplus H_{-1} \oplus H_0 \oplus H_1 \oplus \ldots. \tag{5.1}$$

Conversely, given an orthogonal decomposition of a Hilbert space H of the form (5.1), one can define an associated gauge group Γ by

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n \qquad \lambda \in \mathbb{T}$$

E_n being the orthogonal projection onto H_n.

A Hilbert A-module is said to be *graded* if there is given a distinguished gauge group Γ on H which is related to the canonical operators $T_1, \dots, T_d$ of H by

$$\Gamma(\lambda)T_k\Gamma(\lambda)^{-1} = \lambda T_k, \qquad k = 1, \dots, d, \quad \lambda \in \mathbb{T}. \tag{5.2}$$

Thus, graded Hilbert A-modules are those whose operators admit minimal (i.e., circular) symmetry. Letting H_n be the nth spectral subspace of Γ, (5.2) implies that each operator is of degree one in the sense that

$$T_k H_n \subseteq H_{n+1}, \qquad k = 1, \dots, d, \quad n \in \mathbb{Z}. \tag{5.3}$$

Conversely, given a $\mathbb{Z}$-graded Hilbert space which is also an A-module satisfying (5.3), then it follows that the corresponding gauge group

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n$$

satisfies (5.2), and moreover that the spectral projections E_n of Γ satisfy $T_k E_n = E_{n+1} T_k$ for $k = 1, \dots, d$. Thus it is equivalent to think in terms of gauge groups satisfying (5.2), or of $\mathbb{Z}$-graded Hilbert A-modules with degree-one operators satisfying (5.3). Algebraists tend to prefer the latter description because it generalizes to fields other than the complex numbers. On the other hand, the former description is more convenient for operator theory on complex Hilbert spaces, and in this paper we will work with gauge groups and (5.2).

Let H be a graded Hilbert A-module. A linear subspace $S \subseteq H$ is said to be *graded* if $\Gamma(\lambda)S \subseteq S$ for every $\lambda \in \mathbb{T}$. If $K \subseteq H$ is a graded (closed) submodule of H then K is a graded Hilbert A-module, and the gauge group of K is of course the corresponding subrepresentation of Γ. Similarly, the quotient H/K of H by a graded submodule K is graded in an obvious way. We require the following observation, asserting that several natural hypotheses on graded Hilbert modules are equivalent.

Proposition 5.4. *For every graded finite rank Hilbert A-module H, the following are equivalent.*

(1) *The spectrum of the gauge group Γ is bounded below.*
(2) *H is pure in the sense that its associated completely positive map of $\mathcal{B}(H)$ $\phi(A) = T_1 A T_1^* + \cdots + T_d A T_d^*$ satisfies $\phi^n(\mathbf{1}) \downarrow 0$ as $n \to \infty$.*
(3) *The algebraic submodule*

$$M_H = span\{f \cdot \Delta\zeta : f \in A, \quad \zeta \in \Delta H\}$$

is dense in H.
(4) *There is a finite-dimensional graded linear subspace $G \subset H$ which generates H as a Hilbert A-module.*

Moreover, if (1) through (4) are satisfied then the spectral subspaces of Γ,

$$H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi\} \qquad n \in \mathbb{Z},$$

are all finite dimensional.

proof. We prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). Let E_n be the projection onto the nth spectral subspace H_n of Γ and let $T_1, \dots, T_d$ be the canonical operators of H. From the commutation formula (5.2) it follows that $T_1T_1^* + \dots + T_dT_d^*$ commutes with $\Gamma(\lambda)$ and hence

$$\Gamma(\lambda)\Delta = \Delta\Gamma(\lambda), \qquad \lambda \in \mathbb{T}, \tag{5.5}$$

where $\Delta = (\mathbf{1} - T_1T_1^* - \dots - T_dT_d^*)^{1/2}$.

proof of (1) $\implies$ (2). The hypothesis (1) implies that there is an integer n_0 such that $E_n = 0$ for $n < n_0$. By the preceding remarks we have $T_kE_p = E_{p+1}T_k$ for every $p \in \mathbb{Z}$. Thus $\phi(E_p) = E_{p+1}\phi(\mathbf{1}) \leq E_{p+1}$, and hence $\phi^n(E_p) \leq E_{p+n}$. Writing

$$\phi^n(\mathbf{1}) = \phi^n(\sum_{p=n_0}^{\infty} E_p) = \sum_{p=n_0}^{\infty} \phi^n(E_p) \leq \sum_{p=n_0+n}^{\infty} E_p,$$

the conclusion $\lim_n \phi^n(\mathbf{1}) = 0$ is apparent.

proof of (2) $\implies$ (3). Assuming H is pure, (1.13) implies that the natural map $L \in \hom(H^2 \otimes \Delta H, H)$ defined by $L(f \otimes \zeta) = f \cdot \Delta\zeta$ satisfies $LL^* = \mathbf{1}$, and therefore $\overline{M_H} = L(H^2 \otimes \Delta H) = H$.

proof of (3) $\implies$ (4). Assuming (3), notice that $G = \Delta H$ satisfies condition (4). Indeed, G is finite dimensional because $\operatorname{rank}(H) < \infty$, it is graded because of (5.5), and it generates H as a closed A-module because the A-module M_H generated by G is dense in H.

proof of (4) $\implies$ (1). Let $G \subseteq H$ satisfy (4). The restriction of Γ to G is a finite direct sum of irreducible subrepresentations, and hence there are integers $n_0 \leq n_1$ such that

$$G = G_{n_0} \oplus G_{n_0+1} \oplus \dots \oplus G_{n_1}$$

where $G_k = G \cap H_k$. In particular, $G \subseteq H_{n_0} + H_{n_0+1} + \dots$. Since the space $H_{n_0} + H_{n_0+1} + \dots$ is invariant under the operators $T_1, \dots, T_d$ by (5.3), we have

$$H = \overline{\operatorname{span} A \cdot G} \subseteq H_{n_0} + H_{n_0+1} + \dots.$$

Thus $H = H_{n_0} + H_{n_0+1} + \dots$, hence the spectrum of Γ is bounded below by n_0.

The finite dimensionality of all of the spectral subspaces of Γ follows from condition (4), together with the fact that for every $n = 0, 1, 2, \dots$, the space $\mathcal{P}_n$ of operators $\{f(T_1, \dots, T_d)\}$ where f is a homogeneous polynomial of degree n is finite dimensional and and $\mathcal{P}_n$ maps H_k into H_{k+n}. ∎

Theorem B. *For every finite rank graded Hilbert A-module H satisfying the conditions of Proposition 5.4 we have $K(H) = \chi(H)$.*

proof. Because of the stability properties of $K(\cdot)$ and $\chi(\cdot)$ established in the corollaries of Theorems C and D, it suffices to exhibit a closed submodule $H_0 \subseteq H$ of finite codimension for which $K(H_0) = \chi(H_0)$. H_0 is constructed as follows.

