THE NONCOMMUTATIVE HAHN-BANACH THEOREMS

WILLIAM ARVESON

The Hahn-Banach theorem in its simplest form asserts that a bounded linear functional defined on a subspace of a Banach space can be extended to a linear functional defined everywhere, *without increasing its norm*. There is an order-theoretic version of this extension theorem (Theorem 0.1 below) that is often more useful in context. The purpose of these lecture notes is to discuss the noncommutative generalizations of these two results and their relation to each other. We make use of several standard terms such as *operator space, operator system, n-positive, n-contractive, completely positive, completely contractive*, and refer the reader to [Pau02] for definitions.

The original proof of the extension theorem for completely positive maps is found in [Arv69]. I will sketch a somewhat simplified proof that is based on the following theorem of M. G. Krein (see page 63 of [Nai70]).

Theorem 0.1 (Krein)**.** *Let P be a cone in a real topological vector space X such that the interior of P is nonempty. Let M be a linear subspace of X and let $f : M \to \mathbb{R}$ be a linear functional such that $f(M \cap P) \subseteq [0, \infty)$. Then f can be extended to a linear functional $\tilde{f}$ on X satisfying $\tilde{f}(P) \subseteq [0, \infty)$.*

A straightforward application of Krein's theorem to the cone P of all positive elements of a C^*-algebra leads to the following extension theorem for complex-linear functionals defined on operator systems.

Corollary 0.2. *Let S be an operator system in a unital C^*-algebra A and let $f : S \to \mathbb{C}$ be a complex-linear functional such that $f(S^+) \subseteq [0, \infty)$, S^+ denoting the set of positive elements of S. Then f can be extended to a positive linear functional on A.*

Aside from a compactness argument that will be described in the lecture, the key assertion of the completely positive extension theorem is the following result about extending completely positive maps into matrix algebras.

Theorem 0.3. *Let S be an operator system in a unital C^*-algebra A and let H be a finite-dimensional Hilbert space. Then every completely positive linear map $\phi : S \to \mathcal{B}(H)$ can be extended to a completely positive map of A into $\mathcal{B}(H)$.*

Sketch of proof. It suffices to show that there is a Hilbert space K a representation $\pi : A \to \mathcal{B}(K)$, and a linear operator $V : H \to K$ such that

---

Date: 8 September, 2003.

$\phi(s) = V^*\pi(s)V$, $s \in S$; that is because the map $a \in A \mapsto V^*\pi(a)V$ is rather obviously completely positive.

To that end, let $n = \dim H$, let $\xi_1, \dots, \xi_n$ be a linearly independent set that spans H, and consider the linear functional f defined on $M_n(S)$ by

$$f((s_{ij})) = \sum_{i,j=1}^{n} \langle \phi(s_{ij})\xi_j, \xi_i \rangle, \tag{0.1}$$

(s_{ij}) denoting the $n \times n$ matrix with entries s_{ij}. f is a positive linear functional on $M_n(S)$ because ϕ is n-positive. Corollary 0.2 implies that there is a positive linear functional g on $M_n(A)$ that extends f. By the GNS construction, we obtain a Hilbert space $\bar{K}$, a representation $\bar{\pi}$ of A on K, and a vector $\bar{\eta} \in K$ such that $g(x) = \langle \bar{\pi}(x)\bar{\eta}, \bar{\eta} \rangle$, for all $x \in M_n(A)$. A bit of reflection and a straightforward computation shows that we can realize $\bar{K}$ as a direct sum of n copies of a single Hilbert space K, $\bar{\eta}$ as a column vector with n components $\eta_i \in K$, and a single representation $\pi : A \to \mathcal{B}(K)$ such that $\bar{\pi}((x_{ij}))$ is given by an $n \times n$ operator matrix $(\pi(x_{ij}))$ as follows

$$g((x_{ij})) = \langle (\pi(x_{ij}))\bar{\eta}, \bar{\eta} \rangle = \sum_{ij} \langle \pi(x_{ij})\eta_j, \eta_i \rangle.$$

If we let V be the unique linear map of H to K that satisfies $V\xi_k = \eta_k$, $1 \le k \le n$, and choose $x_{ij} = s_{ij} \in S$, then the above formula implies

$$\sum_{i,j=1}^{n} \langle \phi(s_{ij})\xi_j, \xi_i \rangle = g((s_{ij})) = \sum_{i,j=1}^{n} \langle \pi(s_{ij})V\xi_j, V\xi_i \rangle = \sum_{i,j=1}^{n} \langle V^*\pi(s_{ij})V\xi_j, \xi_i \rangle.$$

Since the s_{ij} can be chosen arbitrarily in S, the latter formula implies that for fixed $s \in S$ and all i, j between 1 and n, we have

$$\langle \phi(s)\xi_j, \xi_i \rangle = \langle V^*\pi(s)V\xi_j, \xi_i \rangle$$

from which the required assertion is evident. □

After a preprint of [Arv69] was circulated, I received a letter from George Elliott outlining the above argument. The original proof of Theorem 0.3 in [Arv69] made no use of Krein's theorem, but was somewhat more involved. The basic idea of both proofs, namely that of using duality to convert statements about matrix-valued maps to statements about functionals, can be embellished. See chapter 6 of [Pau02] for a generalization and a more systematic organization of the details.

