# NONCOMMUTATIVE POISSON BOUNDARIES

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Abstract. In these notes we give a proof of associativity of the the Choi-Effros multiplication, and we apply that to the noncommutative Poisson boundary.

**These are lecture notes and are not intended for publication**

## 1. Background

While the following exposition of idempotent completely positive contractions and the basic properties of noncommutative Poisson boundaries is essentially complete, I have not included adequate references. Moreover, I will occasionally refer to topics that are not discussed here, such as the lifting theorem for UCP maps, nuclearity, and the extension theorems for operator valued maps. The reader can find more discussion of these and related matters – and more references – in the notes for several lectures given in a seminar last fall. The Fall '03 lectures are posted on the same web page that contains this note.

## 2. Completely positive idempotents

Let $A$ be a $C^*$-algebra and let $E : A \to A$ be a completely positive contraction satisfying $E^2 = E$. The range $S = E(A)$ is a norm-closed self-adjoint subspace of $A$, and Choi and Effros showed that $S$ can be made into a $C^*$-algebra with respect to the multiplication $s_1 \circ s_2 = E(s_1 s_2)$; we now prove that result.

We are especially interested in the case where $A$ has a unit $\mathbf{1}$. Some terminology will be convenient. A UCP map $\phi : A \to B$ between unital $C^*$-algebras is a completely positive map such that $\phi(\mathbf{1}_A) = \mathbf{1}_B$. Such maps always have norm 1 (by the Schwarz inequality). If $E : A \to A$ is a UCP idempotent map, then its range $E(A)$ is an *operator system*, i.e., a norm-closed self-adjoint linear subspace of $A$ that contains $\mathbf{1}$.

*Remark* 2.1. Idempotent UCP maps arise frequently and naturally. For example, given an exact sequence of $C^*$-algebras

$$0 \longrightarrow K \longrightarrow A \underset{\pi}{\longrightarrow} B \longrightarrow 0$$

with $A$ and $B$ unital, $\pi(\mathbf{1}) = \mathbf{1}$ and with $B$ nuclear, there is a UCP map $\phi : B \to A$ that lifts $\pi$ in the sense that $\pi(\phi(b)) = b$, $b \in B$. In this

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case, $E = \phi \circ \pi : A \to A$ is a UCP map satisfying $E^2 = E$ whose range $S = E(A)$ is an operator system with the property that $\pi$ maps $S$ completely isometrically onto $B$. In particular, the operator system $S = E(A)$ is *linearly* completely isometrically isomorphic to a $C^*$-algebra.

Let $A$ be a unital $C^*$-algebra and let $S \subseteq A$ be an operator system in $A$. Suppose that $S$ is completely order isomorphic to a $C^*$-algebra $B$. This means that there is a bijective UCP map $\phi : B \to S$ whose inverse $\phi^{-1} : S \to B$ is also completely positive. One can think of this as specifying a multiplication of $S$ that makes it into a $C^*$-algebra. We first want to point out that when such a structure exists, it is unique.

**Proposition 2.2.** *Let $S \subseteq A$ be an operator system, and suppose that we have two $C^*$-algebras $B_k$ and complete order isomorphisms $\phi_k : B_k \to S$, $k = 1, 2$. Then the composition $\phi_2^{-1}\phi_1$ is a $*$-isomorphism of $B_1$ on $B_2$.*

*Proof.* It suffices to show that a UCP map $\phi : B_1 \to B_2$ between two $C^*$-algebras that has a UCP inverse is multiplicative. By the Schwarz inequality, we have $\phi(x^*x) \geq \phi(x^*)\phi(x)$ for all $x \in B_1$. Hence

$$x^*x = \phi^{-1}(\phi(x^*x)) \geq \phi^{-1}(\phi(x^*)\phi(x)) \geq \phi^{-1}(\phi(x^*))\phi^{-1}(\phi(x)) = x^*x,$$

by the Schwarz inequality applied to $\phi^{-1}$. After applying $\phi$ to the preceding inequality we obtain $\phi(x^*x) = \phi(x^*)\phi(x)$ for all $x \in B_1$. Since both $\phi(y^*x)$ and $\phi(y^*)\phi(x)$ are sesquilinear forms in $x, y$, a polarization argument implies that $\phi(y^*x) = \phi(y^*)\phi(x)$ for all $x, y \in B_1$, hence $\phi$ is multiplicative. □