Let $\{E_n : n \in \mathbb{Z}\}$ be the spectral projections of the gauge group

$$\Gamma(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n E_n.$$

Since Δ is a finite rank operator in the commutant of $\{E_n : n \in \mathbb{Z}\}$, we must have $E_n\Delta = \Delta E_n = 0$ for all but a finite number of $n \in \mathbb{Z}$, and hence there are integers $n_0 \leq n_1$ such that

$$\Delta = \Delta_{n_0} + \Delta_{n_0+1} + \cdots + \Delta_{n_1} \tag{5.6}$$

Δ_k denoting the finite rank positive operator ΔE_k.

We claim that for all $n \geq n_1$ we have

$$\phi(E_n) = E_{n+1}. \tag{5.7}$$

Indeed, since H is pure (Proposition 5.4 (2)) we can assert that

$$\mathbf{1}_H = \sum_{p=0}^{\infty} \phi^p(\Delta^2) \tag{5.8}$$

because

$$\sum_{p=0}^{n} \phi^p(\Delta^2) = \sum_{p=0}^{n} \phi^p(\mathbf{1}_H - \phi(\mathbf{1}_H)) = \mathbf{1}_H - \phi^{n+1}(\mathbf{1}_H)$$

converges strongly to $\mathbf{1}_H$ as $n \to \infty$. Multiplying (5.8) on the left with E_n we find that

$$E_n = \sum_{p=0}^{\infty} E_n\phi^p(\Delta^2), \qquad n \in \mathbb{Z}. \tag{5.9}$$

Using (5.6) we have

$$E_n\phi^p(\Delta^2) = \sum_{k=n_0}^{n_1} E_n\phi^p(\Delta_k^2).$$

Now $\Delta_k^2 \leq E_k$ and hence $\phi^p(\Delta_k^2) \leq E_{k+p}$ for every $p = 0, 1, \dots$. Thus for $n \geq n_1$,

$$\sum_{p=0}^{\infty}\sum_{k=n_0}^{n_1} E_n\phi^p(\Delta_k^2) = \sum_{k=n_0}^{n_1} \phi^{n-k}(\Delta_k^2) = \phi^{n-n_1}(\sum_{k=n_0}^{n_1} \phi^{n_1-k}(\Delta_k^2)).$$

This shows that when $n \geq n_1$, E_n has the form

$$E_n = \phi^{n-n_1}(B), \tag{5.10}$$

where B is the operator

$$B = \sum_{k=n_0}^{n_1} \phi^{n_1-k}(\Delta_k^2),$$

and (5.7) follows immediately from (5.10).

Now consider the submodule $H_0 \subseteq H$ defined by

$$H_0 = \sum_{n=n_1}^{\infty} E_n H.$$

Notice that $H_0^\perp$ is finite dimensional. Indeed, that is apparent from the fact that

$$H_0^\perp = \sum_{n=-\infty}^{n_1-1} E_n H$$

because by Proposition 5.4 (1) only a finite number of the projections $\{E_n : n < n_1\}$ can be nonzero (indeed, here one can show that $E_n = 0$ for $n < n_0$), and Proposition 5.4 also implies that E_n is finite dimensional for all n.

Let $\phi_0 : \mathcal{B}(H_0) \to \mathcal{B}(H_0)$ be the completely positive map of $\mathcal{B}(H_0)$ associated with the operators $T_1 \restriction_{H_0}, \dots, T_d \restriction_{H_0}$. Then for every $k = 0, 1, \dots$ we have

$$\phi_0^k(\mathbf{1}_{H_0}) = \sum_{n=n_1}^{\infty} \phi^k(E_n).$$

From (5.7) we have $\phi^k(E_n) = E_{n+k}$ for $n \geq n_1$, and hence

$$\phi_0^k(\mathbf{1}_{H_0}) = \sum_{p=n_1+k}^{\infty} E_p.$$

It follows that

$$\mathbf{1}_{H_0} - \phi_0^{k+1}(\mathbf{1}_{H_0}) = E_{n_1} + E_{n_1+1} + \cdots + E_{n_1+k}$$

is a projection for every $k = 0, 1, \dots$. Thus for every $k \geq 0$,

$$\operatorname{trace}(\mathbf{1}_{H_0} - \phi_0^{k+1}(\mathbf{1}_{H_0})) = \operatorname{rank}(\mathbf{1}_{H_0} - \phi_0^{k+1}(\mathbf{1}_{H_0})),$$

and the desired formula $K(H_0) = \chi(H_0)$ follows immediately from Theorems C and D after multiplying through by $d!/k^d$ and taking the limit on k. ∎

6. Degree.

Theorem D together with its Corollary 2 imply that both the curvature invariant and the Euler characteristic (of a finite rank Hilbert A-module) vanish whenever the rank function $\operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))$ grows relatively slowly with n. In such cases there are other numerical invariants which must be nontrivial and which can be calculated explicitly in certain cases. In this brief section we define these secondary invariants and summarize their basic properties.

Let H be a finite rank Hilbert A-module. Consider the algebraic submodule

$$M_H = \operatorname{span}\{f \cdot \Delta\xi : f \in A, \quad \xi \in H\}$$

and its natural filtration $\{M_n : n = 0, 1, 2, \dots\}$

$$M_n = \operatorname{span}\{f \cdot \Delta\xi : \deg f \leq n, \quad \xi \in H\}.$$

By Theorem 4.2 there are integers $c_0, c_1, \dots, c_d$ such that

$$\dim M_n = c_0 q_0(n) + c_1 q_1(n) + \cdots + c_d q_d(n) \tag{6.1}$$

for sufficiently large n. Let k be the degree of the polynomial on the right of (6.1). We observe first that the pair (k, c_k) depends only on the algebraic structure of M_H.

Proposition 6.2. *Let M be a finitely generated A-module, let $\{M_n : n \geq 1\}$ be a proper filtration of M, and suppose $M \neq \{0\}$. Then there is a unique integer k, $0 \leq k \leq d$, such that the limit*

$$\mu(M) = k! \lim_{n\to\infty} \frac{\dim M_n}{n^k}$$

exists and is nonzero. $\mu(M)$ is a positive integer and the pair $(k, \mu(M))$ does not depend on the particular filtration $\{M_n\}$.

proof. By Theorem 4.2 there are integers $c_0, c_1, \dots, c_d$ such that

$$\dim M_n = c_0 q_0(n) + c_1 q_1(n) + \cdots + c_d q_d(n)$$

for sufficiently large n. Let k be the degree of the polynomial on the right. Noting that $q_r(x)$ is a polynomial of degree r with leading coefficient $1/r!$, it is clear that this k is the unique integer with the stated property and that

$$\mu = k! \lim_{n\to\infty} \frac{\dim M_n}{n^k} = c_k$$

is a (necessarily positive) integer.

