NOTES ON PRODUCT SYSTEMS

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Abstract. We summarize the basic properties of continuous tensor product systems of Hilbert spaces and their role in non-commutative dynamics.

1. Concrete Product Systems

Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$, where as always, H denotes a separable Hilbert space. The *product system* of α gives rise to a classifying structure for cocycle conjugacy, and is defined as follows. For every $t > 0$ let $\mathcal{E}(t)$ be the following linear space of operators

$$\mathcal{E}(t) = \{T \in \mathcal{B}(H) : \alpha_t(X)T = TX, \quad X \in \mathcal{B}(H)\}.$$

The first thing to notice is that there is a natural inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}(t)$ that makes it into a Hilbert space. Indeed, if $S, T \in \mathcal{E}(t)$, then one finds that for every $X \in \mathcal{B}(H)$, one has

$$T^*SX = T^*\alpha_t(X)S = (\alpha_t(X^*)T)^*X = (TX^*)^*S = XT^*S,$$

so that T^*S must be a scalar multiple of the identity operator. Thus we can define an inner product $\langle S, T\rangle$ by

$$T^*S = \langle S, T\rangle \mathbf{1}.$$

This inner product makes the operator space $\mathcal{E}(t)$ into a Hilbert space with the property that the operator norm agrees with the Hilbert space norm, as one sees from

$$T^*T = \langle T, T\rangle \mathbf{1},$$

using the fact that the operator norm satisfies $\|T^*T\| = \|T\|^2$.

The second property of these inner products is also a straightforward consequence of the definition of the various spaces $\mathcal{E}(t)$, namely the following. For every $s, t > 0$, $\mathcal{E}(s)\mathcal{E}(t) \subseteq \mathcal{E}(s+t)$; moreover, for all $S_1, S_2 \in \mathcal{E}(s)$ and $T_1, T_2 \in \mathcal{E}(t)$, one has

$$\langle S_1T_1, S_2T_2\rangle = \langle S_1, S_2\rangle \cdot \langle T_1, T_2\rangle.$$

Note that the inner product $\langle S_1T_1, S_2T_2\rangle$ is formed in the Hilbert space $\mathcal{E}(s+t)$, $\langle S_1, S_2\rangle$ is the inner product of $\mathcal{E}(s)$, and $\langle T_1, T_2\rangle$ is the inner product of $\mathcal{E}(t)$. Finally, it is not hard to show that $\mathcal{E}(s+t)$ is the norm-closed linear span of the set of products $\mathcal{E}(s)\mathcal{E}(t)$.

Date: 5 February, 2006.

The third property of these spaces $\mathcal{E}(t)$ is less obvious, and follows from a result of Dixmier (see [Arv03], pp 36–38). It asserts (in the case of E_0-semigroups) that there is a measurable family $\{U_t : t > 0\}$ of unitary operators in $B(H)$ with the property

$$U_t\mathcal{E}(1) = \mathcal{E}(t), \qquad t > 0. \tag{1.1}$$

We may now assemble this structure into a family $p : \mathcal{E} \to (0, \infty)$ of Hilbert spaces over the interval $(0, \infty)$, by setting

$$\mathcal{E} = \{(t, T) : t > 0, \quad T \in \mathcal{E}(t)\}$$

and taking for p the projection $p(t, T) = t$. If we view $\mathcal{B}(H)$ as a topological space in its weak*-topology, then $\mathcal{E}$ becomes a closed subset of $(0,\infty)\times\mathcal{B}(H)$, and in this way it can be viewed as a *standard Borel space*. The mapping

$$((s, S), (t, T)) \in \mathcal{E} \times \mathcal{E} \mapsto (s + t, ST) \in \mathcal{E}$$

defines an associative multiplication on this structure which (in the precise sense of the above paragraphs) when restricted to a bilinear map of fibers $\mathcal{E}(s) \times \mathcal{E}(t) \to \mathcal{E}(s + t)$, acts as if it were the tensor product operation.

Finally, the unitary operators U_t of (1.1) give rise to a bimeasurable isomorphism θ of $\mathcal{E}$ onto the trivial family $(0, \infty) \times \mathcal{E}(1)$ in which

$$\theta(t, T) = (t, U_t^{-1}T), \qquad t > 0, \quad T \in \mathcal{E}(t).$$

This structure $p : \mathcal{E} \to (0, \infty)$ is called the *concrete product system* of the E_0-semigroup α.

