NOTES ON THE LIFTING THEOREM

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We have seen that the proof of existence of inverses for elements of $\mathrm{Ext}(X)$ can be based on a lifting theorem for (completely) positive maps of $C(X)$ into a quotient C^*-algebra of the form $\mathcal{E}/\mathcal{K}$, where $\mathcal{E} \subseteq \mathcal{B}(H)$ is a C^*-algebra containing the compact operators $\mathcal{K}$. That argument works equally well for arbitrary C^*-algebras in place of $C(X)$ whenever a completely positive lifting exists. Thus we are led to ask if every completely positive linear map ϕ of an arbitrary C^*-algebra A into a quotient C^*-algebra B/K has a completely positive lifting $\phi_0 : A \to B$. The answer is yes if A is nuclear by a theorem of Choi and Effros [CE76], but no in general. We will sketch a proof of the Choi-Effros theorem that is based on the existence of quasicentral approximate units; full details can be found in [Arv77]. Throughout this lecture, all Hilbert spaces are assumed to be separable.

1. Quasicentral Approximate Units

An operator $T \in \mathcal{B}(H)$ is said to be *quasidiagonal* if there is a sequence F_n of finite rank projections such that $F_n \uparrow \mathbf{1}$ and $\|F_nT - TF_n\| \to 0$ as $n \to \infty$. It is not hard to see that this is equivalent to the existence of a sequence of mutually orthogonal finite-dimensional projections $E_1, E_2, \dots$ in $\mathcal{B}(H)$ such that $\sum_n E_n = \mathbf{1}$ and $T = \sum_{n=1}^{\infty} E_nTE_n + K$ where K is a compact operator. Equivalently, T is quasidiagonal if and only if it is a compact perturbation of a *block diagonal* operator – a countable direct sum of finite dimensional operators.

Not all operators are quasidiagonal. Indeed, it is an instructive exercise to show that if a *Fredholm* operator T is quasidiagonal then its index satisfies $\mathrm{ind}\, T = 0$. Thus the simple unilateral shift is not quasidiagonal. Nevertheless, in this section we will show that it is always possible to find a sequence of *positive finite rank* operators F_n such that $F_n \uparrow \mathbf{1}$ and $\|F_nT - TF_n\| \to 0$ as $n \to \infty$. Given that result, it is not hard to deduce that there is a sequence of positive finite rank operators $E_1, E_2, \dots$ such that $\sum_n E_n^2 = \mathbf{1}$ and $\sum_n E_nTE_n$ is a compact perturbation of T. Of course, $\sum_n E_nTE_n$ is not necessarily block diagonal or even quasidiagonal, but one can show that it (and T itself) is always a direct summand of a quasidiagonal operator.

Let K be a two-sided ideal in a C^*-algebra A, not necessarily closed. Recall that an approximate unit for K is is an increasing net u_λ of positive elements of K such that $\|u_\lambda\| \le 1$ and $\lim_\lambda \|u_\lambda k - k\| = 0$. If u_λ also satisfies $\lim_\lambda \|u_\lambda a - au_\lambda\| = 0$ for all $a \in A$ then u_λ will be called *quasicentral*.

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Theorem 1.1. *Every ideal K in a C^*-algebra A has a quasicentral approximate unit. If A is separable, the approximate unit can be chosen to be a sequence $u_1 \leq u_2 \leq \cdots$.*

We sketch the main idea of the proof, which requires two observations.

First, if u_λ is any approximate unit for K and f is any bounded linear functional on A, then we have

$$\lim_\lambda f(u_\lambda a - a u_\lambda) = 0, \qquad a \in A. \tag{1.1}$$

Indeed, since every bounded linear functional on A is a linear combination of four positive linear functionals of norm 1, it suffices to prove (1.1) for the states f; and in that case the proof is a straightforward argument using the GNS representation for f.

Second, given any approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ for K, we point out that one can view its convex hull Λ' as an approximate unit. Indeed, by definition Λ' consists of all finite convex combinations

$$\Lambda' = \{\theta_1 u_{\lambda_1} + \cdots + \theta_n u_{\lambda_n} : \lambda_j \in \Lambda, \quad \theta_j \geq 0, \quad \theta_1 + \cdots + \theta_n = 1\}.$$

Using the fact that the original net u_λ is directed increasing one finds that Λ' is an increasing directed set *with respect to the operator ordering of A*. Thus, we may regard Λ' as an increasing directed net of positive operators, relative to the operator ordering, that indexes itself. One can now show that Λ' is also an approximate unit for K as on page 330 of [Arv77].