To see that (k, μ) does not depend on the filtration, let $\{M_n'\}$ be a second proper filtration. $\{M_n'\}$ gives rise to a polynomial $p'(x)$ of degree k' which satisfies $\dim M_n' = p'(n)$ for sufficiently large n. As in the proof of Proposition 4.5, there is an integer p such that $\dim M_n \leq \dim M_{n+p}'$ for sufficiently large n. Thus

$$0 < k! \lim_{n\to\infty} \frac{\dim M_n}{n^k} \leq k! \limsup_{n\to\infty} \frac{\dim M_{n+p}'}{n^k}.$$

Now if k were greater than k' then the term on the right would be 0. Hence $k \leq k'$ and, by symmetry, $k = k'$.

We may now argue exactly as in the proof of Proposition 4.5 to conclude that the leading coefficients of the two polynomials must be the same, hence $\mu = \mu'$. ∎

Definition 6.3. *Let H be a Hilbert A-module of finite positive rank. The degree of the polynomial (6.1) associated with any proper filtration of the algebraic module M_H is called the degree of H, and is written* $\deg(H)$.

We will also write $\mu(H)$ for the positive integer

$$\mu(H) = \deg(H)! \lim_{n\to\infty} \frac{\dim M_n}{n^{\deg(H)}}$$

associated with the degree of H. If M_H is finite dimensional and not $\{0\}$ then the sequence of dimensions $\dim M_n$ associated with any proper filtration $\{M_n\}$ is eventually a nonzero constant, hence $\deg(H) = 0$ and $\mu(H) = \dim(H)$; conversely, if $\deg(H) = 0$ then M_H is finite dimensional. In particular, $\deg H$ *is a positive integer satisfying* $\deg(H) \leq d$ *whenever the algebraic submodule M_H is infinite dimensional.*

Note too that $\deg(H) = d$ iff the Euler characteristic is positive, and in that case we have $\mu(H) = \chi(H)$. In general, there is no obvious relation between $\deg(H)$ and $\operatorname{rank}(H)$, or between $\mu(H)$ and $\operatorname{rank}(H)$. In particular, $\mu(H)$ can be arbitrarily large. The operator-theoretic significance of the invariant $\mu(H)$ is not well understood. An example for which $1 < \deg(H) < d$ is worked out in section 7.

Finally, let ϕ be the completely positive map associated with the canonical operators $T_1, \dots, T_d$,

$$\phi(A) = T_1AT_1^* + \cdots + T_dAT_d^*, \qquad A \in \mathcal{B}(H).$$

We consider the generating function (more precisely, the formal power series) associated with the sequence of integers $\operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))$, $n = 0, 1, 2, \dots,$

$$\hat{\phi}(t) = \sum_{n=0}^{\infty} \operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1}))t^n. \tag{6.4}$$

We require the following description of $\deg(H)$ and $\mu(H)$ in terms of $\hat{\phi}(t)$.

Proposition 6.5. *The series $\hat{\phi}(t)$ converges for every t in the open unit disk of the complex plane. There is a polynomial $p(t) = a_0 + a_1t + \cdots + a_st^s$ and a sequence $c_0, c_1, \dots, c_d$ of real numbers, not all of which are 0, such that*

$$\hat{\phi}(t) = p(t) + \frac{c_0}{1-t} + \frac{c_1}{(1-t)^2} + \cdots + \frac{c_d}{(1-t)^{d+1}}, \qquad |t| < 1.$$

This decomposition is unique, and c_k belongs to $\mathbb{Z}$ for every $k = 0, 1, \dots, d$. $\deg(H)$ *is the largest k for which $c_k \neq 0$, and $\mu(H) = c_k$.*

proof. The proof of Theorem D shows that $\operatorname{rank}(\mathbf{1} - \phi^{n+1}(\mathbf{1})) = \dim M_n$, where $\{M_n : n = 1, 2, \dots\}$ is the natural filtration of M_H,

$$M_n = \operatorname{span}\{f \cdot \xi : \deg(f) \le n, \quad \xi \in \Delta H\}.$$

Since each $q_r(x)$ is a polynomial of degree r, formula (6.1) implies that there is a constant $K > 0$ such that

$$\dim M_n \le Kn^d, \qquad n = 1, 2, \dots,$$

and this estimate implies that the power series $\sum_n \dim M_n t^n$ converges absolutely for every complex number t in the open unit disk.

Note too that for every $k = 0, 1, \dots, d$ the generating function for the sequence $q_k(n)$, $n = 0, 1, \dots$ is given by

$$\hat{q_k}(t) = \sum_{n=0}^{\infty} q_k(n)t^n = (1-t)^{-k-1}, \qquad |t| < 1. \tag{6.6}$$

Indeed, the formula is obvious for $k = 0$ since $q_0(n) = 1$ for every n; and for positive k the recurrence formula 3.2.2, together with $q_k(0) = 1$, implies that

$$(1-t)\hat{q_k}(t) = \widehat{q_{k-1}}(t),$$

from which (6.6) follows immediately.

Using (6.1) and (6.6) we find that there is a polynomial $f(x)$ such that

$$\hat{\phi}(t) = f(t) + \sum_{k=0}^{d} \frac{c_k}{(1-t)^{k+1}}, \tag{6.7}$$

as asserted.

(6.7) implies that $\hat{\phi}$ extends to a meromorphic function in the entire complex plane, having a single pole at $t = 1$. The uniqueness of the representation of (6.7) follows from the uniqueness of the Laurent expansion of an analytic function around a pole. The remaining assertions of Propostion 6.5 are now obvious from the relation that exists between (6.1) and (6.6). ∎

7. Applications, Examples, Problems.

In this section we establish the existence of inner sequences for invariant subspaces of H^2 which contain at least one nonzero polynomial (Theorem E), and we exhibit a broad class of invariant subspaces of H^2 which define Hilbert modules of infinite rank (Corollary of Theorem F). The latter result stands in rather stark contrast with Hilbert's basis theorem, which implies that submodules of finitely generated $\mathbb{C}[z_1, \dots, z_d]$-modules are finitely generated.