2. Abstract Product Systems

We now put the structures introduced in the preceding section into a general setting. The abstract formulation emphasizes the connection with continuous tensor products of Hilbert spaces [Arv03].

Definition 2.1. A *product system* is a family of separable Hilbert spaces over the open semi-infinite interval $(0, \infty)$

$$p : E \to (0, \infty)$$

with fiber Hilbert spaces $E(t) = p^{-1}(t)$, $t > 0$, which is endowed with an associative multiplication that restricts to a bilinear map on fibers

$$(x, y) \in E(s) \times E(t) \mapsto xy \in E(s + t)$$

that *acts like tensoring* in the sense that for every $u, v \in E(s)$, $x, y \in E(t)$, one has

$$\langle ux, vy\rangle_{E(s+t)} = \langle u, v\rangle_{E(s)}\langle x, y\rangle_{E(t)}, \tag{2.1}$$

together with

$$E(s + t) = \overline{\operatorname{span}}\, E(s)E(t), \qquad s, t > 0. \tag{2.2}$$

In addition, E should be endowed with the structure of a standard Borel space that is compatible with the projection $p : E \to (0, \infty)$, multiplication, the vector space operations and the inner product. Finally, we assume there is a separable Hilbert space H such that E is isomorphic *as a measurable family of Hilbert spaces* to the trivial family $p; (0,\infty) \times H \to (0,\infty)$, p denoting the projection onto the first component $p(t,x) = t$, $t > 0$, $x \in H$.

While it does not make precise mathematical sense, it is often helpful to think of $E(t)$ as a "continuous tensor product"

$$E(t) = \bigotimes_{0 \le s \le t} K_s, \qquad K_s = K$$

of copies of a single Hilbert space K. However, one should keep in mind that for many (if not most) of the important examples of product systems, the "germ" K does not exist!

In the trivial family $p : (0,\infty) \times H \to (0,\infty)$, the fibers $E(t) = H$, $t > 0$ do not vary with t, and a measurable section is simply a Borel-measurable function $f : (0,\infty) \to H$. Given an abstract product system $p : E \to (0,\infty)$, the triviality axiom is equivalent to the assertion that there is a measurable family of orthonormal bases, that is to say, a sequence of Borel-measurable sections

$$e_n : t \in (0,\infty) \mapsto e_n(t) \in E(t), \qquad t > 0, \quad n = 1, 2, \dots,$$

with the property that for every $t > 0$, $\{e_1(t), e_2(t), \dots\}$ is an orthonormal basis for $E(t)$. This requirement is also equivalent to the assertion that there is a sequence of measurable sections

$$f_n : t \in (0,\infty) \mapsto f(t) \in E(t)$$

with the property that $E(t)$ is the closed linear span of $\{f_1(t), f_2(t), \dots\}$ for every $t > 0$. The fact that these two assertions are equivalent follows from a judicious application of the Gram-Schmidt procedure.

3. Alternate Descriptions of Product Systems

It is convenient to refer to a product system $p : E \to (0,\infty)$ with the simpler notation E, in which $E(t)$ denotes the Hilbert space over t. In order to carry out effective analysis with a product system E, one must form certain structures associated with it, such as the "continuous Fock space" $L^2(E)$ that it defines, and the spectral C^*-algebra $C^*(E)$. One also needs to carry out operations on product systems. For example, the tensor product $E \otimes F$ of two product systems E and F corresponds to the tensor product operation of E_0-semigroups (see Section 5). For these and related issues, Definition 2.1 provides the most useful working context.

On the other hand, the measurability axioms of product systems are somewhat redundant. In order to understand that issue it is useful to view a product system in the wrong coordinates, in which it appears as a "flat" family of Hilbert spaces with "curved" multiplication. We now describe this alternate description.

Let E be an abstract product system. By the triviality axiom, there is a separable Hilbert space H such that E can be identified with $(0,\infty)\times H$ *as a measurable family of Hilbert spaces*. Thus, there is a Borel isomorphism $\theta : E \to (0,\infty)\times H$ that restricts to a unitary operator from $E(t)$ to H for every $t>0$, and we may use θ to identify E with $(0,\infty)\times H$, in which $E(t)$ is identified with H for every $t>0$.

