EXTENSIONS AND LIFTINGS

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There are three important steps in the proof the BDF theorem for essentially normal operators having essential spectrum $X \subseteq \mathbb{C}$.

(1) $\mathrm{Ext}(X)$ has a neutral element.
(2) $\mathrm{Ext}(X)$ is a group (i.e., has inverses).
(3) $\mathrm{Ext}(X)$ depends only on the homotopy class of X.

In this lecture we will exhibit the neutral element of $\mathrm{Ext}(X)$ and describe the generalization of that result to noncommutative C^*-algebras (without proof). We then show that $\mathrm{Ext}(X)$ is a group by the method of [Arv75], using a lifting theorem.

The third and most difficult step (homotopy invariance) has not been satisfactorily simplified. An account of the best proof known to date can be found in [Dav96].

1. *Ext*(X) has an identity

We have already pointed out that the direct sum of two operators in $\mathcal{EN}(X)$ is an operator in $\mathcal{EN}(X)$, so that the set $\mathrm{Ext}(X)$ of equivalence classes becomes a commutative semigroup with respect to the addition defined by $[A] + [B] = [A \oplus B]$. In this section we exhibit a neutral element for this operation. It is an instructive exercise to prove:

Proposition 1.1. *For a normal operator N, the following are equivalent.*

(i) *N has no isolated eigenvalues of finite multiplicity.*
(ii) *The C^*-algebra generated by N and $\mathbf{1}$ satisfies $C^*(N) \cap \mathcal{K} = \{0\}$.*
(iii) *$\sigma(N) = \sigma_e(N)$.*

In order to obtain such a normal operator with given spectrum X, one can choose a sequence $\lambda_1, \lambda_2, \dots$ of complex numbers that is dense in X and has the additional property that every isolated point of X occurs infinitely many times, and let N be a diagonal operator with the λ_k as its diagonal terms. The key fact is that the equivalence class of such an operator $[N]$ defines the neutral element of $\mathrm{Ext}(X)$. In more concrete terms, one has the following *absorbption* principle from [BDF77]:

Theorem 1.2 (BDF)**.** *Let X be a compact subset of $\mathbb{C}$ and let N be a normal operator with $\sigma(N) = X$ that satisfies the hypotheses of Proposition 1.1. Then for every $A \in \mathcal{EN}(X)$, the direct sum of operators $A \oplus N$ is unitarily equivalent to a compact perturbation of A.*

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For example, Theorem 1.2 implies that any two unitary operators U, V having full spectrum $\mathbb{T}$ must be approximately equivalent, since by Theorem 1.2 we have $U \sim U \oplus V \sim V \oplus U \sim V$. It also implies that the unilateral shift S *absorbs* every unitary operator U in the sense that $S \oplus U \sim S$.

Responding to a question of Halmos [Hal70] - *Is every operator on a separable Hilbert space a norm limit of reducible operators?* - Voiculescu showed by a very ingenious argument [Voi76] that the answer is yes by establishing a general result about compact perturbations of representaations of separable C^*-algebras. Voiculescu's result was recognized as the assertion that $\mathrm{Ext}(A)$ has a neutral element for arbitrary separable C^*-algebras A in [Arv77], where a more conceptual proof was given by introducing quasicentral approximate units. We now discuss Voiculescu's theorem briefly and show how it generalizes the Brown-Douglas-Fillmore result above. In the statement of this theorem, all Hilbert spaces are understood to be separable and infinite-dimensional.

Theorem 1.3 (Voiculescu). *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a separable C^*-algebra containing the identity operator, let $\pi : \mathcal{A} \to \mathcal{B}(K)$ be a $*$-representation of $\mathcal{A}$ on a separable Hilbert space K with the property that π vanishes on the (possibly trivial) ideal $\mathcal{A} \cap \mathcal{K}$, and let* id *be the identity representation of $\mathcal{A}$, $\mathrm{id}(A) = A$, $A \in \mathcal{A}$.*

Then $\mathrm{id} \oplus \pi \sim \mathrm{id}$ *in the sense that there is a sequence of unitary operators $U_n : H \oplus K \to H$ such that for every $T \in \mathcal{A}$ one has*

$$U_n(T \oplus \pi(T))U_n^* - T \in \mathcal{K}, \quad \text{and} \quad \lim_{n\to\infty} \|U_n(T \oplus \pi(T))U_n^* - T\| = 0.$$

In order to deduce Theorem 1.2 from Theorem 1.3, one chooses an essentially normal operator $A \in \mathcal{B}(H)$ having essential spectrum X, a normal operator $N \in \mathcal{B}(K)$ satisfying the hypotheses of Proposition 1.1, and considers the C^*-algebra $\mathcal{A} = C^*(A) \subseteq \mathcal{B}(H)$. Since the essential spectrum of A is $X = \sigma(N)$, an elementary argument shows that there is a unique $*$-representation $\pi : C^*(A) \to \mathcal{B}(K)$ that vanishes on $C^*(A) \cap \mathcal{K}$ and satisfies $\pi(A) = N$. By Theorem 1.3, there is a sequence of unitary operators $U_n : H \oplus K \to H$ such that $U_n(A \oplus N)U_n^* - A$ is compact for every $n = 1, 2, \dots$ and $\|U_n(A \oplus N)U_n^* - A\| \to 0$ as $n \to \infty$. In particular, one may conclude from this that $A \oplus N \sim A$.