Every algebraic set in complex projective space $\mathbb{P}^{d-1}$ gives rise to a finite rank contractive Hilbert module over $A = \mathbb{C}[z_1, \dots, z_d]$. We will discuss several examples of this construction in some detail (indeed, every finitely generated graded module over A can be "completed" to a finite rank Hilbert A-module, but we restrict attention here to the simplest case of modules arising as the natural coordinate ring of an algebraic set). We give explicit examples of pure rank-one Hilbert modules illustrating (1) the failure of Theorem B for ungraded modules, and (2) the computation of the degree in cases where the Euler characteristic vanishes. We also give examples of pure rank 2 graded Hilbert modules illustrating (3) the computation of nonzero values of $K(H) = \chi(H) = 1 < \operatorname{rank}(H)$.

Let $M \subseteq H^2 = H^2(\mathbb{C}^d)$ be a closed submodule of the rank 1 free Hilbert module. We have seen in section 2 that there are sequences $\phi_1, \phi_2, \dots$ of multipliers of H^2 which satisfy

$$M_{\phi_1} M_{\phi_1}^* + M_{\phi_2} M_{\phi_2}^* + \cdots = P_M,$$

that any such sequence obeys $\sum_n |\phi(z)|^2 \leq 1$ for every $z \in B_d$, and hence the associated sequence of boundary functions $\tilde{\phi}_n : \partial B_d \to \mathbb{C}$ satisfies $\sum_n |\tilde{\phi}_n(z)|^2 \leq 1$ almost everywhere $d\sigma$ on the boundary ∂B_d. Recall that $\{\phi_n\}$ is called an *inner sequence* if equality holds

$$\sum_n |\tilde{\phi}_n(z)|^2 = 1$$

almost everywhere $(d\sigma)$ on ∂B_d.

Problem. Is every nonzero closed submodule $M \subseteq H^2$ associated with an inner sequence?

The following result gives an affirmative answer for many cases of interest.

Theorem E. *Let M be a closed submodule of H^2 which contains a nonzero polynomial. Then every sequence $\phi_1, \phi_2, \dots$ of multipliers satisfying $\sum_n M_{\phi_n} M_{\phi_n}^* = P_M$ is an inner sequence.*

proof. Consider the rank-one Hilbert module $H = H^2/M$. The natural projection $L : H^2 \to H^2/M$ provides the minimal dilation of H (see Lemma 1.4), and the algebraic submodule of H is given by

$$M_H = L(A) = (A + M)/M \cong A/A \cap M,$$

where as usual $A = \mathbb{C}[z_1, \dots, z_d]$. Thus the annihilator of M_H is $A \cap M \neq \{0\}$. A theorem of Auslander and Buchsbaum (Corollary 20.13 of [17], or Theorem 195 of [20]) implies that $\chi(M_H) = 0$. By Corollary 2 of Theorem D we have $K(H) = \chi(H) = \chi(M_H) = 0$, and the assertion now follows from Theorem 2.2. ∎

Every closed invariant subspace of H^2 defines a contractive Hilbert A-module in the obvious way by restricting the d-shift, and it is natural to ask when such submodules are of finite rank. In dimension $d = 1$, every nonzero submodule of H^2 is isomorphic to H^2 itself and thus has rank 1. In dimension $d \geq 2$ at the algebraic level we have Hilbert's basis theorem, which implies that every ideal in the polynomial algebra A is finitely generated. Correspondingly, one might ask if submodules of H^2 must be of finite rank. Certainly there are examples of finite rank submodules of H^2; but the only examples we know are trivial in the sense that the submodules are actually of finite codimension in H^2. Thus we have been led to ask the following question.

Problem. In dimension $d \geq 2$, does there exist a closed submodule $M \subseteq H^2$ of infinite codimension in H^2 such that $\operatorname{rank}(M) < \infty$?

We now show that the answer to this question is no for *graded* submodules $M \subseteq H^2$.

Theorem F. *Let M be a graded proper submodule of H^2 such that $\operatorname{rank}(M) < \infty$. Then M is of finite codimension in H^2 and the canonical operators $T_1, \dots, T_d$ of the quotient H^2/M are all nilpotent.*

proof. The defect operator of M is defined by $\Delta_M = (\mathbf{1}_M - T_1 T_1^* - \dots - T_d T_d^*)^{1/2}$ where $(T_1, \dots, T_d)$ is obtained by restricting the d-shift $(S_1, \dots, S_d)$ to M. Since $T_k T_k^* = S_k P_M S_k^*$, we can identify Δ_M^2 with the following operator on H^2,

$$P_M - S_1 P_M S_1^* - \dots - S_d P_M S_d^* = P_M - \sigma(P_M),$$

σ denoting the completely positive map of $\mathcal{B}(H^2)$ associated with the d-shift $\sigma(X) = S_1 X S_1^* + \dots + S_d X S_d^*$.

Let Γ be the gauge group of H^2, $\Gamma(\lambda) f(z_1, \dots, z_d) = f(\lambda z_d, \dots, \lambda z_d)$, $f \in H^2$, $\lambda \in \mathbb{T}$. Since M is graded we have $\Gamma(\lambda) M = M$ for every λ, hence $\Gamma(\lambda)$ commutes with P_M. Since $\Gamma(\lambda) S_k = \lambda S_k \Gamma(\lambda)$, it follows that for every $X \in \mathcal{B}(H^2)$ we have $\Gamma(\lambda) \sigma(X) \Gamma(\lambda)^* = \sigma(\Gamma(\lambda) X \Gamma(\lambda)^*)$, and hence $\Gamma(\lambda)$ also commutes with $\Delta_M^2 = P_M - \sigma(P_M)$.

We claim first that there is a finite set of polynomials $\phi_1, \dots, \phi_n$ such that each ϕ_k is homogeneous of some degree n_k (i.e. $\Gamma(\lambda) \phi_k = \lambda^{n_k} \phi_k$, $\lambda \in \mathbb{T}$), and

$$\Delta_M^2 = \phi_1 \otimes \overline{\phi_1} + \dots + \phi_n \otimes \overline{\phi_n}, \tag{7.1}$$

$\phi \otimes \overline{\psi}$ denoting the rank one operator defined on H^2 by $\xi \mapsto \langle \xi, \psi \rangle \phi$. To see this, let E_p be the projection of H^2 onto the subspace of homogeneous polynomials of degree p, $p = 0, 1, 2, \dots$. Since

$$\Gamma(\lambda) = \sum_{p=0}^{\infty} \lambda^p E_p$$

and since Δ_M^2 is a finite rank operator commuting with $\Gamma(\mathbb{T})$, we must have $E_p \Delta_M = \Delta_M E_p = 0$ for all but a finite number of p. Thus there is a finite set of integers $0 \leq p_1 < \cdots < p_r$ such that

$$\Delta_M^2 = \Delta_M E_{p_1} + \cdots + \Delta_M E_{p_r}.$$

Each $\Delta_M E_p$ is a finite rank positive operator supported in the space of homogeneous polynomials $E_p H^2$, and by the spectral theorem it can be expressed as a (finite) sum of rank-one operators of the form $f \otimes \overline{f}$ with $f \in E_p H^2$. Formula (7.1) follows.