Here is the key observation.

Lemma 1.2. *Let Λ be a convex approximate unit for an ideal K in a C^*-algebra A. Then for every finite set of elements $a_1, \dots, a_n \in A$ and every $\epsilon > 0$, there is an element $u \in \Lambda$ such that*

$$\|u a_k - a_k u\| \leq \epsilon, \qquad 1 \leq k \leq n. \tag{1.2}$$

Proof of Lemma 1.2. One can immediately reduce to the case $n = 1$ by replacing A with the n-fold direct sum $n \cdot A = A \oplus \cdots \oplus A$ of copies of A, K with $n \cdot K$, Λ with $\{u \oplus \cdots \oplus u : u \in \Lambda\}$ (a convex approximate unit for $n \cdot K$), and then considering the single element $a_1 \oplus \cdots \oplus a_n \in A \oplus \cdots \oplus A$.

For the case of a single element $a \in A$, (1.2) simply asserts that 0 belongs to the norm closure of the set $C = \{ua - au : u \in \Lambda\}$. But if $0 \notin \overline{C}$ then, since C is convex, a standard separation theorem implies that there is a bounded linear functional f on A such that $|f(ua - au)| \geq \epsilon > 0$ for every $u \in \Lambda$, and that contradicts (1.1) above. □

Proof of Theorem 1.1. Choose an arbitrary approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ for K and let Λ' be its convex hull. Choose elements $a_1, \dots, a_n \in A$, $v \in \Lambda'$ and $\epsilon > 0$. Since $\{u \in \Lambda' : u \geq v\}$ is a cofinal convex subnet of Λ', it is also a convex approximate unit for K. Thus, Lemma 1.2 implies that there is an element $w \geq v$ in Λ' such that $\|w a_k - a_k w\| \leq \epsilon$ for $k = 1, \dots, n$.

That assertion is clearly enough to allow us to extract a subnet v_λ of Λ' with the property that $\lim_\lambda \|v_\lambda a - a v_\lambda\| = 0$ for every $a \in A$, and such a subnet $\{v_\lambda\}$ is a quasicentral approximate unit. The proof that $\{v_\lambda\}$ can be chosen as a sequence when A is separable is a straightforward argument that we omit. □

Remark 1.3. Theorem 1.1 was discovered during the writing of [Arv77]; it was discovered independently by Charles Akemann and Gert Pedersen in their work on ideal perturbations of elements of C^*-algebras, at about the same time.

2. Liftable Maps

Let A be and B be unital C^*-algebras. We consider completely positive linear maps $\phi : A \to B$ which preserve units in the sense that $\phi(\mathbf{1}_A) = \mathbf{1}_B$, and we refer to such a ϕ as a UCP map. Given a closed two-sided ideal $K \subseteq B$ in a unital C^*-algebra B, we want to know if there is a UCP map $\phi : B/K \to B$ that provides a lifting for the projection $b \in B \mapsto \dot{b} \in B/K$. More generally, given a UCP map from a given C^*-algebra A into a quotient B/K, the lifting problem for ϕ is the the problem of finding a UCP map $\psi : A \to B$ such that $\dot{\psi}(a) = \phi(a)$, $a \in A$. If such a map ψ exists then ϕ is said to be *liftable* and we write $\phi = \dot{\psi}$.

In this section we discuss some results about liftable maps in general, most of which depend strongly on quasicentral approximate units. Throughout this section and the next, A will denote a *separable* C^*-algebra with unit. Let us fix an ideal K in another unital C^*-algebra B (B need not be separable), and consider the set $\mathrm{UCP}(A, B/K)$ of all UCP maps $\phi : A \to B/K$ as a topological space in its point-norm topology. Thus, a net $\phi_\lambda : A \to B/K$ of linear maps converges to a map $\phi : A \to B/K$ iff

$$\lim_\lambda \|\phi_\lambda(a) - \phi(a)\| = 0, \qquad a \in A.$$

The first key fact is that in general, the set of liftable maps is closed:

Theorem 2.1. *Let A be a separable unital C^*-algebra. Then the set of all liftable UCP maps from A to a quotient B/K is closed in the point-norm topology of* $\mathrm{UCP}(A, B/K)$.

