EXTENSIONS OF $\mathcal{K}$ BY $C(X)$

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An operator $A \in \mathcal{B}(H)$ whose self-commutator $A^*A - AA^*$ is compact is called *essentially normal*. Two operators $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are said to be *approximately equivalent* if they are unitarily equivalent modulo compact operators; more precisely, if there is a unitary operator $U : H \to K$ such that $B - UAU^*$ is compact. This relation is written $A \sim B$, whereas the stronger relation of unitary equivalence will be written $A \cong B$. Roughly speaking, $A \cong B$ means that A and B have the same geometric properties, while $A \sim B$ means that A and B have the same *asymptotic* properties (see Chapter 3 of [Arv01]). We begin by discussing the classification of essentially normal operators, and its generalization to the computation of $\operatorname{Ext}(X)$, originating in work of Brown, Douglas and Fillmore during the mid seventies [BDF77]. In a subsequent lecture we will describe the connection between those results, quasicentral approximate units and the lifting theorem for nuclear C^*-algebras.

1. Essentially Normal Operators and Extensions

Every operator $A \in \mathcal{B}(H)$ has an *essential spectrum* $\sigma_e(A)$, defined as the spectrum of the image of A in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}$. The essential spectrum of A is a nonvoid compact subset of the complex plane, and it provides an invariant for approximate equivalence:

$$A \sim B \implies \sigma_e(A) = \sigma_e(B).$$

On the surface of it, one might guess that the essential spectrum is a *complete* invariant for essentially normal operators. But the unilateral shift S and its adjoint S^* provide a simple example of two essentially normal operators having the same essential spectrum which are not approximately equivalent. Indeed, elementary computations show that both S and S^* are essentially normal operators with essential spectrum the unit circle $\mathbb{T}$. It follows that both S and S^* are Fredholm operators, and one observes that

$$\operatorname{ind} S = -1, \qquad \operatorname{ind} S^* = +1.$$

So if $S^* \sim S$ then S^* would be unitarily equivalent to a compact perturbation of S; but that would imply $\operatorname{ind} S^* = \operatorname{ind} S$ because the Fredholm index is stable under unitary equivalence and compact perturbations (see [Arv01]).

Given a compact subset X of the complex plane, one may consider the class $\mathcal{EN}(X)$ of all essentially normal operators A that act on a separable

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Hilbert space and have essential spectrum $\sigma_e(A) = X$. Strictly speaking, one must be careful in order to avoid set-theoretic anomalies in forming $\mathcal{EN}(X)$, and the easiest way to do that is to select a particular infinite-dimensional separable Hilbert space H_0 and define $\mathcal{EN}(X)$ to be the indicated subset of operators on H_0. With this convention, the direct sum of two operators A, $B \in \mathcal{B}(H_0)$ should be defined as $W^*(A \oplus B)W$, where $W : H_0 \to H_0 \oplus H_0$ is a unitary operator that is, once and for all, fixed. We will systematically ignore such issues, and instead we treat the proper class $\mathcal{EN}(X)$ as if it were a set, and treat the direct sum $A \oplus B$ in the usual way. Such set-theoretically naive conventions will not cause trouble so long as we limit ourselves to countable operations.

For a nonvoid compact subset $X \subseteq \mathbb{C}$, we define $\mathrm{Ext}(X)$ to be the set of equivalence classes $\mathcal{EN}(X)/\sim$. $\mathrm{Ext}(X)$ is in general an honest set, and it carries a natural binary operation + defined by $[A] + [B] = [A \oplus B]$, where $[A]$ denotes the equivalence class of an operator $A \in \mathcal{EN}(X)$. This addition is commutative and associative, making $\mathrm{Ext}(X)$ into a commutative semigroup. The problem of classifying essentially normal operators having essential spectrum X becomes that of a) determining any additional structure that may exist on $\mathrm{Ext}(X)$ and b) describing a set of concrete invariants for distinguishing between the elements of $\mathrm{Ext}(X)$.

We have already alluded to the fact that the Fredholm index provides a nontrivial invariant. More precisely, choose any operator $A \in \mathcal{B}(H)$ having essential spectrum X. For every complex number λ in the complement of X, the operator $A-\lambda = A-\lambda\mathbf{1}$ is a Fredholm operator whose index $\mathrm{ind}(A-\lambda)$ is a function of λ that is constant throughout each connected component of $\mathbb{C} \setminus X$, which vanishes identically on the unbounded component of $\mathbb{C} \setminus X$, and which is stable under compact perturbations. Thus we have defined an integer-valued function from the set of bounded components of $\mathbb{C} \setminus X$ that provides a concrete invariant for approximate equivalence in $\mathcal{EN}(X)$. The theorem of Brown, Douglas and Fillmore for subsets of the plane implies that this is a complete invariant:

Theorem 1.1 (BDF Theorem)**.** *Let A and B be essentially normal operators having the same essential spectrum $X \subseteq \mathbb{C}$. Then $A \sim B$ iff*

$$\mathrm{ind}(A-\lambda) = \mathrm{ind}(B-\lambda), \qquad \lambda \notin X.$$

Moreover, with respect to the operation defined by $[A]+[B] = [A\oplus B]$, $\mathrm{Ext}(X)$ is an abelian group and the map $X \to \mathrm{Ext}(X)$ defines a homotopy-invariant functor from compact subsets of $\mathbb{C}$ to abelian groups.