Let us look at the multiplicative structure. Fixing $s,t>0$, note that the multiplication of E restricts to a bilinear map of $E(s)\times E(t)=H\times H$ onto $E(s+t)=H$. The characteristic property of the tensor product operation in the category of vector spaces implies that there is a unique linear map

$$U_{s,t} : H \odot H \to H$$

such that

$$U_{s,t}(\xi\odot\eta)=\xi\eta, \qquad \xi\in E(s)=H, \quad \eta\in E(t)=H,$$

$H\odot H$ denoting the algebraic tensor product of complex vector spaces. Properties (2.1) and (2.2) imply that $U_{s,t}$ extends uniquely to a unitary operator from the Hilbert space tensor product $H\otimes H$ to H, which we denote by the same letter $U_{s,t}$. The total map associated with these unitary operators is the multiplication of E, namely

$$((s,\xi),(t,\eta))\in E\times E \mapsto (s,\xi)\cdot(t,\eta)=(s+t,U_{s,t}(\xi\otimes\eta))\in E. \tag{3.1}$$

Remark 3.1 (Dimension of H). If a unitary operator from $H\otimes H$ to H exists, then $\dim(H\otimes H)=(\dim H)^2=\dim H$. It follows that H is either infinite-dimensional (and separable, as are all our Hilbert spaces) or it is one-dimensional. The case of product systems E with one-dimensional fibers $E(t)$ is discussed at length in [Arv03], where it was shown that such an E is isomorphic to the trivial product system $(0,\infty)\times\mathbb{C}$, where $\mathbb{C}$ has its usual inner product and where multiplication is defined by

$$(s,z)\cdot(t,w)=(s+t,zw), \qquad s,t>0, \quad z,w\in\mathbb{C}.$$

Thus we can safely rule out this example in the discussion to follow, in which we assume that H *is a separable infinite-dimensional Hilbert space.*

Using standard methods, one verifies easily that the measurability property of the multiplication map reduces to the following measurability requirement on U, namely that for fixed $\zeta_1\in H\otimes H$, $\zeta_2\in H$, the map

$$(s,t)\in(0,\infty)\times(0,\infty)\mapsto\langle U_{s,t}(\zeta_1),\zeta_2\rangle \tag{3.2}$$

should be measurable. Thus, *the multiplication operation gives rise to a Borel-measurable selection $U_{s,t}$ of unitary operators from $H\otimes H$ to H.*

Most significantly, this multiplication must be associative – and in turn, that requirement leads to a functional equation:

Proposition 3.2. *Let $\{U_{s,t} : s,t>0\}$ be an arbitrary family of unitary operators from $H\otimes H$ to H, and let $x\cdot y$ be the binary operation defined on the trivial family $(0,\infty)\times H$ by (3.1). The following are equivalent:*

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *for all* $x, y, z \in (0, \infty) \times H$.
(ii) *The family* $\{U_{s,t} : s, t > 0\}$ *satisfies the functional equation*

$$U_{r,s+t}(\mathbf{1} \otimes U_{s,t}) = U_{r+s,t}(U_{r,s} \otimes \mathbf{1}), \qquad r, s, t > 0. \tag{3.3}$$

Proof. Note that the operator products exhibited on both sides of (3.3) are well-defined unitary operators from $H \otimes H \otimes H$ to H; for example, $\mathbf{1} \otimes U_{s,t}$ is the unitary operator from $H \otimes H \otimes H$ to $H \otimes H$ defined uniquely by

$$\mathbf{1} \otimes U_{s,t} : \xi \otimes \omega \mapsto \xi \otimes U_{s,t}\omega, \qquad \xi \in H, \quad \zeta \in H \otimes H,$$

so that the composition $U_{r,s+t}(\mathbf{1} \otimes U_{s,t})$ belongs to $\mathcal{B}(H \otimes H \otimes H, H)$.