We will say something about what goes into the proof of Theorem 2.1, skipping over the technicalities. The first observation is that the point-norm topology is metrizable when A is separable. Indeed, if we fix a sequence $a_1, a_2, \dots$ of elements that is dense in the unit ball of A then

$$d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \|\phi(a_n) - \psi(a_n)\|$$

is a metric with the stated property. Let us fix a sequence $a_1, a_2, \dots$ throughout the discussion, thereby fixing a single metric $d(\cdot, \cdot)$ on $\mathrm{UCP}(A, B/K)$ that makes it into a *complete* metric space.

Suppose that we are given a pair of UCP maps $\phi, \psi : A \to B$. Then we may compose both maps with the natural projection $b \in B \mapsto \dot{b} \in B/K$ to obtain UCP maps $\dot{\phi}, \dot{\psi} : A \to B/K$ into the quotient. Obviously,

$$d(\dot{\phi}, \dot{\psi}) \leq d(\phi, \psi).$$

What is important here is that the left side of this inequality can almost be realized by perturbing *one* of the two maps $\phi, \psi : A \to B$.

Lemma 2.2. *For any two UCP maps $\phi, \psi : A \to B$ and every $\epsilon > 0$, there is a UCP map $\psi' : A \to B$ such that $d(\phi, \psi') \leq d(\dot{\phi}, \dot{\psi}) + \epsilon$.*

Sketch of proof. Let u_λ be an approximate unit for K that is quasicentral in B. For every λ we can define a UCP map $\psi_\lambda : A \to B$ by

$$\psi_\lambda(a) = u_\lambda^{1/2}\phi(a)u_\lambda^{1/2} + (\mathbf{1} - u_\lambda)^{1/2}\psi(a)(\mathbf{1} - u_\lambda)^{1/2}, \qquad a \in A.$$

Obviously, $\dot{\psi}' = \dot{\psi}$. The main property of these perturbations ψ_λ is:

$$\limsup_\lambda d(\phi, \psi_\lambda) \leq d(\dot{\phi}, \dot{\psi}). \tag{2.1}$$

The estimates required for the proof of (2.1) can be found on pp. 346–347 of [Arv77]. Once one has (2.1), one obtains the assertion of Lemma 2.2 by choosing an appropriately large λ. □

Proof of Theorem 2.1. To prove Theorem 2.1, let $\phi_1, \phi_2, \dots$ be a sequence of liftable maps in $\mathrm{UCP}(A, B/K)$ that converges to a UCP map ϕ_∞. By passing to a subsequence if necessary, we can also arrange that $d(\phi_n, \phi_\infty) < 1/2^{n+1}$.

We claim that there is a sequence $\psi_1, \psi_2, \dots$ in $\mathrm{UCP}(A, B/K)$ satisfying $\dot{\psi}_n = \phi_n$ and $d(\psi_n, \psi_{n+1}) < 1/2^n$, $n = 1, 2, \dots$. Indeed, let ψ_1 be any UCP lifting of ϕ_1. Assuming that $\psi_1, \dots, \psi_n$ have been defined and satisfy the stated conditions, choose any lifting λ of ϕ_{n+1}. Noting that $d(\dot{\psi}_n, \dot{\lambda}) = d(\phi_n, \phi_{n+1}) < 1/2^n$, Lemma 1.2 implies that there is a UCP map ψ_{n+1} satisfying $\dot{\psi}_{n+1} = \dot{\lambda} = \phi_{n+1}$ and $d(\psi_n, \psi_{n+1}) < 1/2^n$.