*Remark* 2.3 (Unital completely isometric maps). It is not hard to show that a unit-preserving linear map $\phi : S_1 \to S_2$ between operator systems is completely positive iff it is completely contractive. Thus, one can deduce the following consequence from Proposition 2.2: *If an operator system $S \subseteq A$ in a unital $C^*$-algebra admits a multiplication $\circ$ that makes it into a $C^*$-algebra with respect to the given norm on $S$, then the resulting $C^*$-algebraic structure of $S$ is uniquely determined.*

What this means is that if $\circ_1$ and $\circ_2$ are two multiplications in $S$ that make it into a $C^*$-algebra, then there is a completely isometric complete order isomorphism $\alpha$ of $S$ onto itself that fixes $\mathbf{1}$ and satisfies $\alpha(u \circ_1 v) = u \circ_2 v$, $u, v \in S$. One should think carefully about why the indicated maps of $C^*$-algebras in the preceding statement are *completely* isometric rather than just isometric.

*Remark* 2.4 (Positive idempotents in $\mathcal{B}(H)$). We remind the reader of the elementary but infrequently cited fact that an idempotent contraction $P \in \mathcal{B}(H)$ must be positive. Indeed, the positive operator $(\mathbf{1} - P^*P)^{1/2}$ restricts to zero on $PH$ because $A = (\mathbf{1} - P^*P)^{1/2}P$ satisfies

$$A^*A = P^*(\mathbf{1} - P^*P)P = P^*P - P^*P = 0.$$

Hence $(\mathbf{1} - P^*P)P = 0$, from which $P = P^*P \geq 0$ follows.

**Lemma 2.5.** *Let $A$ be a $C^*$-algebra and let $E : A \to A$ be a contractive completely positive linear map satisfying $E^2 = E$. Then $E(x^*E(x)) \geq 0$ for every $x \in A$.*

*Proof.* We have to show that $\rho_0(E(x^*E(x))) \geq 0$ for every positive linear functional $\rho_0$ on $A$. Fix such a $\rho_0$ and set $\rho = \rho_0 \circ E$. $\rho$ is a a positive linear functional satisfying $\rho \circ E = \rho$. By the GNS construction there is a representation $\pi$ of $A$ on a Hilbert space $H$ and a cyclic vector $\xi$ for $\pi(A)$ such that $\rho(x) = \langle \pi(x)\xi, \xi \rangle$, $x \in A$. Note that there is a unique contraction $P \in \mathcal{B}(H)$ that maps $\pi(x)\xi$ to $\pi(E(x))\xi$ for every $x \in A$. Indeed, since $E$ satisfies the Schwarz inequality $E(x)^*E(x) \leq E(x^*x)$,

$$\begin{aligned}\|\pi(E(x))\xi\|^2 &= \langle \pi(E(x)^*E(x))\xi, \xi \rangle = \rho(E(x)^*E(x)) \\ &\leq \rho(E(x^*x)) = \rho(x^*x) = \|\pi(x)\xi\|^2,\end{aligned}$$

from which the existence of $P$ follows. $P$ must be idempotent because $E^2 = E$, and an idempotent contraction in $\mathcal{B}(H)$ must be a self-adjoint projection (see Remark 2.4). Hence $\rho(x^*E(x)) = \langle P\pi(x)\xi, \pi(x)\xi \rangle \geq 0$, and $\rho_0(E(x^*e(x))) \geq 0$ follows. □

**Theorem 2.6** (Choi-Effros)**.** *Let $A$ be a $C^*$-algebra and let $E : A \to A$ be an idempotent completely positive contraction with range $E(A)$. Then*

$$E(xE(y)) = E(E(x)y), \qquad x, y \in A. \tag{2.1}$$

*Moreover, $E(A)$ becomes a $C^*$-algebra with respect to the multiplication*

$$x \circ y = E(xy), \qquad x, y \in E(A),$$

*and the norm, involution and vector space structure inherited from $A$.*

*Proof.* The sesquilinear map $x, y \in A \mapsto E(y^*E(x)) \in A$, being positive semidefinite, must be self-adjoint by a familiar polarization argument. It follows that $E(y^*E(x)) = E(x^*E(y))^* = E(E(y)^*x) = E(E(y^*)x)$ for all $x, y \in A$, and the identity (2.1) follows.