To verify the implication (i) $\Longrightarrow$ (ii), choose vectors $\xi, \eta, \zeta \in H$. Using the definition of multiplication (3.1), one calculates

$$\begin{aligned}(r, \xi) \cdot ((s, \eta) \cdot (t, \zeta)) &= (r, \xi) \cdot (s + t, U_{s,t}(\eta \otimes \zeta)) \\ &= (r + s + t, U_{r,s+t}(\xi \otimes U_{s,t}(\eta \otimes \zeta)) \\ &= (r + s + t, U_{r,s+t}(\mathbf{1} \otimes U_{s,t})(\xi \otimes \eta \otimes \zeta),\end{aligned}$$

while on the other hand,

$$\begin{aligned}((r, \xi) \cdot (s, \eta)) \cdot (t, \zeta)) &= (r + s, U_{r,s}(\xi \otimes \eta) \cdot (t, \zeta) \\ &= (r + s + t, U_{r+s,t}(U_{r,s}(\xi \otimes \eta) \otimes \zeta)) \\ &= (r + s + t, U_{r+s,t}(U_{r,s} \otimes \mathbf{1})(\xi \otimes \eta \otimes \zeta),\end{aligned}$$

and (3.3) follows since $H \otimes H \otimes H = (H \otimes H) \otimes H = H \otimes (H \otimes H)$ is spanned by vectors of the form $\xi \otimes \eta \otimes \zeta$. The opposite implication is equally apparent. $\square$

More generally, the basic facts of this description of product systems are summarized as follows:

Proposition 3.3. *Let* $\{U_{s,t} : s, t > 0\}$ *be a family of unitary operators from* $H \otimes H$ *to* H *that is measurable in the sense of (3.2) and satisfies the functional equation (3.3). Then the multiplication defined by (3.1) makes the trivial family of Hilbert spaces* $(0, \infty) \times H$ *into a product system. Conversely, every nontrivial product system is isomorphic to one obtained by this construction.*

Let $\{U_{s,t} : s, t > 0\}$, $\{\tilde{U}_{s,t} : s, t > 0\}$ *be two such families of unitary operators that give rise to product systems* E, $\tilde{E}$ *respectively. Then* E *and* $\tilde{E}$ *are isomorphic as product systems iff there is a Borel-measurable family of unitary operators* $W_t \in \mathcal{B}(H)$, $t > 0$, *such that*

$$\tilde{U}_{s,t} = W_{s+t} U_{s,t} (W_s \otimes W_t)^{-1}, \qquad s, t > 0. \tag{3.4}$$

Remark 3.4 (Remarks on the Equivalence Relation (3.4)). The description of abstract product systems given in Proposition 3.3 is undeniably more concrete than Definition 2.1. For example, it is clear from this description that some of the measurability assertions of Definition 2.1 are redundant. On the other hand, while the so-called type I product systems can be defined relatively easily by specifying such a family of unitaries $\{U_{s,t} : s, t > 0\}$ (see

Section 2.6 of [Arv03] for related discussion), that is not the way examples of type II and III have been constructed. Those examples are obtained by other methods – either using properties of the CAR algebra a la Powers, or via rather deep methods of probability theory a la Tsirelson and Vershik (see Chapters 13 and 14 of [Arv03] and references therin for more detail).

On the other hand, the fundamental problem in this subject is the problem of classifying product systems up to isomorphism – for reasons outlined in the following sections. Equivalently, given two families of unitaries $\{U_{s,t}\}$ and $\{\tilde{U}_{s,t}\}$, one wants to know when they are equivalent in the sense that there is a family of unitaries $\{W_t\}$ that satisfies (3.4). This question has a cohomological flavor, and formulating the isomorphism problem in terms of solving such functional equations is of little help, if any. Rather, what is required here is a set of effective tools - invariants - for distinguishing between different isomorphism classes of product systems. The numerical index is the simplest example of such an invariant. The index is a complete invariant for product systems of type I, it is a nontrivial invariant for examples of type II, but it gives no information about examples of type III. Thus, the basic problem of the subject is to find new invariants that are a) computable, and b) of similar utility as the index.

4. Product Systems and Cocycle Perturbations

In following two sections we give a brief survey of the role of product systems in dynamics. We give no proofs, confining ourselves to brief descriptions of the main ideas, including statements of some of the key results.