Now let $\phi_1, \dots, \phi_n$ be the polynomials of (7.1). We assert next that

$$P_M = M_{\phi_1} M_{\phi_1}^* + \cdots + M_{\phi_n} M_{\phi_n}^*, \tag{7.2}$$

M_ϕ denoting the multiplication operator $\phi(S_1, \dots, S_d) \in \mathcal{B}(H^2)$. For that, notice first that

$$P_M = \lim_{m \to \infty} \sum_{k=0}^{m} \sigma^k(\Delta_M^2).$$

Indeed, since $\Delta_M^2 = P_M - \sigma(P_M)$ the right side telescopes to

$$\lim_{m \to \infty} (P_M - \sigma^{m+1}(P_M)) = P_M,$$

since $\sigma^{m+1}(P_M) \leq \sigma^{m+1}(\mathbf{1}_{H^2}) \downarrow 0$ as $m \to \infty$. Similarly, if ϕ,ψ are any polynomials then we claim

$$M_\phi M_\psi^* = \lim_{m \to \infty} \sum_{k=0}^{m} \sigma^k(\phi \otimes \overline{\psi}).$$

Indeed, since $S_1, \dots, S_d$ (resp. $S_1^*, \dots, S_d^*$) commutes with M_ϕ (resp. M_ψ^*) we have $M_\phi \sigma(X) M_\psi^* = \sigma(M_\phi X M_\psi^*)$, hence

$$\begin{aligned}
\phi \otimes \overline{\psi} &= M_\phi (1 \otimes \overline{1}) M_\psi^* = M_\phi E_0 M_\psi^* = M_\phi (\mathbf{1}_{H^2} - \sigma(\mathbf{1}_{H^2})) M_\psi^* \\
&= M_\phi M_\psi^* - M_\phi \sigma(\mathbf{1}_{H^2}) M_\psi^* = M_\phi M_\psi^* - \sigma(M_\phi M_\psi^*).
\end{aligned}$$

Thus as before we can write

$$\begin{aligned}
M_\phi M_\psi^* &= \lim_{m \to \infty} (M_\phi M_\psi^* - \sigma^{m+1}(M_\phi M_\psi^*)) \\
&= \lim_{m \to \infty} \sum_{k=0}^{m} \sigma^k (M_\phi M_\psi^* - \sigma(M_\phi M_\psi^*)) = \lim_{m \to \infty} \sum_{k=0}^{m} \sigma^k(\phi \otimes \overline{\psi})
\end{aligned}$$

as asserted.

Choosing $\phi_1, \dots, \phi_n$ as in (7.1), the above two formulas imply

$$P_M = \sum_{k=0}^{\infty} \sigma^k(\Delta_M^2) = \sum_{k=0}^{\infty} \sigma^k(\phi_1 \otimes \overline{\phi_1} + \dots + \phi_n \otimes \overline{\phi_n}) = M_{\phi_1} M_{\phi_1}^* + \dots + M_{\phi_n} M_{\phi_n}^*,$$

and (7.2) follows.

We claim next that the polynomials $\phi_1, \dots, \phi_n$ satisfy

$$|\phi_1(z)|^2 + \dots + |\phi_n(z)|^2 \equiv 1, \qquad z \in \partial B_d.$$

Indeed, Theorem E implies that $K(H^2/M) = 0$. By Theorem 2.2 $\phi_1, \dots, \phi_n$ must be an inner sequence, hence there is a Borel set $N \subseteq \partial B_d$ of measure zero such that $|\phi_1(z)|^2 + \dots + |\phi_n(z)|^2 = 1$ for every $z \in \partial B_d \setminus N$. Since the function $z \in \mathbb{C}^d \mapsto |\phi_1(z)|^2 + \dots + |\phi_n(z)|^2 - 1$ is everywhere continuous it must vanish identically on ∂B_d.

Consider now the variety of common zeros

$$V = \{z \in \mathbb{C}^d : \phi_1(z) = \dots = \phi_n(z) = 0\}$$

of $\phi_1, \dots, \phi_n \in A = \mathbb{C}[z_1, \dots, z_d]$. We have just seen that V does not intersect the unit sphere. V cannot be empty because that would imply that the ideal $I = (\phi_1, \dots, \phi_n) \subseteq A$ generated by $\phi_1, \dots, \phi_d$ is all of A (a proper ideal in A must have a nonvoid zero set because of Hilbert's Nullstellensatz) and hence $M = H^2$ would not be a proper submodule. Since V is a nonempty set invariant under multiplication by nonzero scalars which misses the unit sphere, it must consist of just the single point $(0, 0, \dots, 0)$.

By Hilbert's Nullstellensatz there is an integer $p \geq 1$ such that $z_1^p, \dots, z_d^p$ belong to $I = (\phi_1, \dots, \phi_n)$ ([17], Theorem 1.6). Since the A-module A/I has a cyclic vector $1 + I$ and its canonical operators are all nilpotent, it follows that A/I is finite dimensional. Finally, since the natural map $f \mapsto f + M$ of A into H^2/M has dense range and vanishes on I, it induces a linear map of A/I to H^2/M with dense range. Hence H^2/M is finite dimensional and Theorem F follows. ∎

Corollary. *In dimension $d \geq 2$, every graded invariant subspace of infinite codimension in $H^2(\mathbb{C}^d)$ is an infinite rank Hilbert A-module.*

Remarks. We point out that in dimension $d = 1$ the graded submodules of H^2 are simply those of the form $M_n = z^n \cdot H^2$, $n = 0, 1, 2, \dots$. Hence there are no graded submodules of infinite codimension and the preceding corollary is vacuous in dimension 1. On the other hand, in dimension $d \geq 2$ there are many interesting graded submodules of $H^2(\mathbb{C}^d)$. For example, with any projective variety $V \subseteq \mathbb{P}^{d-1}$ we can associate a submodule $M_V \subseteq H^2$ consisting of all H^2 functions which "vanish on V" as in (7.6) below. Theorem F implies that M_V will be of infinite codimension whenever V is nonempty, and $\operatorname{rank}(M_V) = \infty$.