$E(A)$ is clearly a norm-closed linear subspace of $A$, and since a positive linear map must preserve adjoints, it is closed under the $*$-operation as well. What remains to be shown is that the multiplication $\circ$ is associative, and it satisfies $\|u^* \circ u\| = \|u\|^2$ for all $u = E(x) \in E(A)$.

Choose $u, v, w \in E(A)$. Then by (2.1) we have

$$u \circ (v \circ w) = E(uE(vw)) = E(E(u)vw) = E(uvw),$$

and hence

$$(u \circ v) \circ w = E(E(uv)w) = E(uvE(w)) = E(uvw) = u \circ (v \circ w).$$

For the second assertion, choose $u \in E(A)$ and use the Schwarz inequality $E(u^*u) \geq E(u)^*E(u)$ for completely positive contractions to write

$$\|u^* \circ u\| = \|E(u^*u)\| \geq \|E(u)^*E(u)\| = \|u^*u\| \geq \|E(u^*u)\| = \|u^* \circ u\|.$$

Hence $\|u^* \circ u\| = \|u^*u\| = \|u\|^2$, as required. □

## 3. The Poisson Boundary

We now move into the category of von Neumann algebras. We define the Poisson boundary of a noncommutative harmonic space and develop its most basic properties. The reader is referred to Masaki Izumi's exposition [Izu04] which contains a slightly different approach, significant examples, and background material for forthcoming lectures to be given by others.

Let $P : M \to M$ be a normal UCP map of a von Neumann algebra $M$ to itself, and let

$$H(M) = \{x \in M : P(x) = x\}$$

be the space of all harmonic elements of $M$. $H(M)$ is a weak$^*$-closed operator system, and hence it is the dual of a Banach space (namely $H(M)_*$). The purpose of this section is to show that there is a (unique) von Neumann algebra $bH(M)$ that has $H(M)$ as its operator system structure. $bH(M)$ is called the *Poisson boundary* of the noncommutative harmonic space $H(M)$.

Let us assume, for the moment, that there is an associative multiplication $\circ$ defined in $H(M)$ that makes it into a $C^*$-algebra in the sense of the preceding section. Then this $C^*$-algebra is actually a von Neumann algebra. That follows from a theorem of Sakai asserting the following: *if a unital $C^*$-algebra $A$ is isometrically isomorphic to the dual of a Banach space, then there is a faithful representation $\pi : A \to \mathcal{B}(H)$ on a Hilbert space $H$ such that $\pi(A) = \pi(A)''$ is a von Neumann algebra.*

One obtains the von Neumann algebra structure on $H(M)$ in three steps. First, we show that there is a (non-normal) UCP idempotent $E$ mapping $M$ onto $H(M)$. Second, we apply the results of the preceding section to deduce that there is a $C^*$-algebraic structure on $H(M)$, and then we make use of Sakai's result to conclude that this is actually a von Neumann algebra structure.

Consider the unit ball $\mathcal{B}_1(M)$ in the Banach space $\mathcal{B}(M)$ of all bounded linear maps of $M$ into itself. There is a natural topology on $\mathcal{B}(M)$ that makes $\mathcal{B}_1(M)$ into a compact convex set. Let's call this the BW topology (the term BW refers to "bounded weak" topology for historical reasons). The BW topology is defined by the family of finite sums of seminorms of the form

$$|L| = |\rho(L(x))|, \qquad \rho \in M_*, \quad x \in M.$$

A net $\{L_\alpha : \alpha \in I\}$ in $\mathcal{B}(M)$ converges to zero in the BW topology iff for every $x \in M$, $L_\alpha(x)$ converges to zero in the weak$^*$-topology of $M$. It is a now-classical result that the unit ball of $\mathcal{B}(M)$ is compact in its BW topology (a proof can be found in [Arv69]).