Two E_0-semigroups $\alpha = \{\alpha_t : t \geq 0\}$ (acting on $\mathcal{B}(H)$) and $\beta = \{\beta_t : t \geq 0\}$ (acting on $\mathcal{B}(K)$) are said to be *conjugate* if there is a $*$-isomorphism $\theta : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\theta \circ \alpha_t = \beta_t \circ \theta$ for all $t \geq 0$. After noting that $*$-isomorphism such as θ are implemented by unitary operators, it follows that conjugate E_0-semigroups are indistinguishable: one can be brought into the other by a "change of coordinates". There are too many conjugacy classes of E_0-semigroups to hope for an effective classification up to conjugacy; indeed, there are concrete theorems which imply that such a classification is impossible. Instead, one seeks to classify E_0-semigroups up to a weaker notion of equivalence, which we now discuss.

Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$. By an *α-cocycle* we mean a strongly continuous family of unitary operators $\{U_t : t \geq 0\}$ in $\mathcal{B}(H)$ that satisfy the cocycle equation

$$U_{s+t} = U_s \alpha_s(U_t), \qquad s, t \geq 0.$$

Given an α-cocycle $U = \{U_t : t \geq 0\}$, one can form a second E_0-semigroup β as follows

$$\beta_t(X) = U_t \alpha_t(X) U_t^*, \qquad t \geq 0, \quad X \in \mathcal{B}(H).$$

It is a simple but worthwhile exercise to verify that the cocycle condition is precisely what is required to show that β satisfies the semigroup property

$\beta_{s+t} = \beta_s\beta_t$, $s, t \geq 0$. An E_0-semigroup β obtained in this way from an α-cocycle is called a *cocycle perturbation* of α.

Definition 4.1. Two E_0-semigroups α, β are said to be *cocycle conjugate* if β is conjugate to a cocycle perturbation of α.

One verifies easily that cocycle conjugacy is an equivalence relation. The fundamental problem of the dynamics of E_0-semigroups is to obtain effective invariants for cocycle conjugacy. The following result shows that the internal structure of the concrete product system of an E_0-semigroup serves to classify it up to cocycle conjugacy.

Theorem 4.2. *Let α and $\tilde{\alpha}$ be E_0-semigroups acting on $\mathcal{B}(H)$ and $\mathcal{B}(\tilde{H})$ respectively, with concrete product systems $\mathcal{E}$ and $\tilde{\mathcal{E}}$. Then $\tilde{\alpha}$ is conjugate to a cocycle perturbation of α iff $\tilde{\mathcal{E}}$ is isomorphic to $\mathcal{E}$.*

Given Theorem 4.2, it is natural to ask if the problem of classifying E_0-semigroups up to cocycle conjugacy is actually *equivalent* to the problem of classifying product systems; in other words, is every abstract product system associated with an E_0-semigroup? The following result answers this in the affirmative.

Theorem 4.3. *For every abstract product system E, there is an E_0-semigroup α whose concrete product system is isomorphic to E.*

On the surface, this would appear to be a solution to the problem of classifying E_0-semigroups. However, Theorems 4.2 and 4.3 simply provide a reduction of the problem to a somewhat more concrete one - the reason being that an effective and general classification of product systems up to isomorphism is unknown. In the following section we will summarize some of the key results that have been obtained concerning the latter problem.

5. Units and Index

For most of us, the concept of index is most naturally formulated in the more concrete dynamical context of E_0-semigroups - the context in which the concept first emerged. But since these notes concern product systems, we use that context to introduce the index, referring the reader to [Arv03] for a discussion of the equivalence of the two notions.

Let E be a product system. A *unit* of E is a measurable cross section

$$u : t \in (0, \infty) \mapsto u(t) \in E(t)$$

of the natural projection $p : E \to (0, \infty)$ which is not the zero section $u \equiv 0$ and which satisfies

$$u(s + t) = u(s)u(t), \qquad s, t > 0. \tag{5.1}$$

The set of units of E will be denoted $\mathcal{U}_E$. It is significant that $\mathcal{U}_E$ can be empty. There is a rough classification of product systems into types, in which a product system E is said to be of type I if there are sufficiently many units to generate E in a certain sense (see [Arv03]), type II if it is not type I but $\mathcal{U}_E \neq \emptyset$, and type III if $\mathcal{U}_E = \emptyset$.