One may broaden this class of examples by choosing a set $\{\phi_1, \phi_2, \dots, \phi_n\}$ of *homogeneous* polynomials in A (perhaps of different degrees) and by taking for M the closed submodule of H^2 generated by $\{\phi_1, \phi_2, \dots, \phi_n\}$. M is a graded submodule, and hence rank $(M) = \infty$ whenever M is of infinite codimension in H^2.

We now discuss the limits of Theorem B by presenting a class of examples for which $K(H) < \chi(H)$ (Proposition 7.3); a concrete example of such a Hilbert A-module is given in Example 7.4. Then we will elaborate on the method alluded to in the preceding paragraphs which associates a graded Hilbert A-module with an algebraic variety in complex projective space $\mathbb{P}^{d-1}$, and we show that for *some* examples one can calculate all numerical invariants of their associated Hilbert modules.

Remark 7.3. We make use of the fact that if K_1 and K_2 are two closed submodules of the free Hilbert module H^2 for which H^2/K_1 is isomorphic to H^2/K_2, then $K_1 = K_2$. In particular, no nontrivial quotient of H^2 of the form H^2/K with $K \neq \{0\}$ can be a free Hilbert A-module (see Corollary 2 of Theorem 7.5 in [1]).

Proposition 7.4. *Let $K \neq \{0\}$ be a closed submodule of H^2 which contains no nonzero polynomials, and consider the pure rank-one module $H = H^2/K$. Then*

$$0 \leq K(H) < \chi(H) = 1.$$

proof. We show first that $\chi(H) = 1$ by proving that the algebraic submodule M_H of H is free. Let $L \in \hom(H^2, H)$ be the natural projection onto $H = H^2/K$. The kernel of L is K, and L maps the dense linear subspace $A \subseteq H^2$ of polynomials onto M_H, $L(A) = M_H$. Since $A \cap K = \{0\}$, the restriction of L to A gives an isomorphism of A-modules $A \cong M_H$, and hence $\chi(H) = \chi(A) = 1$.

On the other hand, if $K(H)$ were to equal $1 = \operatorname{rank}(H)$ then by the extremal property (4.13) H would be isomorphic to the free Hilbert module H^2 of rank-one, which is impossible because of Remark 7.3. ∎

Problem. Is the curvature invariant $K(H)$ of a *pure* finite rank Hilbert A-module H always an integer?

Theorem B implies that this is the case for graded Hilbert modules, but Proposition 7.4 shows that Theorem B does not always apply. In particular, it is not known if $K(H) = 0$ for the ungraded Hilbert modules H of Prop. 7.4. In such cases, the equation $K(H) = 0$ is equivalent to the existence of an "inner sequence" for the invariant subspace K (see Theorem 2.2).

Example 7.5. It is easy to give concrete examples of submodules K of H^2 satisfying the hypothesis of Proposition 7.4. Consider, for example, the graph of the exponential function $G = \{(z, e^z) : z \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Take $d = 2$, let $H^2 = H^2(\mathbb{C}^2)$, and let K be the submodule of all functions in H^2 which vanish on the intersection of G with the unit ball

$$K = \{f \in H^2 : f \restriction_{G \cap B_d} = 0\}.$$

Since $f \in H^2 \mapsto f(z) = \langle f, u_z \rangle$ is a bounded linear functional for every $z \in B_d$ it follows that K is closed, and it is clear that $K \neq \{0\}$ (the function $f(z_1, z_2) = e^{z_1} - z_2$ belongs to H^2 and vanishes on $G \cap B_d$). After noting that the open unit disk about $z = -1/2$ maps into $G \cap B_d$,

$$\{(z, e^z) : |z + 1/2| < 1\} \subseteq G \cap B_d$$

an elementary argument (which we omit) establishes the obvious fact that no nonzero polynomial can vanish on $G \cap B_d$.

An algebraic set in complex projective space $\mathbb{P}^{d-1}$ can be described as the set of common zeros of a finite set of *homogeneous* polynomials $f_1, \dots, f_n \in \mathbb{C}[z_1, \dots, z_d]$,

$$V = \{z \in \mathbb{C}^d : f_1(z) = \cdots = f_n(z) = 0\}$$

[17], pp 39–40. One can associate with V a graded rank-one Hilbert A-module in the following way. Let M_V be the submodule of $H^2 = H^2(\mathbb{C}^d)$ defined by

$$M_V = \{f \in H^2 : f \restriction_{V \cap B_d} = 0\}. \tag{7.6}$$

As in example 7.5, M_V is a closed submodule of H^2. Moreover, since $\lambda V \subseteq V$ for complex scalars λ, M_V is invariant under the action of the gauge group of H^2 and hence it is a graded submodule of H^2. Thus, $H = H^2/M_V$ is a graded, pure, rank-one Hilbert A-module.

We will show how to explicitly compute H^2/M_V and its numerical invariants in certain cases, using operator-theoretic methods. The simplest member of this class of examples is the variety V defined by the range of the quadratic polynomial

$$F : (x, y) \in \mathbb{C}^2 \mapsto (x^2, y^2, \sqrt{2}xy) \in \mathbb{C}^3,$$

that is,

$$V = \{(x^2, y^2, \sqrt{2}xy) : x, y \in \mathbb{C}\} \subseteq \mathbb{C}^3.$$

However, one finds more interesting behavior in the higher dimensional example

$$V = \{(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz) : x, y, z \in \mathbb{C}\} \subseteq \mathbb{C}^6, \tag{7.7}$$

and we will discuss the example (7.7) in some detail.

Notice first that V can be described in the form (7.6) as the set

$$V = \{z \in \mathbb{C}^6 : f_1(z) = f_2(z) = f_3(z) = f_4(z) = 0\} \tag{7.8}$$

of common zeros of the four homogeneous polynomials $f_k : \mathbb{C}^6 \to \mathbb{C}$,

$$\begin{aligned} f_1(z) &= z_4^2 - 2z_1z_2 = 0 \\ f_2(z) &= z_5^2 - 2z_1z_3 = 0 \\ f_3(z) &= z_6^2 - 2z_2z_3 = 0 \\ f_4(z) &= z_4z_5z_6 - 2^{3/2}z_1z_2z_3 = 0. \end{aligned}$$

The equivalence of (7.7) and (7.8) is an elementary computation which we omit. Note, however, that the fourth equation $f_4(z) = 0$ is necessary in order to exclude points such as $z = (1, 1, 1, -\sqrt{2}, \sqrt{2}, \sqrt{2})$, which satisfy the first three equations $f_1(z) = f_2(z) = f_3(z) = 0$ but which do not belong to V. Note too that f_1, f_2, f_3 are quadratic but that f_4 is cubic.