Since $\sum_n d(\psi_n, \psi_{n+1}) < \infty$, $\{\psi_n\}$ is a Cauchy sequence relative to the d-metric, and we can define a UCP map ψ_∞ as the limit $\lim_n \psi_n$. Since $\dot{\psi}_n = \phi_n$ converges to ϕ_∞, ψ_∞ is a lifting of ϕ_∞. □

The second key fact that we require is that the lifting problem can always be solved for matrix algebras $M_n = M_n(\mathbb{C})$, $n = 1, 2, \dots$:

Proposition 2.3 (M.-D. Choi)**.** *Every UCP map $\phi : M_n \to B/K$ is liftable.*

Sketch of Proof. Let $\{e_{pq} : 1 \leq p, q \leq n\}$ be a system of matrix units for M_n. Thus, $e_{pq}e_{rs} = \delta_{qr}e_{ps}$, $e_{pq}^* = e_{qp}$, and M_n is spanned by $\{e_{pq}\}$. Define an array of elements $f_{pq} \in B/K$ by $f_{pq} = \phi(e_{pq})$. The $n \times n$ matrix (f_{pq}) can be considered an element of $M_n \otimes (B/K) \cong (M_n \otimes B)/(M_n \otimes K)$. It is positive because ϕ is n-positive. An elementary exercise with the functional calculus shows that every positive element of a quotient of C^*-algebras can be lifted to a positive element of the ambient C^*-algebra. Applying this to the ideal

$M_n \otimes K$ in $M_n \otimes B$ we obtain a positive $n \times n$ matrix (F_{pq}) of elements of B such that F_{pq} projects to f_{pq}, via the map $B \to B/K$, $1 \le p, q \le n$.

Now let $\phi_0 : M_n \to B$ be the unique linear map satisfying $\phi_0(e_{pq}) = F_{pq}$, $1 \le p, q \le n$. Since the $n \times n$ matrix $(\phi_0(e_{pq})) = (F_{pq})$ is positive, it follows that ϕ_0 must be completely positive (see Lemma 3.2 of [Arv77]). Obviously ϕ_0 is a lifting of ϕ. ϕ_0 need not carry unit to unit, but a simple argument shows that it can be perturbed into another completely positive lifting that does (see p. 350 of [Arv77]). □

3. Liftings and Nuclearity

We now show that the results of the preceding section imply the lifting theorem for nuclear C^*-algebras. Let A and B be unital C^*-algebras. A UCP map $\phi : A \to B$ is called *factorable* if it can be factored through some matrix algebra M_n, $n = 1, 2, \dots$ in the sense that there are UCP maps $\sigma : A \to M_n$ and $\tau : M_n \to B$ such that $\phi = \tau \circ \sigma$. ϕ is called a *nuclear* map if it belongs to the point-norm closure of the set of all factorable maps in $\mathrm{UCP}(A, B)$. Finally, a C^*-algebra A is called *nuclear* if the identity map of A is a nuclear map. The notion of nuclearity is equivalent to several natural and useful properties when A is separable, including:

(i) For any C^*-algebra B, the natural $*$-homomorphism of $A \otimes_{\max} B$ onto $A \otimes_{\min} B$ is an isomorphism.
(ii) A is amenable.
(iii) The weak closure of A in any representation is injective.
(iv) The weak closure of A in any factor representation is hyperfinite.

The following result was originally proved in [CE76] by a rather different method.

Theorem 3.1 (Choi-Effros)**.** *Every nuclear UCP map from a separable C^*-algebra A into a quotient B/K is liftable.*

Proof. Let $\phi : A \to B/K$ be a nuclear UCP map. By Theorem 2.1, the set of liftable maps in $\mathrm{UCP}(A, B/K)$ is closed in the point-norm topology. Since ϕ is the point-norm limit of a set of factorable UCP maps, it suffices to show that every factorable UCP map is liftable.

But if ϕ is a composition $\tau \circ \sigma$ where $\sigma : A \to M_n$ and $\tau : M_n \to B/K$ are UCP maps, then Proposition 2.3 implies that τ can be lifted to a UCP map $\tau_0 : M_n \to B$, and clearly $\tau_0 \circ \sigma$ is a lifting of $\phi = \tau \circ \sigma$. □

Corollary 3.2. *Every UCP map of a separable nuclear C^*-algebra A into a quotient B/K is liftable.*

References

[Arv77] W. Arveson. Notes on extensions of C^*-algebras. *Duke Math. J.*, 44:329–355, 1977.

[CE76] M.-D. Choi and E. Effros. The completely positive lifting problem for C^*-algebras. *Ann. Math.*, 104(3):585–609, 1976.