All product systems have a numerical index that is defined as follows. Given two units $u, v \in \mathcal{U}_E$, and given $t > 0$, we can form the inner product $\langle u(t), v(t)\rangle$ in the Hilbert space $E(t)$, and it is a nontrivial fact that the function $f(t) = \langle u(t), v(t)\rangle$ is continuous. Property (5.1), together with the tensor product structure of E, imply that $f(s+t) = f(s)f(t)$. Hence there is a unique complex number $c(u, v)$ that satisfies

$$\langle u(t), v(t)\rangle = f(t) = e^{c(u,v)t}, \qquad t > 0.$$

The bivariate function $c : \mathcal{U}_E \times \mathcal{U}_E \to \mathbb{C}$ is called the *covariance function* of the product system E. The covariance function is conditionally positive definite, hence one can use it to construct a Hilbert space H_E.

In cases where $\mathcal{U}_E \neq \emptyset$, the *index* of E is defined by

$$\operatorname{index}(E) = \dim(H_E).$$

Another nontrivial fact is that H_E is a separable Hilbert space when $\mathcal{U}_E \neq \emptyset$, so in such cases the index takes values in the set $\{0, 1, 2, \dots, \infty\}$, ∞ denoting the denumerable cardinal $\aleph_0$. All values can occur, even for type I product systems. This defines a numerical invariant for product systems E for which $\mathcal{U}_E \neq \emptyset$, and it is convenient to extend the index to cover the remaining cases by setting $\operatorname{index}(E) = c$ to be the cardinality of the continuum when $\mathcal{U} = \emptyset$.

There is a natural way of forming the tensor product of two E_0-semigroups α (acting on $\mathcal{B}(H)$) and β (acting on $\mathcal{B}(K)$), and in fact $\alpha \otimes \beta$ is the unique E_0-semigroup that acts on $\mathcal{B}(H \otimes K)$ and satisfies

$$(\alpha \otimes \beta)_t(A \otimes B) = \alpha_t(A) \otimes \beta_t(B), \qquad A \in \mathcal{B}(H),\ B \in \mathcal{B}(K),\ t \geq 0.$$

There is also a natural notion of tensor product $E \otimes F$ in the category of product systems which we leave for the reader to discover, and which has the property that the concrete product system of $\alpha \otimes \beta$ is isomorphic to $E \otimes F$, where E and F are the concrete product systems of α and β respectively. The first key property of the index is that it is logarithmically additive in general:

Theorem 5.1. *For any two product systems E, F, one has*

$$\operatorname{index}(E \otimes F) = \operatorname{index}(E) + \operatorname{index}(F). \tag{5.2}$$

The difficult element in the proof of Theorem 5.2 is establishing the fact that, in general, every unit of $E \otimes F$ decomposes into a tensor product of units $u \otimes v$, where $u \in \mathcal{U}_E$ and $v \in \mathcal{U}_F$. In particular, (5.2) implies that $E \otimes F$ is of type III if and only if either E or F is of type III.

The second key property of the index is that it is a complete invariant for the simplest class of product systems:

Theorem 5.2. *Two product systems E, F of type I are isomorphic if and only if* $\operatorname{index}(E) = \operatorname{index}(F)$.

Not surprisingly, there is a corresponding notion of index in the category of E_0-semigroups; indeed, that is the context in which the concept of index first appeared (see Sections 2.5 and 2.10 of [Arv03]). As one would expect, the index of an E_0-semigroup agrees with the index of its concrete product system in all cases. Thus, when combined with Theorem 4.2, Theorem 5.2 gives the following information about the classification problem of noncommutative dynamics:

Corollary 5.3. *Let α and β be two type I E_0-semigroups. Then α and β are cocycle-conjugate iff they have the same index.*

The index is certainly not a complete invariant for type II examples, and it contains no information whatsoever about product systems of type III. The classification problem for such systems remains mysterious: *we have not seen all possible examples of such E_0-semigroups, and we lack appropriate invariants for distinguishing between the ones we have seen.*

References

[Arv03] W. Arveson. *Noncommutative Dynamics and E-semigroups.* Monographs in Mathematics. Springer-Verlag, New York, 2003.