We will describe the Hilbert module $H = H^2(\mathbb{C}^6)/M_V$ by identifying its associated 6-contraction $(T_1, \dots, T_6)$. These operators act on the *even* subspace H of $H^2(\mathbb{C}^3)$, defined as the closed linear span of all homogeneous polynomials

$f(z_1, z_2, z_3)$ of even degree $2n$, $n = 0, 1, 2, \dots$. Let $S_1, S_2, S_3 \in \mathcal{B}(H^2(\mathbb{C}^3))$ be the 3-shift. The even subspace H is not invariant under the S_k, but it is invariant under any product of two of these operators S_iS_j, $1 \leq i, j \leq 3$. Thus we can define a 6-tuple of operators $T_1, \dots, T_6 \in \mathcal{B}(H)$ by

$$(7.9) \quad (T_1, \dots, T_6) = (S_1^2 \restriction_H, S_2^2 \restriction_H, S_3^2 \restriction_H, \sqrt{2}S_1S_2 \restriction_H, \sqrt{2}S_1S_3 \restriction_H, \sqrt{2}S_2S_3 \restriction_H).$$

$(T_1, \dots, T_6)$ is a 6-contraction because

$$\sum_{k=1}^{6} T_k T_k^* = \sum_{i,j=1}^{3} S_iS_j(P_H)S_j^*S_i^* \leq P_H,$$

and in fact H becomes a pure Hilbert $\mathbb{C}[z_1, \dots, z_6]$-module.

If f is a sum of homogeneous polynomials of even degrees then

$$\Gamma(e^{i\theta})f(z_1, z_2, z_3) = f(e^{i\theta/2}z_1, e^{i\theta/2}z_2, e^{i\theta/2}z_3)$$

gives a well-defined unitary action of the circle group on the subspace $H \subseteq H^2(\mathbb{C}^3)$, and H becomes a graded Hilbert module.

Proposition 7.10. *H is a rank-one graded Hilbert $\mathbb{C}[z_1, \dots, z_6]$-module which is isomorphic to $H^2(\mathbb{C}^6)/M_V$. The invariants of H are given by $K(H) = \chi(H) = 0$, $\deg(H) = 3$, $\mu(H) = 4$.*

proof. Let $\phi(A) = T_1AT_1^* + \dots + T_6AT_6^*$ be the canonical completely positive map of $\mathcal{B}(H)$ and, considering H as a subspace of $H^2(\mathbb{C}^3)$, let $\sigma : \mathcal{B}(H^2) \to \mathcal{B}(H^2)$ be the map associated with the 3-shift

$$\sigma(B) = S_1BS_1^* + S_2BS_2^* + S_3BS_3^*.$$

ϕ and σ are related in the following simple way: for every $A \in \mathcal{B}(H)$ we have

$$(7.11) \qquad \phi(A) = \sum_{k=1}^{6} T_kAT_k^* = \sum_{i,j=1}^{3} S_iS_jAP_HS_j^*S_i^* = \sigma^2(AP_H).$$

If $E_n \in \mathcal{B}(H^2)$ denotes the projection onto the subspace of homogeneous polynomials of degree n, then

$$\phi(\mathbf{1}_H) = \sigma^2(\sum_{n=0}^{\infty} E_{2n}) = \sum_{n=0}^{\infty} E_{2n+2}.$$

It follows that

$$\Delta^2 = \mathbf{1}_H - \phi(\mathbf{1}_H) = E_0$$

is the one-dimensional projection onto the space of constants. Since

$$\phi^n(\mathbf{1}_H) = \sigma^{2n}(\sum_{p=0}^{\infty} E_{2p}) = \sum_{p=n}^{\infty} E_{2p}$$

obviously decreases to 0 as $n \to \infty$, we conclude that H is a pure Hilbert module of rank one.

Hence the minimal dilation $L : H^2(\mathbb{C}^6) \to H$ of H is given by

$$L(f) = f \cdot \Delta 1 = f(T_1, \dots, T_6)\Delta 1.$$

If we evaluate this expression at a point $z = (z_1, z_2, z_3) \in B_3$ we find that

$$L(f)(z_1, z_2, z_3) = f(z_1^2, z_2^2, z_3^2, \sqrt{2}z_1 z_2, \sqrt{2}z_1 z_3, \sqrt{2}z_2 z_3).$$

The argument on the right is a point in the ball B_6, and thus the preceding formula extends immediately to all $f \in H^2(\mathbb{C}^6)$. Notice too that L is a *graded* morphism in that $L\Gamma_0(\lambda) = \Gamma(\lambda)L$, $\lambda \in \mathbb{T}$, where Γ_0 is the gauge group of $H^2(\mathbb{C}^6)$. The precding formula shows that the kernel of L is M_V, and thus we conclude that H is isomorphic to $H^2(\mathbb{C}^6)/M_V$, as asserted in Proposition 7.9.

It remains to calculate the power series $\hat{\phi}(t)$ of Proposition 7.7 which determines the numerical invariants of H. Since $\mathbf{1}_H - \phi^{n+1}(\mathbf{1}_H)$ is the projection

$$\mathbf{1}_H - \phi^{n+1}(\mathbf{1}_H) = E_0 + E_2 + \cdots + E_{2n},$$

it follows that

$$\hat{\phi}(t) = \sum_{n=0}^{\infty} \dim(E_0 + E_2 + \cdots + E_{2n})t^n,$$

and therefore

$$(1 - t)\hat{\phi}(t) = \sum_{n=0}^{\infty} \dim E_{2n} t^n. \tag{7.12}$$

Setting

$$\hat{\sigma}(t) = \sum_{p=0}^{\infty} \dim E_p t^p,$$

we find that for $0 < t < 1$

$$\begin{aligned}\sum_{n=0}^{\infty} \dim E_{2n} t^n &= 1/2(\sum_{p=0}^{\infty} \dim E_p(\sqrt{t})^p + \sum_{p=0}^{\infty} \dim E_p(-\sqrt{t})^p) \\ &= 1/2(\hat{\sigma}(\sqrt{t}) + \hat{\sigma}(-\sqrt{t})),\end{aligned}$$

and hence from (7.12) we have

$$\hat{\phi}(t) = \frac{\hat{\sigma}(\sqrt{t}) + \hat{\sigma}(-\sqrt{t})}{2(1 - t)}, \qquad 0 < t < 1. \tag{7.13}$$