To say that $\mathrm{Ext}(X)$ is an abelian group involves two concrete assertions:

(i) There is an essentially normal operator N having essential spectrum X which acts as a neutral element in that $N \oplus A \sim A$ for every $A \in \mathcal{EN}(X)$.

(ii) For every $A \in \mathcal{EN}(X)$ there is a $B \in \mathcal{EN}(X)$ such that $A \oplus B \sim N$, where N is the "neutral" operator of (i).

Indeed, the main results of [BDF77] addressed a more general problem, in which $X \to \text{Ext}(X)$ was shown to be a homotopy-invariant functor from the category of compact metric spaces X to abelian groups, that in fact gives a concrete realization of K-homology by way of the theory of extensions of commutative C^*-algebras by the compact operators. It is significant that in the broader category of compact metric spaces X (or even compact C^∞ manifolds), there are invariants for $\text{Ext}(X)$ that are more subtle than those associated with the Fredholm index. For example, the group $\text{Ext}(X)$ can have torsion - while on the other hand, any invariant associated with the Fredholm index cannot detect torsion elements of $\text{Ext}(X)$. Thus, the realization of $\text{Ext}(X)$ as the K-homology of X provided essentially new information about almost commuting sets of operators on Hilbert spaces.

We now describe some of the key issues in the more general BDF theorem. Let X be a compact metrizable space and let $C(X)$ be the commutative C^*-algebra of complex-valued continuous functions on X. We will write $\mathcal{K}$ for the ideal of all compact operators on a given separable infinite-dimensional Hilbert space H. By an *extension of* $\mathcal{K}$ *by* $C(X)$ we mean a $*$-*monomorphism* $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ such that $\sigma(\mathbf{1}) = \mathbf{1}$. Given two Hilbert spaces H_1, H_2 and extensions $\sigma_k : C(X) \to \mathcal{B}(H_k)/\mathcal{K}$ we write $\sigma_1 \sim \sigma_2$ if there is a unitary operator $U : H_1 \to H_2$ such that

$$\theta_U(\sigma_1(f)) = \sigma_2(f), \qquad f \in C(X)$$

where θ_U is the $*$-isomorphism of Calkin algebras induced by the spatial $*$-isomorphism $T \in \mathcal{B}(H_1) \to UTU^* \in \mathcal{B}(H_2)$. $\text{Ext}(X)$ is defined as the set of equivalence classes of such maps σ. It is a good exercise to show that when X is a compact subset of the complex plane $\mathbb{C}$, a) extensions of $C(X)$ by $\mathcal{K}$ correspond to essentially normal operators with essential spectrum X, b) two operators determine the same extension iff they differ by a compact operator, and c) equivalence of extensions corresponds to approximate equivalence of operators.

It is useful to view extensions as short exact sequences of C^*-algebras in the following way. Given a $*$-monomorphism $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ as above, let $T \in \mathcal{B}(H) \mapsto \dot{T} \in \mathcal{B}(H)/\mathcal{K}$ be the natural projection onto the Calkin algebra, and consider the associated short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow_\pi C(X) \longrightarrow 0, \tag{1.1}$$

where $\mathcal{E} = \{T \in \mathcal{B}(H) : \dot{T} \in \sigma(C(X))\}$, the map of $\mathcal{K}$ to $\mathcal{E}$ is inclusion, and $\pi : \mathcal{E} \to C(X)$ is given by composing the natural map of $\mathcal{E}$ to the Calkin algebra with the inverse of σ, $\pi(T) = \sigma^{-1}(\dot{T})$. Conversely, every exact sequence of the form (1.1) arises from a uniquely determined extension $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ as defined in the preceding paragraph.

Notice that an exact sequence of the form (1.1) is defined uniquely by specifying a pair $(\mathcal{E}, \pi)$ consisting of a C^*-algebra $\mathcal{E}$ of operators on H that contains $\mathcal{K}$ together with a surjective $*$-homomorphism $\pi : \mathcal{E} \to C(X)$ that satisfies $\ker \pi = \mathcal{K}$. Two sequences such as (1.1) are said to be *equivalent* if their associated pairs $(\mathcal{E}_k, \pi_k)$ are related as follows: there is a $*$-isomorphism $\theta : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\pi_2 \circ \theta = \pi_1$. This equivalence relation can also be viewed as an equivalence relation existing between short exact sequences of the form (1.1), and it is denoted $\sim$. Since both $\mathcal{E}_1$ and $\mathcal{E}_2$ contain the compact operators and $\ker \pi_k = \mathcal{K}$, it is a straightforward exercise to show that both the equivalence map θ and its inverse must carry compact operators to compact operators, and is therefore implemented by a unitary operator $U : H_1 \to H_2$ by way of $\theta(T) = UTU^*$, $T \in \mathcal{E}_1$. In this way one sees that two short exact sequences of the form (1.1) with pairs $(\mathcal{E}_1, \pi_1)$ and $(\mathcal{E}_2, \pi_2)$ are equivalent iff their associated extensions σ_1 and σ_2 are equivalent.

References

[Arv01] W. Arveson. *A Short Course on Spectral Theory*, volume 209 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[BDF77] L.G. Brown, R. Douglas, and P. Fillmore. Extensions of C^*-algebras and K-Homology. *Ann Math.*, 105(2):265–324, March 1977.