For every $n = 1, 2, \dots$, let $A_n$ be the $n$th average

$$A_n(x) = \frac{1}{n}(x + P(x) + \cdots + P^{n-1)}(x)), \qquad x \in M,$$

and let $\mathcal{K}_n$ be the BW-closed set of maps

$$\mathcal{K}_n = \overline{\{A_n, A_{n+1}, A_{n+2}, \dots\}}^{\mathrm{BW}}.$$

We have $\mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \cdots$, and since each $\mathcal{K}_n$ is compact and nonempty, the intersection

$$\mathcal{K}_\infty = \cap_{n=1}^\infty \mathcal{K}_n = \cap_{n=1}^\infty \overline{\{A_n, A_{n+1}, A_{n+2}, \dots\}}^{\text{BW}}$$

consisting of all BW-limit points of $A_1, A_2, A_3, \dots$, must be nonempty.

Choose any element $E \in \mathcal{K}_\infty$. We claim that $E$ is a UCP idempotent with range $H(M)$. Indeed, $E$ is obviously a unital completely positive map of $M$ into itself, and since

$$\begin{aligned} PA_n(x) - A_n(x) &= \frac{1}{n}(P(x) + \cdots + P^n(x)) - \frac{1}{n}(x + P(x) + \cdots + P^{n-1}(x)) \\ &= \frac{1}{n}P^n(x) - \frac{1}{n}x \end{aligned}$$

we find that

$$\|PA_n - A_n\| \leq \frac{2}{n} \to 0$$

as $n \to \infty$. Hence $\|PA_n(x) - A_n(x)\| \to 0$ as $n \to \infty$, for each $x \in M$. Since $P$ is normal, this implies that $PE = E$, hence $E(M) \subseteq H(M)$. On the other hand, it is obvious that each $P^k$, and therefore each $A_n$, must fix every element of $H(M)$. Hence $E$ must fix the elements of $H(M)$ as well, and the claim is established.

By Theorem 2.6, $H(M)$ admits an associative multiplication making it into a $C^*$-algebra, and by Sakai's theorem this $C^*$-algebra is a von Neumann algebra. To summarize, we have proved the following result, which establishes the existence and uniqueness of the noncommuative Poisson boundary:

**Theorem 3.1.** *Let $P : M \to M$ be a normal UCP map on a von Neumann algebra and let $H(M) = \{x \in M : P(x) = x\}$ be the space of harmonic elements. Then there is a von Neumann algebra $N$ and a unit preserving linear bijection $\phi : N \to H(M)$ with the following properties:*

(i) *Both $\phi$ and $\phi^{-1}$ are completely positive.*
(ii) *Both $\phi$ and $\phi^{-1}$ are weak$^*$-continuous.*

*Moreover, if $N_1$, $N_2$ are von Neumann algebras and $\phi_k : N_k \to H(M)$ are maps satisfying (i) and (ii), then $\phi_2^{-1}\phi_1 : N_1 \to N_2$ is a normal $*$-isomorphism of von Neumann algebras.*

Theorem 3.1 concerns discrete semigroups $\{P^n : n = 0, 1, 2, \dots\}$ of normal UCP maps acting on $M$. However, the classical contexts involving harmonic functions and the heat flows of complete Riemannian manifolds, and their noncommutative generalizations, involve semigroups $\{P_t : t \geq 0\}$ of normal UCP maps. The reader should have no difficulty in finding appropriate variations on the proof of Theorem 3.1 that lead to its continuous time counterpart for von Neumann algebras $M$:

**Theorem 3.2.** *Let $\{P_t : t \geq 0\}$ be a semigroup of normal UCP maps accting on $M$ and let $H(M) = \{x \in M : P_t(x) = x,\ t \geq 0\}$ be its space of harmonic elements. Then $H(M)$ is completely order isomorphic to a uniquely determined von Neumann algebra $bH(M)$.*

## References

[Arv69] W. Arveson. Subalgebras of $C^*$-algebras. *Acta Math.*, 123:141–224, 1969.

[Izu04] M. Izumi. Non-commutative poisson boundaries. In M. Kotani, T. Shirai, and T. Sunada, editors, *Discrete Geometric Analysis*, volume 347 of *Contemporary Mathematics*. Amer. Math. Soc., 2004.