2. $Ext(X)$ is a group

Let $X \subseteq \mathbb{C}$ be compact. We have just seen that the identity element of $\mathrm{Ext}(X)$ is the equivalence class $[N]$ of any normal operator N satisfying the conditions of Proposition 1.1. Thus, the assertion that $\mathrm{Ext}(X)$ is a group reduces to the following one:

Theorem 2.1. *For every operator $A \in \mathcal{EN}(X)$ there is an operator $B \in \mathcal{EN}(X)$ such that $A \oplus B$ is unitarily equivalent to an operator $N + K$ where K is compact and N is a normal operator satisfying the hypotheses of Proposition 1.1.*

The original BDF proof of Theorem 2.1 was difficult. We now sketch a simpler proof from [Arv75], which makes use of the following lifting theorem of T.-B. Andersen and J. Vesterstrøm (see [And74] and [Ves73]).

Theorem 2.2 (Andersen-Vesterstrøm)**.** *Let J be a closed ideal in a unital C^*-algebra A with the property that A/J is commutative and let $\pi : A \to A/J$ be the natural projection. There is a positive linear map $\phi : A/J \to A$ satisfying $\phi(\mathbf{1}) = \mathbf{1}$ and $\pi \circ \phi(x) = x$ for every $x \in A/J$.*

Sketch of proof of Theorem 2.1. Fix an operator $A \in \mathcal{EN}(X)$. The critical assertion of Theorem 2.1 is that there is a $B \in \mathcal{EN}(X)$ such that $A \oplus B$ is a compact perturbation $N + K$ of a normal operator N; one can then change N into another normal operator that satisfies the hypothesis of Proposition 1.1 by making use of Theorem 1.2 in a straightforward way (we omit that argument, see [Arv75]).

The operator B is obtained as follows. Consider the C^*-algebra $\mathcal{A} = C^*(A) + \mathcal{K}$ generated by A, $\mathbf{1}$, and the compact operators. We are given a faithful $*$-homomorphism $\theta : C(X) \to \mathcal{A}/\mathcal{K}$ that satisfies $\theta(\zeta) = A + \mathcal{K}$, ζ denoting the current variable $\zeta(\lambda) = \lambda$, $\lambda \in X$. Theorem 2.2 provides a unit-preserving positive linear map of $\mathcal{A}/\mathcal{K}$ into $\mathcal{A}$ and, composing that map with θ, we obtain a positive linear map $\phi : C(X) \to \mathcal{A}$ having the following properties:

(i) $\phi(\mathbf{1}) = \mathbf{1}$,
(ii) $\phi(fg) - \phi(f)\phi(g) \in \mathcal{K}$ for all $f, g \in C(X)$,
(iii) $\phi(\zeta) = A + K$, for some compact operator K.

Since a positive linear map of $C(X)$ must be completely positive, we can apply Stinespring's theorem to find a $*$-representation π of $C(X)$ on some other Hilbert space K and a bounded linear map $V : H \to K$ such that $\phi(f) = V^*\pi(f)V$, $f \in C(X)$. We may also assume that the pair (π, V) is *minimal* in the sense that K is spanned by the set of vectors $\pi(C(X))VH$, so that K is separable and π is nondegenerate, i.e., $\pi(\mathbf{1}) = \mathbf{1}$. Since $V^*V = \phi(\mathbf{1}) = \mathbf{1}$, V must be an isometry; so we can use V to identify H with a subspace of K in such a way that the Stinespring representation becomes $\phi(f) = P\pi(f) \restriction_H$, $f \in C(X)$, P denoting the projection of K onto its subspace H.

If we write K as a direct sum $K = H \oplus L$, then we obtain a 2×2 matrix representation for operators $\pi(f)$ of the form

$$\begin{pmatrix} \phi(f) & * \\ * & \psi(f) \end{pmatrix} \tag{2.1}$$

where $\psi : C(X) \to \mathcal{B}(L)$ is the unital CP map $\psi(f) = P^\perp\pi(f) \restriction_{H^\perp}$ and the off-diagonal terms are linear maps of $C(X)$ into appropriate spaces of linear operators from one space to another.