As I have already pointed out, a compactness argument allows one to generalize Theorem 0.3 to the case of infinite dimensional Hilbert spaces, and the latter result is a noncommutative generalization of the Hahn-Banach theorem in its *order-theoretic* form, namely Krein's Theorem 0.1. I will sketch this compactness argument in the lecture (it is reproduced in Chapter 7 of [Pau02]). More than ten years went by before anyone looked seriously for a version of the extension theorem for operator spaces. Finally, in 1981, Gerd Wittstock proved the following result [Wit81].

Theorem 0.4 (Wittstock)**.** *Let S be a linear subspace of a unital C^*-algebra A and let $\phi : S \to \mathcal{B}(H)$ be a completely bounded linear map. Then ϕ can be extended to a linear map $\tilde{\phi} : A \to \mathcal{B}(H)$ such that $\|\tilde{\phi}\|_{\rm cb} = \|\phi\|_{\rm cb}$.*

Wittstock's proof of Theorem 0.4 was somewhat involved, by way of generalizing the Hahn-Banach theorem to set-valued maps into $\mathcal{B}(H)$. Soon afterward, Paulsen discovered a simple device that allows one to deduce Theorem 0.4 from the extension theorem for completely positive maps [Pau82]. We now discuss Paulsen's trick, then we sketch the proof of Theorem 0.4.

Lemma 0.5 (Paulsen)**.** *Let S be an operator space in a unital C^*-algebra A, and let $\phi : S \to \mathcal{B}(H)$ be an operator-valued linear map. Consider the operator system $\tilde{S} \subseteq M_2(A)$ defined by*

$$\tilde{S} = \{\begin{pmatrix} a\mathbf{1} & s \\ t^* & b\mathbf{1} \end{pmatrix} : s, t \in S, \quad a, b \in \mathbb{C}\},$$

and the operator-valued linear map $\Phi : \tilde{S} \to M_2(\mathcal{B}(H))$ defined by

$$\Phi(\begin{pmatrix} a\mathbf{1} & s \\ t^* & b\mathbf{1} \end{pmatrix}) = \begin{pmatrix} a\mathbf{1} & \phi(s) \\ \phi(t)^* & b\mathbf{1} \end{pmatrix}.$$

If ϕ is completely contractive, then Φ is completely positive.

Sketch of proof. Assuming that ϕ is n-contractive for some $n = 1, 2, \dots$, we will show that $\Phi_n : M_n(\tilde{S}) \to M_n(\mathcal{B}(H))$ is positive. For this, we identify $M_n(M_2(A))$ with the C^*-algebra of all 2×2 operator matrices of the form

$$\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}, \qquad A, B, X, Y \in M_n(A).$$

This follows because the natural map of $M_n(M_2(A))$ onto $M_2(M_n(A))$ is a $*$-isomorphism. In this identification, $M_n(\tilde{S})$ becomes the set of matrices

$$\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix},$$

where A, B belong to $M_n(\mathbb{C})$ and X, Y belong to $M_n(S)$. Let us choose a positive element T of this form in $M_n(\tilde{S})$. Then A and B are positive $n \times n$ matrices and $Y = X$, so that for every $\epsilon > 0$, $A_\epsilon = A + \epsilon\mathbf{1}$ and $B_\epsilon = B + \epsilon\mathbf{1}$ are invertible positive elements of $M_n(\mathbb{C})$, and we have

$$T + \epsilon\mathbf{1} = \begin{pmatrix} A_\epsilon & X \\ X^* & B_\epsilon \end{pmatrix} = \begin{pmatrix} A_\epsilon^{1/2} & 0 \\ 0 & B_\epsilon^{1/2} \end{pmatrix} \begin{pmatrix} \mathbf{1} & Y_\epsilon \\ Y_\epsilon^* & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_\epsilon^{1/2} & 0 \\ 0 & B_\epsilon^{1/2} \end{pmatrix},$$

where $Y_\epsilon = A_\epsilon^{-1/2} X B_\epsilon^{-1/2}$ is an element of $M_n(S)$. Since $T + \epsilon\mathbf{1}$ is positive it follows that $\begin{pmatrix} \mathbf{1} & Y_\epsilon \\ Y_\epsilon^* & \mathbf{1} \end{pmatrix} \geq 0$, which in turn is equivalent to $\|Y_\epsilon\| \leq 1$.