The dimensions $\dim E_p$ were computed in Appendix A of [1], where it was shown that $\dim E_p = q_2(p)$, $q_2(x)$ being the polynomial defined in (3.7). Thus

$$\hat{\sigma}(t) = \sum q_2(n)t^n = \frac{1}{(1 - t)^3};$$

and finally (7.13) becomes

$$\hat{\phi}(t) = \frac{(1-\sqrt{t})^{-3} + (1+\sqrt{t})^{-3}}{2(1-t)} = \frac{(1+\sqrt{t})^3 + (1-\sqrt{t})^3}{2(1-t)^4} = \frac{1+3t}{(1-t)^4}.$$

The right side of the last equation can be rewritten

$$\hat{\phi}(t) = \frac{-3}{(1-t)^3} + \frac{4}{(1-t)^4},$$

hence the coefficients $(c_0, c_1, \dots, c_6)$ of Prop. 6.5 are given by $(0, 0, -3, 4, 0, 0, 0)$. One now reads off the numerical invariants listed in Proposition 7.9. ∎

Finally, we compute nontrivial values of the curvature invariant $K(H)$ for certain examples of pure rank-two graded Hilbert modules H. Let ϕ be a homogeneous polynomial of degree $N = 1, 2, \dots$ in $A = \mathbb{C}[z_1, \dots, z_d]$ and let M be the graph of its associated multiplication operator

$$M = \{(f, \phi \cdot f) : f \in H^2\} \subseteq H^2 \oplus H^2.$$

M is a closed submodule of the free Hilbert module $F = H^2 \oplus H^2$, and $H = F/M$ is a pure Hilbert module of rank 2 whose minimal dilation $L : F \to H$ is given by the natural projection of F onto the quotient Hilbert module $H = F/M$.

We make H into a graded Hilbert module as follows. Let Γ be the gauge group defined on $F = H^2 \oplus H^2$ by

$$\Gamma(\lambda)(f, g) = (\Gamma_0(\lambda)f, \lambda^{-N}\Gamma_0(\lambda)g), \qquad f, g \in H^2,$$

where Γ_0 is the natural gauge group of H^2 defined by

$$\Gamma_0(\lambda)f(z_1, \dots, z_d) = f(\lambda z_1, \dots, \lambda z_d).$$

One verifies that $\Gamma(\lambda)M \subseteq M$, $\lambda \in \mathbb{T}$. Thus the action of Γ can be promoted naturally to the quotient $H = F/M$, and H becomes a graded rank 2 pure Hilbert module whose gauge group has spectrum $\{-N, -N+1, \dots\}$. $L : F \to H$ becomes a graded dilation in that $L\Gamma(\lambda) = \Gamma(\lambda)L$ for all $\lambda \in \mathbb{T}$.

Proposition 7.14. *For these rank 2 examples we have $K(H) = \chi(H) = 1$.*

proof. By Theorem B, $K(H) = \chi(H)$, and it suffices to show that $\chi(H) = 1$.

Let $H_n = \{\xi \in H : \Gamma(\lambda)\xi = \lambda^n \xi\}$, $n \in \mathbb{Z}$, be the spectral subspaces of H. It is clear that $H_n = \{0\}$ if $n < -N$, and since $L : H^2 \oplus H^2 \to H$ is the minimal dilation of H, the algebraic submodule M_H is given by $M_H = L(A \oplus A)$, $A = \mathbb{C}[z_1, \dots, z_d]$. Hence M_H is the (algebraic) sum

$$M_H = \sum_{n=-\infty}^{\infty} H_n.$$

Consider the proper filtration $M_1 \subseteq M_2 \subseteq \dots$ of M_H defined by

$$M_k = \sum_{n \le k} H_n, \qquad k = 1, 2, \dots.$$

By the Corollary of Proposition 3.10 we have

$$\chi(M_H) = d! \lim_{k\to\infty} \frac{\dim M_k}{k^d}, \tag{7.15}$$

and thus we have to calculate the dimensions

$$\dim M_k = \dim(\sum_{n\le k} H_n) = \dim H_{-N} + \cdots + \dim H_{k-1} + \dim H_k \tag{7.16}$$

for $k = 1, 2, \dots$.

In order to calculate the dimension of H_n it is easier to realize H as the orthogonal complement $M^\perp \subseteq F$, with canonical operators $T_1, \dots, T_d$ given by compressing the natural operators of $F = H^2 \oplus H^2$ to $M^\perp$. Since M is the graph of the multiplication operator $M_\phi f = \phi \cdot f$, $f \in H^2$, $M^\perp$ is given by

$$M^\perp = \{(-M_\phi^* g, g) : g \in H^2\}.$$

We compute

$$H_n = (M^\perp)_n = \{\xi \in M^\perp : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}\}.$$

Since $\Gamma_0(\lambda) M_\phi^* \Gamma_0(\lambda)^{-1} = (\Gamma_0(\lambda) M_\phi \Gamma_0(\lambda)^{-1})^* = (\lambda^N M_\phi)^* = \lambda^{-N} M_\phi^*$, we have

$$\Gamma(\lambda)(-M_\phi^* g, g) = (-\Gamma_0(\lambda) M_\phi^* g, \lambda^{-N}\Gamma_0(\lambda) g) = (-\lambda^{-N} M_\phi^* \Gamma_0(\lambda) g, \lambda^{-N}\Gamma_0(\lambda) g),$$

thus $\Gamma(\lambda)(-M_\phi^* g, g) = \lambda^n(-M_\phi^* g, g)$ iff $\Gamma_0(\lambda) g = \lambda^{n+N} g$, $\lambda \in \mathbb{T}$. For $n < -N$ there are no nonzero solutions of this equation, and for $n \ge -N$ the condition is satisfied iff g is a homogeneous polynomial of degree $n + N$.

We conclude that $\dim H_n = 0$ if $n < -N$ and $\dim H_n = \dim A_{n+N} = q_{d-1}(n+N)$ if $n \ge -N$. Thus for $k \ge -N$ we see from (7.16) that

$$\dim M_k = \sum_{n=-N}^{k} H_n = \sum_{n=-N}^{k} q_{d-1}(n+N).$$

The recurrence formula $q_{d-1}(x) = q_d(x) - q_d(x-1)$ of (3.6) implies that the right side of the preceding formula telescopes to $q_d(k+N) - q_d(-1) = q_d(k+N)$. Thus (7.15) implies that

$$\chi(H) = \chi(M_H) = d! \lim_{k\to\infty} \frac{q_d(k+N)}{k^d} = \lim_{k\to\infty} \frac{(k+N+1)\dots(k+N+d)}{k^d} = 1,$$

as asserted. ∎

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