The key observation is that the off-diagonal terms of (2.1) are compact, or equivalently, that $P\pi(f) - \pi(f)P$ is compact for every $f \in C(X)$. To see that, choose $f, g \in C(X)$ and write

$$\begin{aligned} P\pi(f)P^{\perp}\pi(g)P &= P\pi(f)\pi(g)P - P\pi(f)P\pi(g)P \\ &= P\pi(fg)P - P\pi(f)P\pi(g)P = \phi(fg) - \phi(f)\phi(g). \end{aligned}$$

By property (ii) above, it follows that $P\pi(f)P^{\perp}\pi(g)P$ is compact; since f and g are arbitrary in $C(X)$ the compactness of $P\pi(f)-\pi(f)P$ for $f \in C(X)$ follows.

Now let N be the normal operator on $K = H \oplus L$ defined by $N = \pi(\zeta)$. Using the fact that $\phi(\zeta)$ is a compact perturbation of A, we find that

$$N = \begin{pmatrix} \phi(\zeta) & 0 \\ 0 & \psi(\zeta) \end{pmatrix} + \text{compact} = \begin{pmatrix} A & 0 \\ 0 & \psi(\zeta) \end{pmatrix} + \text{compact}.$$

So if we let $B = \psi(\zeta) \in \mathcal{B}(L)$ then we have shown that $A \oplus B$ is a compact perturbation of the normal operator N.

Note that B is an essentially normal operator satisfying $\sigma_e(B) \subseteq X$. Indeed, essential normality of B follows from the fact that both A and $A \oplus B$ are essentially normal, and a similar argument shows that $\sigma_e(B)$ is contained in the spectrum of N, namely X. Finally, if we replace B with the direct sum $B' = B \oplus M$ of B with a normal operator M having essential spectrum X, then B' will belong to $\mathcal{EN}(X)$ and will satisfy $A \oplus B' = \text{normal} + \text{compact}$, as required. □

Remark 2.3. We have only defined $\operatorname{Ext}(A)$ when $A = C(X)$ is a separable commutative C^*-algebra with unit. But there is a natural way to generalize the definitions we have given in these lectures to define Ext in this more general context. For example, one can define an extension of the compact operators by A to be a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow A \longrightarrow 0, \tag{2.2}$$

where $\mathcal{E} \subseteq \mathcal{B}(H)$ is a unital separable concrete C^*-algebra containing the compact operators, the map of $\mathcal{K}$ into $\mathcal{E}$ is inclusion, and the map of $\mathcal{E}$ to A is a $*$-homomorphism having kernel $\mathcal{K}$. Once that is done, we have the three questions (1), (2), (3) of the introduction. The third question relating to homotopy invariance remains somewhat mysterious in this more general setting. But the first two questions relating to the existence of neutral elements and inverses are by now well-understood, as we now describe.

As pointed out in [Arv77], Theorem 1.3 can be used to show that $\operatorname{Ext}(A)$ has a neutral element whenever A is a separable unital C^*-algebra. The existence of inverses was recalcitrant. The proof of Theorem 2.1 sketched above obviously will work for any C^*-algebra A for which one has an analogue of the lifting theorem of Andersen and Vesterstrøm, and for several years that problem was publicized. A solution was found for *nuclear* C^*-algebras by Choi and Effros in [CE76], and that will be the topic of my next (and hopefully last) lecture. In subsequent work, Choi and Effros [CE77] showed that that there are quotients of separable C^*-algebras whose natural projection cannot be lifted, and Joel Anderson built on that example to show that there is a separable unital C^*-algebra A such that $\mathrm{Ext}(A)$ is *not* a group [And78]. But the nature of Anderson's counter example was somewhat mysterious. Very recently, it was shown by Haagerup and Thorbjornsen that $\mathrm{Ext}(A)$ is not a group when A is the reduced C^*-algebra of the free group on two generators.

Finally, the reader may have already noted that there is a further generalization that is possible, namely by choosing a pair of separable C^*-algebras A and B, B being nonunital, and replacing extensions of $\mathcal{K}$ by A with extensions of B by A, giving rise to an object $\mathrm{Ext}(B, A)$. In fact, it appears that one should allow B to be unital, but should replace the sequence (2.2) with a sequence of the form

$$0 \longrightarrow \mathcal{K} \otimes B \longrightarrow \mathcal{E} \longrightarrow A \longrightarrow 0.$$

Final note: A currently popular way of approaching such issues is by way of Kasparov's bivariate KK functor. However, homotopy invariance is built into the *definition* of $KK(A, B)$, so that KK-theory in itself cannot shed light on the issue of homotopy-invariance for generalizations of the theory of extensions as we have described them above.

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