To show that $\Phi_n(T)$ is positive, it suffices to show that $\Phi_n(T + \epsilon\mathbf{1}) \geq 0$ for every $\epsilon > 0$. After noting that A_ϵ and B_ϵ are scalar matrices we find that $\phi_n(Y_\epsilon) = A_\epsilon^{-1/2}\phi_n(X)B_\epsilon^{-1/2}$, and moreover

$$\Phi_n(T+\epsilon\mathbf{1}) = \begin{pmatrix} A_\epsilon & \phi_n(X) \\ \phi_n(X)^* & B_\epsilon \end{pmatrix}$$
$$= \begin{pmatrix} A_\epsilon^{1/2} & 0 \\ 0 & B_\epsilon^{1/2} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \phi_n(Y_\epsilon) \\ \phi_n(Y_\epsilon)^* & \mathbf{1} \end{pmatrix} \begin{pmatrix} A_\epsilon^{1/2} & 0 \\ 0 & B_\epsilon^{1/2} \end{pmatrix}.$$

Now $\|\phi_n(Y_\epsilon)\| \le 1$ because ϕ is n-contractive, and $\|\phi_n(Y_\epsilon)\| \le 1$ is equivalent to the assertion $\begin{pmatrix} \mathbf{1} & \phi_n(Y_\epsilon) \\ \phi_n(Y_\epsilon)^* & \mathbf{1} \end{pmatrix} \ge 0$. It follows that the right side of the preceding equation is positive, hence $\Phi_n(T+\epsilon\mathbf{1}) \ge$ as required. □

While we will not require the fact, we remark that the converse of Lemma 0.5 is true as well; indeed, the reader can easily adapt the above argument to show that for every $n = 1, 2, \dots$, ϕ is n-contractive iff Φ is n-positive.

We now indicate how one deduces Theorem 0.4 from the extension theorem for completely positive linear maps via Lemma 0.5.

Proof of Theorem 0.4. In order to prove Theorem 0.4 it is enough to show that that every linear map $\phi : S \to \mathcal{B}(H)$ that is completely contractive (i.e., $\|\phi\|_{\rm cb} \le 1$) has a completely contractive linear extension to a map of A into $\mathcal{B}(H)$. By Lemma 0.5, the associated map $\Phi : \tilde{S} \to M_2(\mathcal{B}(H))$ is completely positive. By the extension theorem for completely positive maps, Φ can be extended to a completely positive map $\tilde{\Phi}$ of $M_2(A)$ into $M_2(\mathcal{B}(H))$. Let $\tilde{\phi}$ be the linear map of A into $\mathcal{B}(H)$ defined by

$$\tilde{\Phi}(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} * & \tilde{\phi}(x) \\ * & * \end{pmatrix}, \qquad x \in A.$$

Obviously, $\tilde{\phi}$ is an extension of ϕ. Now $\tilde{\Phi}$ may be viewed as a completely positive unit-preserving operator valued map defined on a C^*-algebra $M_2(A)$, and therefore it has a Stinespring decomposition of the form

$$\tilde{\Phi}(X) = V^*\pi(X)V, \qquad X \in M_2(A)$$

where π is a representation of $M_2(A)$ and V is a linear operator between appropriate Hilbert spaces satisfying $V^*V = \tilde{\Phi}(\mathbf{1}) = \mathbf{1}$. Hence V is an isometry and $\tilde{\Phi}$ is a *completely contractive* map. It is now a routine matter to check that $\tilde{\phi}$ must also be completely contractive (see p. 100 of [Pau02]). □

References

[Arv69] W. Arveson. Subalgebras of C^*-algebras. *Acta Math.*, 123:141–224, 1969.

[Nai70] M. A. Naimark. *Normed Rings*. Wolters–Noordhoff, Groningen, 1970.

[Pau82] V. Paulsen. Completely bounded maps on C^*-algebras and invariant opertor ranges. *Proc. AMS*, 86:91–96, 1982.

[Pau02] V. Paulsen. *Completely bounded maps and Operator Algebras*. Cambridge, Cambridge, UK, 2002.

[Wit81] G. Wittstock. Ein operatorwertiger Hahn-Banach satz. *J. Funct. Anal.*, 40:127–150, 1981.